On $\mathbb{Z}_d$-symmetry of spectra of linear operators in Banach spaces

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M. I. Zelikin's Remark

All the main results below have their beginning in the following remark of M.I. Zelikin (Moscow State Uniersity):

Remark

The spectrum of a linear operator $A : \mathbb{R}^k \to \mathbb{R}^k$ *is central-symmetric iff the trace of any odd power of A equals zero:*

$$\text{trace } A^{2n-1} = 0, \; n \in \mathbb{N}.$$

Zelikin's Theorem

To formulate the theorem, we need a definition:

- The spectrum of A is central-symmetric, if together with any eigenvalue $\lambda \neq 0$ it has the eigenvalue $-\lambda$ of the same multiplicity.
  It was proved in a paper by M. I. Zelikin

M. I. Zelikin, A criterion for the symmetry of a spectrum", Dokl. Akad. Nauk 418 (2008), no. 6, 737-740

- **Theorem.** The spectrum of a nuclear operator A acting on a separable Hilbert space is central-symmetric iff $trace\, A^{2n-1} = 0,\ n \in \mathbf{N}$.

Mityagin's $\mathbb{Z}_d$-symmetry

Definition

Let T be an operator in X, all non-zero spectral values of which are eigenvalues of finite multiplicity and have no limit point except possibly zero. For a fixed $d = 2, 3, \dots$ and for the operator T, the spectrum of T is called $\mathbb{Z}_d$-symmetric, if $0 \neq \lambda \in \mathrm{sp}\,(T)$ implies $t\lambda \in \mathrm{sp}\,(T)$ for every $t \in \sqrt[d]{1}$ and of the same nultiplicity.

If $d = 2$, then one has the central symmetry.

Mityagin's Theorem

Theorem

Let X be a Banach space and $T : X \to X$ is a compact operator. Suppose that some power of T is nuclear. The spectrum of T is $\mathbb{Z}_d$-symmetric iff there is an integer $K \geq 0$ such that for every $l > Kd$ the value trace T^l *is well defined and*

$$\operatorname{trace} \; T^{kd+r} = 0$$

$$\textit{for all } \; k = K, K+1, K+2, \ldots \;\; \textit{and} \;\; r = 1, 2, \ldots, d-1.$$

In the proof, the Riesz theory of compact operators is used.

B. S. Mityagin, *A criterion for the $\mathbb{Z}_d$-symmetry of the spectrum of a compact operator*, J. Operator Theory, **76**:1 (2016), 57–65.

Our Generalization of Mityagin's Theorem

Theorem

Let X be a Banach space and $T : X \to X$ is a linear continuous operator. Suppose that some power of T is nuclear. The spectrum of T is $\mathbb{Z}_d$-symmetric iff there is an integer $K \geq 0$ such that for every $l > Kd$ the value $\operatorname{trace} T^l$ is well defined and

$$\operatorname{trace} T^{kd+r} = 0$$

for all $k = K, K+1, K+2, \dots$ *and* $r = 1, 2, \dots, d-1$.

Oleg Reinov, *Some remarks on spectra of nuclear operators*, SPb. Math. Society Preprint 2016-09, 1-9

In the proof we use the Fredholm theory of A. Grothendieck:

A. Grothendieck, *La théorie de Fredholm*, Bull. Soc. Math. France, **84** (1956), 319–384.

Simplest Examples

Remark

Let Π_p be the ideal of absolutely p-summing operators. Then for some n one has $\Pi_p^n \subset N$. In particular, $\Pi_2^2(C[0,1]) \subset N(C[0,1])$, but not every absolutely 2-summing operator in $C[0,1]$ is compact.

The p-summing operators in such spaces provide instances of operators to which the last theorem may be applied even though Mityagin's Theorem is not always applicable.

General notation

X, Y Banach spaces.
$L(X, Y)$ — linear continuous operators.
For $T : X \to Y$,

$$||T|| = \sup\{||T(x)|| : \ x \in X, \ ||x|| \leq 1\}.$$

$X^* = L(X, \mathbb{C})$.
For $0 < p < \infty$,

$$l^p = \{(a_k) : \ a_k \in \mathbb{C}, \sum_{k=1}^{\infty} |a_k|^p \leq \infty\},$$

$||(a_k)||_{l^p} = \{\sum_{k=1}^{\infty} |a_k|^p\}^{1/p}$.

$$l^\infty = \{(a_k) : \ ||(a_k)||_{l^\infty} = \sup_k |a_k| < \infty\};$$

$$c_0 = \{(a_k) \subset l^\infty; \ a_k \to 0\}.$$

Preliminaries

$$\mathcal{F}(X, Y) = \{T \in L(X, Y) : \ \mathrm{rank}\, T < \infty\}$$

$$T \in \mathcal{F}(X, Y) \implies T(x) = \sum_{k=1}^{n} x'_k(x) y_k,$$

where $x'_k \in X^*$, $y_k \in Y$.
If $T \in \mathcal{F}(X, X)$, then $T(x) = \sum_{k=1}^{n} x'_k(x) x_k$ $(x'_k \in X^*,\ x_k \in X)$
and

$$\text{trace } T := \sum_{k=1}^{n} x'_k(x_k).$$

"Trace" does not depend on a representation of T and

$$\text{trace } T = \sum \textit{eigenvalues}\,(T)$$

(written according their multuplicities).

Nuclear representations

Also, a finite rank $T \in L(X,X)$. Consider a *nuclear* representation

$$Tx = \sum_{k=1}^{\infty} x'_k(x)x_k, \; \sum_{k=1}^{\infty} ||x'_k||\, ||x_k|| < \infty$$

and

$$\alpha := \sum_{k=1}^{\infty} x'_k(x_k).$$

- **Question:** $\alpha = \text{trace } T$?
- Generally, NO.

Enflo P. , A counterexample to the approximation property in Banach spaces, Acta Math., Volume 130, 1973, 309–317

Nuclear operators

Definition

$T: X \to Y$ is nuclear, if

$$\exists\ (x'_k) \subset X^*, (y_k) \subset Y:\ \sum_{k=1}^{\infty} ||x'_k||\,||y_k|| < \infty,$$

$$T(x) = \sum_{k=1}^{\infty} x'_k(x) y_k,\ \ \forall\ x \in X.$$

Remark: If T is nuclear, then $T: X \to c_0 \xrightarrow{\Delta} l_1 \to Y$. $\Delta \in l^1$.

s-Nuclear operators

Generally:

Definition

$T : X \to Y$ is s-nuclear $(0 < s \leq 1)$, if

$$\exists\ (x'_k) \subset X^*, (y_k) \subset Y :\ \sum_{k=1}^{\infty} ||x'_k||^s\, ||y_k||^s < \infty,$$

$$T(x) = \sum_{k=1}^{\infty} x'_k(x) y_k,\ \ \forall\ x \in X.$$

Remark: If T is s-nuclear, then $T : X \to c_0 \overset{\Delta}{\to} l_1 \to X,\ \Delta \in l^s$.

Nuclear operators: Trace and AP

Definition

Let $T \in L(X, X)$ be nuclear with

$$T(x) = \sum_{k=1}^{\infty} x'_k(x) x_k, \ \ \forall\, x \in X.$$

If $\sum_{k=1}^{\infty} x'_k(x_k)$ **does not depend** *on a representation, then it is the (nuclear) trace of T. Notation:* trace T.

Definition

If every nuclear $T : X \to X$ has a trace, then X has the AP.

Grothendieck's Definition:

Definition

X has the AP if id_X is in the closure of $\mathcal{F}(X,X)$ in the topology of compact convergence:

$$\forall\ \varepsilon > 0,\ \forall \text{ compact } K \subset X\ \exists\ R \in \mathcal{F}(X,X):\ \sup_{x\in K} ||Rx - x|| < \varepsilon.$$

A. Grothendieck: *Produits tensoriels topologiques et éspaces nucléaires*, Mem. Amer. Math. Soc., **16**(1955).

Enflo P. , A counterexample to the approximation property in Banach spaces, Acta Math., Volume 130, 1973, 309–317

AP : Examples

Examples

- $AP : C(K), L_p(\mu), A, L_\infty/H^\infty$ *etc;*
- $\forall\ p \in [1, \infty] \setminus \{2\}\ \exists\ X \subset l_p : \quad X \notin AP$;
- $L(H) \notin AP$, H^∞ — *not known;*

A. Grothendieck:

Theorem

The following are equivalent:
1) Every Banach space has the approximation property.
2) If a nuclear operator $U : c_0 \to c_0$ *is such that* $\operatorname{trace} U = 1$*, then* $U^2 \neq 0$.

By Enflo:

Theorem

There exists a nuclear operator $U : c_0 \to c_0$ *such that* $\operatorname{trace} U = 1$ *and* $U^2 = 0$.

Bad nuclear operators in l^1

Can be obtain from Davie's

Davie A.M., The approximation problem for Banach spaces, Bull. London Math. Soc., Vol 5, 1973, 261–266

Theorem

There exists a nuclear operator T in l^1 :
(i) T is s-nuclear for every $s \in (2/3, 1]$.
(ii) trace $T = 1$.
(iii) $T^2 = 0$.

A proof can be found in

A. Pietsch, Operator ideals, North-Holland, 1978.

Positive results

On the other hand:

A. Grothendieck:

Theorem

If T is 2/3-nuclear (in any X), then trace T *is well-defined. Moreover, if* trace $T \neq 0$, *then* $T^2 \neq 0$.

V. B. Lidskiĭ:

Theorem

If $T : l^2 \to l^2$ is 1-nuclear, then trace T *is well-defined. Moreover, if* trace $T \neq 0$, *then* $T^2 \neq 0$.

Can be found in

V. B. Lidskiĭ, *Nonselfadjoint operators having a trace*, Dokl. Akad. Nauk SSSR, **125**(1959), 485–487.

or in A. Pitsch's book.

Our aim

Thus, the cases of nuclear operatots in c_0, l^1 and l^2 were considered above, and these are all the cases (in the scale of l^p-spaces) that have been known to us so far.

We are going to consider the cases where $1 < p < \infty$ and *to get an optimal results* (also in case of c_0).

Main result: Our Generalization of Zelikin's Theorem

Our main theorems:

Theorem

Let Y is a subspace of a quotient (or a quotient of a subspace) of some $L_p(\mu)$-space, $1 \leq p \leq \infty$ and $1/r = 1 + |1/2 - 1/p|$. If $T : Y \to Y$ is r-nuclear, then trace T *is well-defined. For a fixed $d = 2, 3, \ldots,$ the spectrum of T is $\mathbb{Z}_d$-symmetric iff*

$$\operatorname{trace} T^{kd+j} = 0 \ \textit{for all} \ k = 0, 1, 2, \ldots \ \textit{and} \ j = 1, 2, \ldots, d-1.$$

In particular, if trace $T \neq 0$, *then* $T^2 \neq 0$.

Main result: Sharpness

Theorem is optimal with respect to p and r :

Theorem

Let $p \in [1, \infty], p \neq 2$, $1/r = 1 + |1/2 - 1/p|$. There exists a nuclear operator V in l^p (in c_0 for $p = \infty$) such that
1) *V is s-nuclear for each $s \in (r, 1]$;*
2) *V is not r-nuclear;*
3) trace $V = 1$ *and* $V^2 = 0$.

Note that for $p = \infty$ we have $r = 2/3$ and for $p = 2$ we have $r = 1$.

Our auxiliary theorem

$L_c(X, Y)$ — $L(X, Y)$ with topology of compact convergence.
Main ingredient for getting V above:

Theorem

Let $r \in [2/3, 1), p \in (2, \infty], 1/r = 3/2 - 1/p$. There exist a subspace Y_p of the space l_p (c_0 if $p = \infty$), a linear continuous functional Ψ on $L_c(Y_p, Y_p)$ and systems $(y_k) \subset Y^*$, $(y_k) \subset Y$ such that

$$\sum_{k=1}^{\infty} ||y_k'||^s \, ||y_k||^s < \infty \ \forall \ s > r,$$

$$\Psi(U) = \sum_{k=1}^{\infty} y_k'(Uy_k) \ \forall \ U \in L(Y_p, Y_p),$$

$$\Psi(R) = 0 \ \forall \ R \in \mathcal{F}(Y_p, Y_p).$$

Moreover, such situation is impossible for $s = r$.

Thank you for your attention!