Free boundary problem of magnetohydrodynamics for two liquids

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1. Introduction

We consider the free boundary problem of magnetohydrodynamics in the bounded domain $\Omega \subset R^3$. It describes the motion of a finite isolated mass of viscous incompressible electrically conducting capillary liquid inside the other viscous incompressible liquid under the action of magnetic field. The interface between the liquids is unknown. Let the bounded variable domain Ω_{1t} is filled by the liquid of density d_1 and viscosity ν_1. The domain Ω_{1t} is surrouded by the bounded variable domain $\Omega_{2t} = \Omega \setminus \overline{\Omega}_{1t}$, filled by the liquid of density d_2 and viscosity ν_2. The boundary of Ω_{2t} consists of two disjoint components: the free boundary Γ_t and the fixed boundary $S = \partial\Omega$. We assume that both Γ_0 and S are homeomorphic to a sphere, $dist\{\Gamma_0, S\} \geq \delta > 0$.

The problem consists of determination for $t > 0$ the variable domains Ω_{it}, $i = 1, 2$ together with the velocity vector field $\mathbf{v^{(i)}}$, the pressure $p^{(i)}$, and the magnetic field $\mathbf{H^{(i)}}$. Equations in Ω_{it} have the form

$$\begin{aligned} \mathbf{v^{(i)}}_t + (\mathbf{v^{(i)}} \cdot \nabla)\mathbf{v^{(i)}} - \nabla \cdot T(\mathbf{v^{(i)}}, p^{(i)}) - \nabla \cdot T_M(\mathbf{H^{(i)}}) = 0, \\ \mu_i \mathbf{H^{(i)}}_t + \alpha_i^{-1} rotrot\mathbf{H^{(i)}} - \mu_i rot(\mathbf{v^{(i)}} \times \mathbf{H^{(i)}}) = 0, \\ \nabla \cdot \mathbf{v^{(i)}} = 0, \qquad \nabla \cdot \mathbf{H^{(i)}} = 0, \qquad x \in \Omega_{it}, \end{aligned} \tag{1.1}$$

where μ_i, - magnetic permeability, ν_i - kinematic viscosity, α_i - conductivity , d_i - density. We assume that $\nu_i, \alpha_i, d_i, \mu_i$ are positive constants. $T_M(\mathbf{H}) = \mu(\mathbf{H} \otimes \mathbf{H} - \frac{1}{2} I |\mathbf{H}|^2)$ - magnetic stress tensor.

$$T(\mathbf{v}, p) = -\frac{1}{d_i} pI + \nu S(\mathbf{v})$$

is the viscous stress tensor,

$$S(\mathbf{v}) = \nabla\mathbf{v} + (\nabla\mathbf{v})^T = \Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)_{i,j=1,2,3}$$

is the doubled rate-of-strain tensor.

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On the free surface Γ_t, which is subject to capillary forces, we have the following boundary conditions

$$\begin{gathered}
\big([T(\mathbf{v},p)] + [T_M(\mathbf{H})]\big)\mathbf{n} = \sigma\mathbf{n}\mathcal{H}, \\
\mathbf{V}_n = \mathbf{v}\cdot\mathbf{n}, \qquad [\mathbf{v}] = 0, \\
[\tfrac{1}{\alpha}(rot\mathbf{H})_\tau] = [\mu(\mathbf{v}\times\mathbf{H})_\tau], \\
[\mu\mathbf{H}\cdot\mathbf{n}] = 0, \quad [\mathbf{H}_\tau] = 0, \quad x\in\Gamma_t,
\end{gathered} \tag{1.2}$$

where σ - coefficient of the surface tension, $\mathcal{H}$ - is the doubled mean curvature of Γ_t, $\mathbf{V}_n$ is the velocity of evolution of the surface Γ_t in the direction of the normal $\mathbf{n}$ to Γ_t, which is exterior with respect to the domain Ω_{1t}. By $(rot\mathbf{H})_\tau$ we mean the tangential part of the rotor. By $[f]$ we denote the jump on Γ_t: $[f] = f^{(1)} - f^{(2)}$. Condition on the jump of the tangential part of $rot\mathbf{H}$ follows from the fact that on the interface tangential part of electric field is continuous and Maxwell equations.

We assume that the fixed boundary S is a perfectly conducting bounded closed surface. Boundary conditions on S have the form

$$\mathbf{H}\cdot\mathbf{n} = 0, \quad (rot\mathbf{H})_\tau = 0, \quad \mathbf{v} = 0, \quad x\in S. \tag{1.3}$$

Finally, we add the initial conditions

$$\mathbf{v}(x,0) = \mathbf{v}_0(x), \qquad \mathbf{H}(x,0) = \mathbf{H}_0(x), \quad x\in\Omega_{10}\cup\Omega_{20}. \tag{1.4}$$

Free boundary problem governing the motion of a finite isolated mass of electrically conducting capillary liquid in vacuum has been studied in [1-3]. In particular, local in time solvability is proved in [1]. The solution is obtained in Sobolev-Slobodetskii spaces $W_2^{2+l,1+l/2}$, $1/2 < l < 1$. We obtain the similar result for the problem $(1.1) - (1.4)$.

2. Coordinate transform

In order to reduce the problem (1.1)-(1.4) to a problem set in a domain with a fixed boundary, we use a modification of Hanzawa coordinate transform.

We assume that the initial position of the free boundary Γ_0 can be regarded as a small normal perturbation of the given smooth closed surface G

$$\Gamma_0 = \{x = y + \mathbf{N}(y)\rho_0(y), \quad y\in G\},$$

where $N(y)$ is the external normal to the surface G, $\rho_0 \in W_2^{2+l}(G)$ is a given function, and $|\rho_0| \le \frac{\delta}{4}$. Moreover, we are looking for the free boundary in the similar form

$$\Gamma_t = \{x = y + \mathbf{N}(y)\rho(y,t), \quad y\in G\},$$

where the function $\rho(y,t)$ is unknown.

We denote by $\mathcal{F}_1$ the domain bounded by G, $\mathcal{F}_2 = \Omega\setminus\overline{\mathcal{F}}_1$. We construct the mapping which transforms $\Omega = \mathcal{F}_1\cup G\cup\mathcal{F}_2$ to $\Omega = \Omega_{1t}\cup\Gamma_t\cup\Omega_{2t}$. To this end, we extend N and ρ into Ω. By N^* we mean a smooth non-vanishing vector field in Ω which coincides with N on G. By $\rho^*(y,t)$ we denote an extension of unknown function $\rho(y,t)$ from G into Ω with preservation of the class, which vanishes in a $\frac{\delta_0}{4}$ neighborhood of the surface S and satisfies the condition $\frac{\partial \rho^*(y,t)}{\partial N}\Big|_G = 0$. We introduce this mapping by the relation

$$x = y + \mathbf{N}^*(y)\rho^*(y,t) = e_\rho(y). \tag{2.1}$$

When ρ is sufficiently small (which is certainly the case for small t), transform (2.1) establishes one-to-one correspondence between $\mathcal{F}_i$ and Ω_{it}, $i = 1, 2$. We denote by $\mathcal{L}(y, \rho^*)$ the Jacobi matrix of the transformation (2.1), $L = det\mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$ is the cofactor matrix. The normal $\mathbf{n}$ to the free boundary corresponds to

$$\mathbf{n}(e_\rho(y)) = \frac{\widehat{\mathcal{L}}\mathbf{N}(y)}{|\widehat{\mathcal{L}}\mathbf{N}(y)|}. \tag{2.2}$$

Let

$$\mathbf{v}(e_\rho, t) = \mathbf{u}(y,t), \qquad p(e_\rho, t) = q(y,t).$$

To simplify the calculations, we introduce the new unknown function

$$\mathbf{h} = \widehat{\mathcal{L}}\mathbf{H}(e_\rho, t).$$

As it is demonstrated in [1], $\mathbf{h}$ is a solenoidal vector field and satisfies the homogeneous condition $[\mu\mathbf{h}\cdot\mathbf{N}] = 0$, $y \in G$. Transformation (2.1) converts the problem (1.1) – (1.4) to a nonlinear problem in the fixed domain $\Omega = \mathcal{F}_1 \cup S_{R_0} \cup \mathcal{F}_2$, for the unknown functions $\mathbf{u}(y,t)$, $q(y,t)$, $\mathbf{h}(y,t)$. We separate linear and nonlinear parts in this problem and write the boundary condition $(1.2)_1$ for the tangential and normal parts separately, then it can be written in the following form:

$$\begin{aligned}
&\mathbf{u}_t^{(i)} - \nu_i\nabla^2\mathbf{u}^{(i)} + \frac{1}{d_i}\nabla q^{(i)} = \mathbf{l}_1^{(i)}(\mathbf{u}^{(i)}, q^{(i)}, \mathbf{h}^{(i)}, \rho), \quad y \in \mathcal{F}_i\\
&\nabla\cdot\mathbf{u}^{(i)} = l_2^{(i)}(\mathbf{u}^{(i)}, \rho), \quad y \in \mathcal{F}_i,\\
&[\nu\Pi_0 S(\mathbf{u})\mathbf{N}] = \mathbf{l}_3^{(i)}(\mathbf{u}, \rho), \quad y \in G,\\
&-[\frac{1}{d}q] + [\nu\mathbf{N}\cdot S(\mathbf{u})\mathbf{N}(y)] + \sigma B\rho = l_4(\mathbf{u}, \mathbf{h}, \rho), \quad y \in G,\\
&\rho_t - \mathbf{u}\cdot\mathbf{N} = l_5(\mathbf{u}, \rho), \quad [\mathbf{u}] = 0, \quad y \in G,\\
&\mu_i\mathbf{h}_t^{(i)} + \alpha_i^{-1} rotrot\mathbf{h}^{(i)} = \mathbf{l}_6^{(i)}(\mathbf{h}^{(i)}, \mathbf{u}^{(i)}, \rho), \quad y \in \mathcal{F}_i,\\
&\nabla\cdot\mathbf{h}^{(i)} = 0, \quad y \in \mathcal{F}_i,\\
&[\mu\mathbf{h}\cdot\mathbf{N}] = 0, \quad [\mathbf{h}_\tau] = \mathbf{l}_7(\mathbf{h}, \rho), \quad [\frac{1}{\alpha}(rot\mathbf{h})_\tau] = \mathbf{l}_8(\mathbf{h}, \mathbf{u}, \rho) \quad y \in G,\\
&\mathbf{h}^{(2)}\cdot\mathbf{n} = 0, \quad (rot\mathbf{h}^{(2)})_\tau = 0, \quad \mathbf{u}^{(2)} = 0 \quad y \in S,\\
&\mathbf{u}^{(i)}(y,0) = \mathbf{u}_0^{(i)}(y), \quad \mathbf{h}^{(i)}(y,0) = \mathbf{h}_0^{(i)}(y), \quad y \in \mathcal{F}_i, \quad \rho(y,0) = \rho_0(y), \quad y \in G.
\end{aligned} \tag{2.3}$$

Here $\Pi_0\mathbf{u} = \mathbf{u} - \mathbf{N}(\mathbf{u}\cdot\mathbf{N})$ is the tangential part of the vector field $\mathbf{u}$, $-B\rho$ is the first variation of $\mathcal{H}$ with respect to ρ and has the form $B\rho = -\Delta_G\rho + b\rho$, where Δ_G is the Laplace-Beltrami operator on G. The nonlinear terms $\mathbf{l}_1^{(i)} - \mathbf{l}_7$ are similar to the nonlinear terms calculated in [1]. The nonlinear term $\mathbf{l}_8$ has the form

$$\begin{aligned}
\mathbf{l}_8 = [\tfrac{1}{\alpha}(rot\mathbf{h})_\tau] &= [\tfrac{1}{\alpha}\left(rot\mathbf{h} - (rot\mathbf{h}\cdot\mathbf{N})\mathbf{N}\right)]\\
&= [\tfrac{1}{\alpha}\left(rot\mathbf{h} - \tfrac{1}{L}\mathcal{L}rot\mathcal{L}^T\tfrac{1}{L}\mathcal{L}\mathbf{h}\right)]\\
&+[\tfrac{1}{\alpha}\left((\tfrac{1}{L}\mathcal{L}rot\mathcal{L}^T\tfrac{1}{L}\mathcal{L}\cdot\mathbf{n}(e_\rho)\mathbf{n})(e_\rho) - (rot\mathbf{h}\cdot\mathbf{N})\mathbf{N}\right)]\\
&+[\mu\left(\mathcal{L}^{-1}\mathbf{u}\times\mathbf{h} - ((\mathcal{L}^{-1}\mathbf{u}\times\mathbf{h})\cdot\mathbf{n}(e_\rho)\mathbf{n}(e_\rho))\right)],
\end{aligned}$$

where $\mathbf{n}(e_\rho)$ is given in (2.2).

3. Main result

Theorem 1. Let $\mathbf{u}_{0i} \in W_2^{1+l}(\mathcal{F}_i)$, $\mathbf{H_{0i}} \in W_2^{1+l}(\mathcal{F}_i)$, $i = 1,2$, $\rho_0 \in W_2^{2+l}(G)$ with a certain $l \in (1/2, 1)$ and the following compatibility conditions

$$\begin{gathered}
\nabla \cdot \mathbf{u}_0^{(i)} = l_2^{(i)}(\mathbf{u}_0^{(i)}, \rho_0), \quad y \in \mathcal{F}_i, \\
[\nu \Pi_0 S(\mathbf{u}_0)\mathbf{N}] = \mathbf{l}_3(\mathbf{u}_0, \rho_0), \quad y \in G, \\
\nabla \cdot \mathbf{h}_0^{(i)} = 0, \quad y \in \mathcal{F}_i, \\
[\mu \mathbf{h}_0 \cdot \mathbf{N}] = 0, \quad [(\mathbf{h}_0)\tau] = \mathbf{l}_7(\mathbf{h}_0, \rho_0), \quad [\tfrac{1}{\alpha}(rot\mathbf{h}_0)_\tau] = \mathbf{l}_8(\mathbf{h}_0, \mathbf{u}_0, \rho_0), \quad [\mathbf{u}_0] = 0 \quad y \in G, \\
\mathbf{h}_0^{(2)} \cdot \mathbf{n} = 0, \quad (rot\mathbf{h}_0^{(2)})_\tau = 0, \quad \mathbf{u}_0^{(2)} = 0 \quad y \in S
\end{gathered} \tag{3.1}$$

are hold. We assume that the smallness conditions

$$\|\rho_0\|_{W_2^{2+l}(G)} \le \varepsilon \qquad \|\mathbf{U}_0 - \mathbf{u}_0\|_{W_2^{l+1/2}(G) \le \varepsilon}, \tag{3.2}$$

where $\mathbf{U}_0 \in W_2^{l+3/2}(G)$ is a given vector field, be satisfied. Then problem (2.3) has a unique solution on a certain small time interval $(0,T)$ with the following regularity properties

$$\rho \in W_2^{5/2+l,0}(G_T) \cap W_2^{l/2}((0,T), W_2^{5/2}(G)), \quad \rho_t \in W_2^{3/2+l,3/4+l/2}(G_T),$$

$$\mathbf{u}^{(i)} \in W_2^{2+l,1+l/2}(\mathcal{F}_i \times (0,T)), \qquad \mathbf{h}^{(i)} \in W_2^{2+l,1+l/2}(\mathcal{F}_i \times (0,T)),$$

$$q \in W_2^{1/2+l,0}(G_T) \cap W_2^{l/2}((0,T); W_2^{1/2}(G)), \qquad \nabla q \in W_2^{l,l/2}(\mathcal{F}_i \times (0,T)).$$

Scheme of the proof. It is clear that problem (2.3) can be decomposed in two parts: the hydrodynamical part with linear terms depending on $\mathbf{u}$, q, and ρ and the magnetic part with linear terms depending on $\mathbf{h}$. Linearized hydrodynamical problem is as follows

$$\begin{aligned}
&\mathbf{u}_t^{(i)} - \nu^{(i)}\nabla^2\mathbf{u}^{(\mathbf{i})} + \frac{\mathbf{1}}{\mathbf{d}^{(\mathbf{i})}}\nabla\mathbf{p}^{(\mathbf{i})} = \ \mathbf{f}^{(\mathbf{i})}, \quad \nabla \cdot \mathbf{u}^{(\mathbf{i})} = \nabla \cdot \mathbf{F}^{(\mathbf{i})}, \quad \mathbf{y} \in \mathcal{F}_\mathbf{i}, \\
&[\nu \Pi_0 S(\mathbf{u})]\mathbf{N} = \Pi_0 \mathbf{A}, \\
&-[\frac{1}{d}p] + [\nu \mathbf{N} \cdot S(\mathbf{u})\mathbf{N}] + \sigma B \rho = \mathbf{A} \cdot \mathbf{N}, \\
&\rho_t - \mathbf{u} \cdot \mathbf{N} = g(y,t), \quad [\mathbf{u}] = 0, \quad y \in G, \\
&\mathbf{u}^{(2)} = 0, \quad y \in S, \\
&\mathbf{u}^{(i)}(y,0) = \mathbf{u}_0^{(i)}(y), \quad y \in \mathcal{F}_i, \quad \rho(x,0) = \rho_0(y), \quad y \in G.
\end{aligned} \tag{3.3}$$

Problem (3.3) similar to the linearized problem in two phase free boundary problem describes the motion of two liquids without action of magnetic field. This linear problem has been studied in [6], [4]. In particular unique solvability in Sobolev-Slobodetskii spaces is obtained.

The part with linear terms depending on $\mathbf{h}$ is as follows

$$\begin{aligned}
&\mu_i \mathbf{H}_t^{(i)} + \frac{1}{\alpha_i} rotrot\mathbf{H}^{(i)} = \mathbf{f}^{(i)}, \quad \nabla \cdot \mathbf{H}^{(i)} = 0, \quad y \in \mathcal{F}_i, \\
&[\mu \mathbf{H} \cdot \mathbf{N}]\big|_G = 0, \quad [\mathbf{H}_\tau]\big|_G = \mathbf{a}, \\
&[\frac{1}{\alpha}(rot\mathbf{H})_\tau]\Big|_G = \mathbf{g}, \\
&\mathbf{H}^{(2)} \cdot \mathbf{n} = 0, \qquad (rot\mathbf{H}^{(2)})_\tau = 0, \quad y \in S, \\
&\mathbf{H}^{(i)}(y,0) = \mathbf{H}_0^{(i)}(y), \quad y \in \mathcal{F}_i.
\end{aligned} \tag{3.4}$$

Problem (3.4) can be reduced to the similar problem with $\mathbf{g} = 0,$, $\mathbf{a} = 0$ in the same way as in [1], where the solution to the auxilliary problem

$$\begin{aligned}
&rot\mathbf{h}(y) = \mathbf{j}(y), \qquad \nabla \cdot \mathbf{h}(y) = 0, \qquad y \in \mathcal{F}_i, \\
&[\mu \mathbf{h} \cdot \mathbf{n}]\big|_G = 0, \quad [\mathbf{h}_\tau]\big|_G = \mathbf{a}, \\
&\mathbf{h} \cdot \mathbf{N}(y) = 0, \qquad y \in S
\end{aligned} \tag{3.5}$$

has been constructed.

Theorem 2.[5] Let in (3.4) $\mathbf{a} = 0$, $\mathbf{g} = 0$, $\mathbf{f}^{(i)} \in W_2^{l,l/2}(Q_T^{(i)})$, $\mathbf{H}_0^{(i)} \in W_2^{l+1}(\mathcal{F}_i)$, $l \in [0,1)$ and the following compatibility conditions be satisfied

$$\begin{gathered}
\nabla \cdot \mathbf{f}^{(i)} = 0, \qquad \nabla \cdot \mathbf{H}_0^{(i)} = 0, \qquad y \in \mathcal{F}_i, \\
[\mu \mathbf{H}_0 \cdot \mathbf{N}]\Big|_G = 0, \qquad [\mathbf{H}_{0\tau}]\Big|_G = 0, \qquad [\tfrac{1}{\alpha} rot_\tau \mathbf{H}_0]\Big|_G = 0, \qquad [\mathbf{f} \cdot \mathbf{N}]\Big|_G = 0 \\
\mathbf{H}_0 \cdot \mathbf{n}\Big|_S = 0, \qquad (rot\mathbf{H}_0^{(2)})_\tau\Big|_S = 0, \qquad \mathbf{f}^{(2)} \cdot \mathbf{n}\Big|_S = 0.
\end{gathered}$$

(Condition $\nabla \cdot \mathbf{f}^{(i)} = 0$ holds in a week sense. Compatibility conditions on the tangential part of rotor at the boundary and for $\mathbf{f}$ on the boundary are set only when $l \geq 1/2$.)

Then problem (3.4) has a unique solution $\mathbf{H}^{(i)} \in W_2^{l+2,l/2+1}(Q_T^{(i)})$, $Q_T^{(i)} = \mathcal{F}_i \times (0,T)$, $i = 1,2$. For this solution the following estimate

$$\sum_{i=1}^{2} \parallel \mathbf{H}^{(i)} \parallel_{W_2^{l+2,l/2+1}(Q_T^{(i)})} \leq c \sum_{i=1}^{2} \Big(\parallel \mathbf{f}^{(i)} \parallel_{W_2^{l,l/2}(Q_T^{(i)})} + \parallel \mathbf{H}_0^{(i)} \parallel_{W_2^{l+1}(\mathcal{F}_i)} \Big) \tag{3.6}$$

holds.

Solvability of the nonlinear problem is proved by the successive approximations method, based on solvability results for linear problems (3.3), (3.4) and estimates of nonlinear terms. Assumption $(3.2)_1$ is stronger as the corresponding assumption in [1] ($\|\rho_0\|_{W_2^{3/2+l}(G)} \leq \varepsilon$). It gives us the opportunity to obtain for the magnetic field the same regularity properties as for the velocity vector field. Detailed proof will be given in subsequent publications.

References

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