Finite dimensional aspect of existence of non-nuclear operators with s-nuclear adjoints

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General Reference for Definitions, main Questions under consideration etc is

A. Grothendieck: *Produits tensoriels topologiques et éspaces nucléaires*, Mem. Amer. Math. Soc., **16**(1955).

Our considerations will revolve around the following question of A. Grothendieck:

Question

Suppose T is a (bounded linear) operator acting in a Banach space X. Is it true that if T^* is nuclear then T is nuclear too?

- A. Grothendieck has shown that

Theorem

The answer is positive if X^* possesses the approximation property (AP).

Introduction — AP_s

$T \in \mathcal{F}(X) \subset L(X)$, $Tx = \sum_{k=1}^m \langle f_k', x\rangle g_k$; trace $T = \sum_{k=1}^m \langle f_k', g_k\rangle$ — does not depend on a representation.
Also, $T \in L(X)$. Consider a *nuclear* representation

$$Tx = \sum_{k=1}^{\infty} \langle x_k', x\rangle x_k, \ \sum_{k=1}^{\infty} ||x_k'|| \, ||x_k|| < \infty$$

and

$$\alpha := \sum_{k=1}^{\infty} \langle x_k', x_k\rangle .$$

- **Question:** $\alpha =$ trace T?
- Generally, NO.

Enflo P. , A counterexample to the approximation property in Banach spaces, Acta Math., Volume 130, 1973, 309–317

Remark: If T is nuclear, then $T : X \to c_0 \to l_1 \to X$.

Remark: $\alpha = \text{trace } T$ for all T and all nuclear representations of T means $X \in AP$.

Now, let in a nuclear representation of T

$$Tx = \sum_{k=1}^{\infty} \langle x'_k, x \rangle x_k,$$

we have, for $s \leq 1$,

$$\sum_{k=1}^{\infty} ||x'_k||^s \, ||x_k||^s < \infty$$

(T is an *s-nuclear operator*).
Again:

- **Question:** $\alpha = \text{trace } T$?
- Generally, NO — for every $s \in (2/3, 1]$.

Davie A.M., The approximation problem for Banach spaces, Bull. London Math. Soc., Vol 5, 1973, 261–266

$s = 2/3$ and AP_s

BUT:

- if $s = 2/3$, then for any above s-nuclear representation of T

$$\text{trace } T = \sum_{k=1}^{\infty} \langle x'_k, x_k \rangle.$$

A. Grothendieck: *Produits tensoriels topologiques et éspaces nucléaires*, Mem. Amer. Math. Soc., **16**(1955).

- *Definition*: If $\alpha = \text{trace } T$ for all T and all s-nuclear representations of T, then $X \in AP_s$.

Note that $AP = AP_1$. Every Banach space has the $AP_{2/3}$.

Retreat for Examples

Examples

- AP : $C(K), L_p(\mu), A, L_\infty/H^\infty$ *etc;*
- $\forall\ p \in [1, \infty] \setminus \{2\}\ \exists\ X \subset l_p : \quad X \notin AP$;
- $L(H) \notin AP$, H^∞ — *not known;*
- AP_s : *if* $s \in [2/3, 1], 1/p + 1/2 = 1/s$, *then* $\forall\ X \subset L_p/E$ $(E \subset L_p)$;
- $AP_{2/3}$: *all.*

Comparing nuclearity of T and T^*

Now, let $T : X \to X$, $\pi : X \hookrightarrow X^{**}$, so

$$\pi T : X \to X \hookrightarrow X^{**}$$

(T is f. r. or any).

- **Question:** Suppose

$$\pi Tx = \sum_{k=1}^{\infty} \langle x'_k, x\rangle x''_k, \ \sum_{k=1}^{\infty} ||x'_k||\, ||x''_k|| < \infty.$$

  Is it true:

$$Tx = \sum_{k=1}^{\infty} \langle y'_k, x\rangle y_k,$$

  with $\sum_{k=1}^{\infty} ||y'_k||\, ||y_k|| < \infty$?
- Generally, NOT. If T is f. rank, then YES (evidently).

Figiel T., Johnson W.B., The approximation property does not imply the bounded approximation property, Proc. Amer. Math. Soc. 41, 197–200 (1973)

BUT ("quantitative" question):

- Let, *for a finite rank* T,

$$\pi Tx = \sum_{k=1}^{\infty} \langle x'_k, x \rangle x''_k,$$

$$\sum_{k=1}^{\infty} ||x'_k|| \, ||x''_k|| < 1.$$

Is it true:

$$Tx = \sum_{k=1}^{\infty} \langle y'_k, x \rangle y_k,$$

$$\sum_{k=1}^{\infty} ||y'_k|| \, ||y_k|| < 1?$$

(i.e., if the nuclear norm $\nu(\pi T) < 1$, then $\nu(T) < 1$?)

Let us give an estimation of type $\nu(T) \leq C\, \nu(\pi T)$.

- **An estimation.** Let $T \in \mathcal{F}(X), \dim T(X) = N.$

$$\pi T : X \to X \hookrightarrow X^{**}, \ \ T(X) \subset X^{**}.$$

$\exists\ P : X^{**} \overset{\text{onto}}{\to} T(X)$ with $||P|| \leq \sqrt{N}.$

M. J. Kadec, M. G. Snobar: *Certain functionals on the Minkowski compactum* (Russian), Mat. Zametki **10** (1971), 453-458.

We have:

$$T = iP\pi T : X \overset{T}{\to} X \overset{\pi}{\hookrightarrow} X^{**} \overset{P}{\to} T(X) \overset{i}{\hookrightarrow} X.$$

Therefore,

$$\nu(T) = \nu(iP\pi T) \leq ||iP||\, \nu(\pi T) \leq \sqrt{N}\, \nu(\pi T).$$

OR:

- *If* $\nu(\pi T) < 1$, *then* $\nu(T) < \sqrt{N}$.

It is sharp.

Comparing nuclearity of T and T^* — a theorem

Theorem

(O. Reinov)

- *If T is an n-dimensional operator in a Banach space with nuclear norm $\nu(T) = 1$, then $\nu(T^*) \geq n^{-\frac{1}{2}}$.*
- *There exist a separable Banach space X with the AP, a sequence of operators $(z_n), z_n : X \to X$, and a constant $C > 0$ such that $\dim z_n = n, \nu(z_n) \geq \operatorname{trace} z_n = 1$ and $\nu(z_n^*) \leq C\, n^{-\frac{1}{2}}$.*

Remark: The last space X has the AP, but does not have the BAP (roughly speaking, BAP is a property of a space X meaning that the second statement is not true). The first example was given in

Figiel T., Johnson W.B, The approximation property does not imply the bounded approximation property, Proc. Amer. Math. Soc., 41 (1973), 197–200

If we will have time, we will give a sketch of the proof.

s-nuclear operators – Applications de puissance p.éme sommable

- Recall that an operator $T : X \to Y$ is s-nuclear $(0 < s \leq 1)$ if it is of the form
$$Tx = \sum_{k=1}^{\infty} \langle x'_k, x \rangle y_k$$
for all $x \in X$, where
$(x'_k) \subset X^*, (y_k) \subset Y$, $\sum_k ||x'_k||^s \, ||y_k||^s < \infty$. We use the notations $N_s(X, Y)$ and $\nu_s(T)$ for $\inf(\sum_k ||x'_k||^s \, ||y_k||^s)^{1/s}$.
In 2014, answering a question of A. Hinrichs and A. Pietsch (2010, [Problem 10.1]), we have found some sharp conditions for a operator in Banach spaces to be nuclear, if its adjoint is s-nuclear $(0 < s < 1)$.

- A. Hinrichs, A. Pietsch, *p*-nuclear operators in the sense of Grothendieck, Math. Nachr., Volume 283, No. 2 (2010), 232–261.

Recall a part of the conditions.

When s-nuclearity of T^* implies nuclearity of T?

Theorem

Let $s \in (0, 1]$, $T \in L(X)$ *and assume that* $X^* \in AP_s$. *If* $\pi T \in N_s(X, X^{**})$, *then* $T \in N_1(X, X)$.

In other words, under these conditions, from the s-nuclearity of the conjugate operator T^*, it follows that the operator T is nuclear.

O. I. Reinov, On linear operators with s-nuclear adjoints, $0 < s \leq 1$, J. Math. Anal. Appl., Volume 415 (2014) 816-824.

It was shown also that the above condition is sharp. Now, we present some finite dimensional analogues of these results.
But before, we give a new "quantitative" version of the above theorem.

When s-nuclearity of T^* implies nuclearity of T?

Theorem

Let $s \in (0,1]$, $T \in L(X)$ *and assume that* $X^* \in AP_s$. *If* $\pi T \in N_s(X, X^{**})$ *and* $\nu_s(\pi T) < 1$, *then* $T \in N_1(X, X)$ *and* $\nu(T) < 1$.

In other words, under these conditions,

- From the fact that the conjugate operator T^* lies in the "unit" ball of the space of all s-nuclear operators, it follows that the operator T is nuclear and belongs to the unit ball of the space of all nuclear operators, too..

Finite rank operators

The last theorem can be applied, in particular, to the case of finite rank operators. So, in the case where X^* has the AP_s, the situation, like a $n^{-1/2}$-situation in one of the above theorem, is not possible. However, in general case things are not so good. Indeed, we can prove (getting one way estimation):

Theorem

Let $s \in [2/3, 1]$.

- *If T is an n-dimensional operator in a Banach space with nuclear norm $\nu(T) = 1$, then $\nu_s(T^*) \geq n^{1/s-3/2}$.*

The proof is not so simple as in the case $s = 1$ (above). We do not give it, but consider the limit cases $s = 1$ and $s = 2/3$. But before this, let us mention what we use in the general proof.

Finite rank operators (continued)

In the proof we use, in particular,

- factorization Grothendieck technique;
- duality results for operator ideals;
- finite dimensional estimations of p-summing operators;
- duality due to A. Grothendieck.

Now, let $s = 1$. Then we have $1/s - 3/2 = -1/2$, and this is the case, which was considered before.

Finally, let $s = 2/3$, so that $1/s - 3/2 = 0$ (in Estimation, $n^0 = 1$).

Write πT as

$$\pi T = \sum_{k=1}^{\infty} \mu_k \, x'_k \otimes x''_k,$$

where $\mu_k \geq 0$, $(x'_k) \subset X^*, (x''_k) \subset X^{**}$, $\sum_k \mu_k^{2/3} < 1$, $||x'_k|| = 1$, $||y_k|| = 1$.

Putting $\alpha_k := \mu_k^{2/3}$ and $\beta_k := \mu_k^{1/3}$, we factorize πT as

$$\pi T : X \xrightarrow{A} l_\infty \xrightarrow{\Delta_\alpha} l_1 \xrightarrow{j} l_2 \xrightarrow{\Delta_\beta} l_1 \xrightarrow{B} X^{**}.$$

Finite rank operators - $\nu_{2/3}$ (continued)

Putting $\alpha_k := \mu_k^{2/3}$ and $\beta_k := \mu_k^{1/3}$, we factorize πT as

$$\pi T : X \xrightarrow{A} l_\infty \xrightarrow{\Delta_\alpha} l_1 \xrightarrow{j} l_2 \xrightarrow{\Delta_\beta} l_1 \xrightarrow{B} X^{**}.$$

Here,

- $Ax := (\langle x_k', x \rangle) \in l_\infty$;
- $\Delta_\alpha(a_k) := (\alpha_k \, a_k) \in l_1$;
- j is the natural embedding;
- $\Delta_\beta(b_k) := \sum \beta_k b_k \, x_k'' \in X^{**}$.

$\nu(j\Delta_\alpha A) < 1$; $\pi T(X) \subset X \subset X^{**}$,
Take a projector $P : l_2 \underset{\text{onto}}{\rightarrow} E := (B\Delta_\beta)^{-1}(T(X))$ with norm 1.
Then

$$T : X \xrightarrow{A} l_\infty \xrightarrow{\Delta_\alpha} l_1 \xrightarrow{j} l_2 \xrightarrow{P} E \xrightarrow{B\Delta_\beta|_E} X.$$

Finite rank operators - ν_s

Remark: One can consider the last theorem as an "interpolation theorem" between $s = 2/3$ and $s = 1$.

The following can be seen as preparation for getting the sharpness of the last theorem.

Theorem

There exist a subspace Y of the space c_0 and a finite rank operators $z_n, n = 1, 2, \dots,$ in Y such that

- *Y does not have the AP_r for every $r \in (2/3, 1]$;*
- $\dim z_n = n$ *and* $\operatorname{trace} z_n = 1$, $n = 1, 2, \dots;$
- *for all $s \in (2/3, 1]$, $\delta > 0$ $\exists\ C_\delta > 0 : \nu_s(z_n) \leq C_\delta\, n^{1/s - 3/2 + \delta}$.*

[RAPPEL: If T is an n-dimensional operator in a Banach space with nuclear norm $\nu(T) = 1$, then $\nu_s(T^*) \geq n^{1/s-3/2}$.]

Finite rank operators - ν_s (continued)

Theorem

Let $s \in (2/3, 1], q \in [2, \infty), 1/q = 3/2 - 1/s$. *There exist a separable reflexive Banach space* Y *and a finite rank operators* $z_n, n = 1, 2, \dots,$ *in* Y *such that*

- Y *(as well as* Y^**) has the* AP_r *for every* $r < s$;
- Y *does not have the* AP_s;
- $\dim z_n = n$ *and* $\operatorname{trace} z_n = 1$, $n = 1, 2, \dots$;
- $\nu_s(z_n) \leq \frac{C}{log(n+1)}$.

Moreover, $Y \subset \left(\sum_N l_{q_N}^{3 \cdot 2^N} \right)_{l_q}$, *where* $q_N \searrow q$.

Remark: We have a nice "by-product consequence" of Theorem.

An unexpected application

For $q = 2$ (that is, $s = 1$), the space Y is a subspace of the space of type $\left(\sum_j l_{p_j}^{k_j}\right)_{l_2}$ with $p_j \searrow 2$ and $k_j \nearrow \infty$. Every such space is an asymptotically Hilbertian space (for definitions and some discussion, see

P. G. Casazza, C. L. García, W. B. Johnson, An example of an asymptotically Hilbertian space which fails the approximation property, Proc. Amer. Math. Soc., Volume 129, No. 10 (2001), 3017-3024.

). So, we got:

Corollary

There exists an asymptotically Hilbertian space without the Grothendieck approximation property.

Reminding.

RAPPEL:
A Banach space X is said to be *asymptotically Hilbertian* provided there is a constant K so that for every m there exists n so that X satisfies: there is an n-codimensional subspace X_m of X so that every m-dimensional subspace of X m is K-isomorphic to l_2^m.

An unexpected application (continued)

First example of an asymptotically Hilbertian space without the Grothendieck approximation property was constructed (by O. Reinov) in 1982 in

O. I. Reinov, Banach spaces without approximation property, Functional Analysis and Its Applications, Volume 16, No. 4 (1982), 315-317.

where A. Szankowski's results were used

Later, in 2000, by applying Per Enflo's example in a version of A.M. Davie, P. G. Casazza, C. L. García and W. B. Johnson gave another example of an asymptotically Hilbertian space which fails the approximation property.

We here, not being searching for an example of such a space, have got it (accidentally) by using the construction from

A. Pietsch, Operator ideals, North-Holland, 1978.

ν_s("adjoint operator")

Recalling:

- Let $s \in (0,1]$, $T \in L(X)$ and assume that $X^* \in AP_s$. If $\pi T \in N_s(X, X^{**})$ and $\nu_s(\pi T) < 1$, then $T \in N_1(X, X)$ and $\nu(T) < 1$.
- Let $s \in (2/3, 1]$. If T is an n-dimensional operator in a Banach space with nuclear norm $\nu(T) = 1$, then $\nu_s(T^*) \geq n^{1/s-3/2}$.

We have, finishing a talk and in particular:

Theorem

There exist a Banach space W and a finite rank operators $z_n, n = 1, 2, \dots,$ in W such that

- $W \in AP$;
- W^* *does not have the* AP_r *for every* $r \in (2/3, 1]$;
- $\dim z_n = n$ *and* $\operatorname{trace} z_n = 1$, $n = 1, 2, \dots;$
- *for all* $s \in (2/3, 1]$, $\delta > 0$ $\exists\ C_\delta > 0 : \nu_s(z_n^*) \leq C_\delta\, n^{1/s-3/2+\delta}$.

Thank you for your attention!