Remark on fractional Laplacians

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Abstract

We compare norms of u and $|u|$ generated by fractional Lapalcian (restricted or spectral) of order $s \in (1, 3/2)$, in a bounded domain $\Omega \subset \mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Recall that the spectral fractional Dirichlet Laplacian (or the "Navier" fractional Laplacian) of order $m \in \mathbb{R}_+$ on Ω is given by the formula

$$(-\Delta)^m_{sD} u = \sum_{j=1}^{\infty} \lambda_j^m \Big(\int\limits_{\Omega} u\varphi_j \Big) \varphi_j \,,$$

where λ_j and φ_j, $j \geq 1$, are, respectively, the eigenvalues and (orthonormal in $L_2(\Omega)$) eigenfunctions of the conventional Dirichlet Laplacian $-\Delta_D$ in Ω. Here the series converges in the sence of distributions.

It is well known (see, e.g., classical monograph [3]), that for $0 < m < 3/2$ the domain of the corresponding quadratic form $Q^{sD}_m[u] = \langle (-\Delta)^m_{sD}\, u, u\rangle$ is the space

$$\widetilde{H}^m(\Omega) := \{u \in H^m(\mathbb{R}^n) \mid u \equiv 0 \ \text{ on } \ \mathbb{R}^n \setminus \overline{\Omega}\}.$$

Next, the restricted fractional Dirichlet Laplacian of order $m \in \mathbb{R}_+$ is defined via the Fourier transform by

$$\mathcal{F}[(-\Delta)^m_{rD}\, u](\xi) = |\xi|^{2m} \mathcal{F}[u](\xi) = \frac{|\xi|^{2m}}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx \,.$$

The domain of its quadratic form $Q^{rD}_m[u] = \langle (-\Delta)^m_{rD}\, u, u\rangle$ is also the space $\widetilde{H}^m(\Omega)$ (for all $m \in \mathbb{R}_+$).

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From a general result of [1] it follows that for $0 < m < 3/2$ the nonlinear operator $u \mapsto |u|$ maps $\widetilde{H}^m(\Omega)$ into itself. So, we can try to compare $Q_m^{sD}[|u|]$ and $Q_m^{sD}[u]$ ($Q_m^{rD}[|u|]$ and $Q_m^{rD}[u]$) for $u \in \widetilde{H}^m(\Omega)$.

Obviously, for $m = 1$ we have

$$Q_1^{sD}[|u|] \equiv Q_1^{sD}[u] \equiv Q_1^{rD}[|u|] \equiv Q_1^{rD}[u] \equiv \int\limits_{\Omega} |\nabla u|^2 \ , \qquad u \in \widetilde{H}^1(\Omega) = H_0^1(\Omega).$$

In contrast, the following result was established in [2, Theorem 3]:

Let $0 < m < 1$. For any $u \in \widetilde{H}^m(\Omega)$ such that $u \neq |u|$ (i.e. u changes sign) we have $Q_m^{sD}[|u|] < Q_m^{sD}[u]$ and $Q_m^{rD}[|u|] < Q_m^{rD}[u]$.

Here we show that for $1 < m < 3/2$ the reverse inequalities hold.

1 Restricted operators

Lemma 1. *Let $u, v \in \mathcal{C}_0^\infty$ be nonnegative functions, and $\operatorname{supp}(u) \cap \operatorname{supp}(v) = \emptyset$. Then for $m > 0$, $m \notin \mathbb{N}$ we have*

$$\begin{aligned} \langle (-\Delta)^m_{rD}\, u, v\rangle &< 0, \quad \text{if} \quad \lfloor m \rfloor \quad \text{is} \quad \text{even};\\ \langle (-\Delta)^m_{rD}\, u, v\rangle &> 0, \quad \text{if} \quad \lfloor m \rfloor \quad \text{is} \quad \text{odd}. \end{aligned}$$

Proof. We write

$$\begin{aligned} &\langle (-\Delta)^m_{rD}\, u, v\rangle = \langle (-\Delta)^{m-\lfloor m \rfloor}_{rD}\, u, (-\Delta_D)^{\lfloor m \rfloor}\, v\rangle \\ &= C_{n,m} \cdot \int\limits_{\Omega} V.P. \int\limits_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2(m-\lfloor m \rfloor)}}\, dy \ (-\Delta_D)^{\lfloor m \rfloor}\, v(x)\, dx, \end{aligned}$$

Here $V.P.$ stands for the principal value of the integral while $C_{n,m} = 2^{2m+\frac{n}{2}} \frac{\Gamma(m+\frac{n}{2})}{\Gamma(-m)}$.

Since the supports of u and v are separated, we have

$$\langle (-\Delta)^m_{rD}\, u, v\rangle = -C_{n,m} \cdot \int\limits_{\Omega} \int\limits_{\Omega} \frac{u(y)\ (-\Delta_D)^{\lfloor m \rfloor}\, v(x)}{|x-y|^{n+2(m-\lfloor m \rfloor)}}\, dydx.$$

We integrate by parts and use the relation $-\Delta_D |x|^{-(n+a)} = -(n+a)(a+2)|x|^{-(n+a+2)}$, and the statement follows. $\square$

Theorem 1. *Let $m > 0$, $m \notin \mathbb{N}$. Suppose that u is such that $u_+, u_- \in \widetilde{H}^m(\Omega)$, $u_\pm \neq 0$. Then*

$$Q_m^{rD}[|u|] < Q_m^{rD}[u], \quad \text{if} \quad \lfloor m \rfloor \quad \text{is} \quad \text{even};$$
$$Q_m^{rD}[|u|] > Q_m^{rD}[u], \quad \text{if} \quad \lfloor m \rfloor \quad \text{is} \quad \text{odd}.$$

Proof. For $u_+, u_- \in \mathcal{C}_0^\infty$ with separated supports, the statement follows from Lemma and relations $u = u_+ - u_-$, $|u| = u_+ + u_-$. In general case we proceed by approximation. □

Corollary. *Let $1 < m < 3/2$. Then $Q_m^{rD}[|u|] > Q_m^{rD}[u]$ for all $u \in \widetilde{H}^m(\Omega)$ such that $u \neq |u|$.*

Proof. This statement follows from [1] and Theorem 1. □

2 Spectral operators

Lemma 2. *Let $m < 0$ or $m \in (1, 3/2)$. Then $Q_m^{sD}[|\varphi_k|] > Q_m^{sD}[\varphi_k]$ for any $k \geq 2$.*

Proof. Denote by $a_j = \int\limits_\Omega |\varphi_k|\varphi_j$ the Fourier coefficients of the function $|\varphi_k|$. Then for arbitrary $m < 3/2$

$$Q_m^{sD}[|\varphi_k|] = \sum_{j=1}^{\infty} \lambda_j^m \, |a_j|^2; \qquad Q_m^{sD}[\varphi_k] = \lambda_k^m.$$

It is well known that φ_k changes sign. Hence $a_j \neq 0$ for some $j \neq k$, and therefore

$$\frac{d^2}{dm^2} \left(\frac{Q_m^{sD}[|\varphi_k|]}{Q_m^{sD}[\varphi_k]} \right) = \sum_{j=0}^{\infty} \left(\frac{\lambda_j}{\lambda_k} \right)^m |a_j|^2 \ln^2 \left(\frac{\lambda_j}{\lambda_k} \right) > 0,$$

Thus, the quotient in the left-hand side is strictly convex in m. Since $Q_0^{sD}[|\varphi_k|] = Q_0^{sD}[\varphi_k]$ and $Q_1^{sD}[|\varphi_k|] = Q_1^{sD}[\varphi_k]$, the statement follows. □

Theorem 2. *Let $1 < m < 3/2$. Then $Q_m^{sD}[|u|] > Q_m^{sD}[u]$ for all $u \in \widetilde{H}^m(\Omega)$ such that $u \neq |u|$.*

Proof. First, let $u = b_1\varphi_1 + b_i\varphi_i + b_k\varphi_k$. Denote by a_j the Fourier coefficients of the function $|u|$. Then

$$Q_m^{sD}[|u|] = \sum_{j=1}^{\infty} \lambda_j^m \, |a_j|^2; \qquad Q_m^{sD}[u] = \lambda_1^m \, |b_1|^2 + \lambda_i^m \, |b_i|^2 + \lambda_k^m \, |b_k|^2.$$

Note that $a_1 > |b_1|$ since $u \neq |u|$. Suppose that $|b_i| > |a_i|$ and $|b_k| > |a_k|$ (other possibilities only simplify the proof). Then we can choose $c'_j, c''_j \geq 0$ such that

$$c'_1 + c''_1 = |a_1|^2 - |b_1|^2; \qquad c'_i = c''_i = c'_k = c''_k = 0; \qquad c'_j + c''_j = |a_j|^2, \quad j \neq 1, i, k;$$

$$\sum_{j=1}^{\infty} c'_j = |b_i|^2 - |a_i|^2; \qquad \sum_{j=1}^{\infty} c''_j = |b_k|^2 - |a_k|^2;$$

$$\sum_{j=1}^{\infty} \lambda_j c'_j = \lambda_i (|b_i|^2 - |a_i|^2); \qquad \sum_{j=1}^{\infty} c''_j = \lambda_k (|b_k|^2 - |a_k|^2).$$

By the same argument as in previous Lemma, we have for $1 < m < 3/2$

$$\sum_{j=1}^{\infty} \lambda_j^m c'_j > \lambda_i^m (|b_i|^2 - |a_i|^2); \qquad \sum_{j=1}^{\infty} \lambda_j^m c''_j > \lambda_k^m (|b_k|^2 - |a_k|^2),$$

and thus $Q_m^{sD}[|u|] > Q_m^{sD}[u]$.

For arbitrary finite linear combination of eigenfunctions the arguments are the same. In general case we proceed by approximation. □

References

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