# Approximation properties associated with quasi-normed operator ideals of $(r, p, q)$-nuclear operators

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Abstract. We consider quasi-normed tensor products lying between Lapresté tensor products and spaces of $(r, p, q)$-nuclear operators. We define and investigate the corresponding approximation properties for Banach spaces. An intermediate aim is to answer a question of Sten Kaijser.

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This is, essentially, a continuation of the author's paper **[4]** and the first part of the work on the approximation properties connected with the quasi-normed ideals of so-called $(r, p, q)$-nuclear operators.

## 0. Notation, preliminaries

Throughout, we denote by $X, Y, \dots, G, F, W \dots$ Banach spaces over a field $\mathbb{K}$ (which is either $\mathbb{R}$ or $\mathbb{C}$); $X^*, Y^*, \dots$ are Banach dual to $X, Y, \dots$. By $x, y, \dots, x', \dots$ (maybe with indices) we denote elements of $X, Y, \dots, Y^* \dots$ respectively. $\pi_Y : Y \to Y^{**}$ is a natural isometric imbedding.

Notations $l_p$, $l_p^n$ $(0 < p \leq \infty, n = 1, 2, \dots)$, $c_0$ are standard; $e_k$ $(k = 1, 2, \dots)$ is the k-th unit vector in $l_p$ or $c_0$ (when we consider the unit vectors as the linear functionals, we use notation $e'_k$). We use $\mathrm{id}_X$ for the identity map in $X$.

It is denoted by $F(X, Y)$ a vector space of all linear continuous mappings from $X$ to $Y$. By $X \otimes Y$ we denote the algebraic tensor product of the spaces $X$ and $Y$. $X \otimes Y$ can be considered as a subspace of the vector space $F(X^*, Y)$ (namely, as a vector space of all linear weak$^*$-to-weak continuous finite rank operators). We can identify also the tensor product (in a natural way) with a corresponding subspace of $F(Y^*, X)$. If $X = W^*$, then $W^* \otimes Y$ is identified with $F(X, Y^{**})$ (or with $F(Y^*, X^*)$). If $z \in X \otimes Y$, then $\widetilde{z}$ is the corresponding finite rank operator. If $z \in X^* \otimes X$ and e.g. $z = \sum_{k=1}^n x'_k \otimes x_k$, then trace $z := \sum_{k=1}^n \langle x'_k, x_k \rangle$ does not depend on

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representation of $z$ in $X^* \otimes X$. $L(X,Y)$ is a Banach space of all linear continuous mappings ("operators") from $X$ to $Y$ equipped with the usual operator norm.

If $A \in L(X,W)$, $B \in L(Y,G)$ and $z \in X\otimes Y$, then a linear map $A\otimes B : X\otimes Y \to W \otimes G$ is defined by $A \otimes B((x \otimes y) := Ax \otimes By$ (and then extended by linearity). Since $\widetilde{A \otimes B(z)} = B\tilde{z}A^*$ for $z \in X \otimes Y$, we will use notation $B \circ z \circ A^* \in W \otimes G$ for $A \otimes B(z)$. In the case where $X$ is a dual space, say $F^*$, and $T \in L(W,F)$ (so, $A = T^* : F^* \to W^*$), one considers a composition $B\tilde{z}T$; in this case $T^* \otimes B$ maps $F^* \otimes Y$ into $W^* \otimes Y$ and we use notation $B \circ z \circ T$ for $T^* \otimes B(z)$.

If $\nu$ is a tensor quasi-norm (see [**5**, 0.5]), then $\nu(A \otimes B(z)) \le ||A||\,||B||\,\nu(z)$ and we can extend the map $A\otimes B$ to the completions of the tensor products with respect to the quasi-norm $\nu$, having the same inequality. The natural map $(X \otimes Y, \nu) \to L(X^*, Y)$ is continuous and can be extended to the completion $\widehat{X \otimes_\nu Y}$; for a tensor element $z \in \widehat{X \otimes_\nu Y}$, we still denote by $\tilde{z}$ the corresponding operator. The natural mapping $\widehat{X \otimes_\nu Y} \to L(X^*, Y)$ need not to be injective; *if it is injective for a fixed $Y$ and for all $X$, then we say that $Y$ has the $\nu$-approximation property.*

A projective tensor product $X\widehat{\otimes}Y$ of Banach spaces $X$ and $Y$ is defined as a completion of $X \otimes Y$ with respect to the norm $||\cdot||_\wedge$ : if $z \in X \otimes Y$, then

$$||z||_\wedge := \inf \sum_{k=1}^{n} ||x_k||\,||y_k||,$$

where infimum is taken over all representation of $z$ as $\sum_{k=1}^n x_k \otimes y_k$. We can try to consider $X\widehat{\otimes}Y$ also as operators $X^* \to Y$ or $Y^* \to X$, but this correspondence is, in general, not one-to-one. Note that $X\widehat{\otimes}Y = Y\widehat{\otimes}X$ in a sence. If $z \in X\widehat{\otimes}Y, \varepsilon > 0$, then one can represent $z$ as $z = \sum_{k=1}^\infty x_k \otimes y_k$ with $\sum_{k=1}^\infty ||x_k||\,||y_k|| < ||z||_\wedge + \varepsilon$. For $z \in X^*\widehat{\otimes}X$ with a "projective representation" $z = \sum_{k=1}^\infty x'_k \otimes x_k$, trace of $z$, trace $z := z = \sum_{k=1}^\infty \langle x_k, y_k\rangle$, does not depend of representation of $z$. The Banach dual $(X\widehat{\otimes}Y)^* = L(Y, X^*)$ by $\langle T, z\rangle = \text{trace } T \circ z$.

Some more notations: If $\mathfrak{A}$ is an operator ideal then $\mathfrak{A}^{reg}(X,Y) := \{T \in L(X,Y) :\ \pi_Y T \in \mathfrak{A}(X, Y^{**})\}$, $\mathfrak{A}^{dual}(X,Y) := \{T \in L(X,Y) :\ T^* \in \mathfrak{A}(Y^*, X^*)\}$.

Finally,

$$l_p(X) := \{(x_i) \subset X :\ ||(x_i)||_p := \big(\sum ||x_i||^p\big)^{1/p} < \infty\},$$

$$l_\infty(X) := \{(x_i) \subset X :\ ||(x_i)||_\infty := \sup_i ||x_i|| < \infty\},$$

$$l_p^w(X) := \{(x_i) \subset X :\ ||(x_i)||_{w,p} := \sup_{||x'||\le 1} \big(\sum |\langle x', x_i\rangle|^p\big)^{1/p} < \infty\},$$

$$l_\infty^w(X) := \{(x_i) \subset X :\ ||(x_i)||_{w,\infty} := \sup_i ||x_i|| < \infty\}.$$

Note that if $p \le q$, then $||\cdot||_q \le ||\cdot||_p$ and $||\cdot||_{w,q} \le ||\cdot||_{w,p}$. If $0 < p \le \infty$, then $p'$ is a conjugate exponent: $1/p + 1/p' = 1$ if $p \ge 1$ and $p' = \infty$ if $p \in (0,1]$.

**Below** $0 < r, s \le 1,\ 0 < p, q \le \infty$ **and** $1/r + 1/p + 1/q = 1/\beta \ge 1$.

Let us note that "Remarks" in the paper can contain sometimes quite important information comparable to the information presented in Theorems and Propositions.

## 1. The tensor products $X\widehat{\otimes}_{r,p,q}Y$

We use partially notations from [**5**]. For $z \in X \otimes Y$ we put

$$\mu_{r,p,q}(z) := \inf\{||(\alpha_k)||_r||(x_k)||_{w,p}||(y_k)||_{w,q} : \ z = \sum_{k=1}^{n} \alpha_k x_k \otimes y_k\};$$

$X \otimes_{r,p,q} Y$ is the tensor product, equipped with this quasi-norm $\mu_{r,p,q}$. Note that $\mu_{1,\infty,\infty}$ is the projective tensor norm of A. Grothendieck [**1**].

Let us denote by $X \widehat{\otimes_{r,p,q}} Y$ the completion of $X \otimes Y$ with respect to this quasi-norm $\mu_{r,p,q}$ (in [**5**] — $X \underset{r,p,q}{\widehat{\otimes}} Y$). Every tensor element $z \in X \widehat{\otimes_{r,p,q}} Y$ admits a representation of type $z = \sum_{k=1}^{\infty} \alpha_k x_k \otimes y_k$, where $||(\alpha_k)||_r||(x_k)||_{w,p}||(y_k)||_{w,q} < \infty$, and

$$\mu_{r,p,q}(z) := \inf ||(\alpha_k)||_r||(x_k)||_{w,p}||(y_k)||_{w,q}$$

(infimum is taken over all such finite or infinite representations) [**5**, Proposition 1.3, p. 52]. Note that $X \widehat{\otimes_{1,\infty,\infty}} Y = X\widehat{\otimes}Y$.

**Lemma 1.1** Let 1) $0 < r_1 \le r_2 \le 1$, $p_1 \le p_2$ and $q_1 \le q_2$ or 2) $0 < r_1 < r_2 \le 1$, $p_1 \ge p_2$, $q_1 \ge q_2$ and $1/r_2 + 1/p_2 + 1/q_2 \le 1/r_1 + 1/p_1 + 1/q_1$. If $z \in X \otimes Y$, then $\mu_{r_2,p_2,q_2}(z) \le \mu_{r_1,p_1,q_1}(z)$. In particular, $\mu_{1,\infty,\infty}(z) \le \mu_{r_1,p_1,q_1}(z)$. Consequently, a natural mappings $X \otimes_{r_1,p_1,q_1} Y \to X \otimes_{r_2,p_2,q_2} Y \to X\widehat{\otimes}Y$ can be extended to the (natural) continuos maps

$$X \widehat{\otimes_{r_1,p_1,q_1}} Y \to X \widehat{\otimes_{r_2,p_2,q_2}} Y \to X\widehat{\otimes}Y.$$

Proof. Case 1): If $z = \sum_{k=1}^{n} \alpha_k x_k \otimes y_k$, then $||(\alpha_k)||_{r_2}||(x_k)||_{w,p_2}||(y_k)||_{w,q_2} \le ||(\alpha_k)||_{r_1}||(x_k)||_{w,p_1}||(y_k)||_{w,q_1}$.

Case 2: The proof is standard (cf. [**6**, 18.1.5, p. 246-247]). Take $r$ such that $1/r_1 = 1/r+1/p+1/q$, where $1/p := 1/p_2-1/p_1$ and $1/q := 1/q_2-1/q_1$. Then $r \le r_2$ and $r_1/r + r_1/p + r_1/q = 1$. If $z = \sum_{k=1}^{n} \alpha_k x_k \otimes y_k$, then $z = \sum_{k=1}^{n} \alpha_k^{r_1/r}(\alpha_k^{r_1/p}x_k) \otimes (\alpha_k^{r_1/q}y_k)$ and

$$||(\alpha_k^{r_1/r})||_{r_2} \le ||(\alpha_k^{r_1/r})||_r = ||(\alpha_k)||_{r_1}^{r_1/r}.$$

Since $p_2 \le p_1$ and $1 - p_2/p_1 = p_2/p \le 1$, we can apply Golder inequality to get

$$||(\alpha_k^{r_1/p}x_k)||_{w,p_2} \le \big(\sum_{k=1}^{n} |\alpha_k|^{\frac{p}{p_2}\cdot\frac{r_1 p_2}{p}}\big)^{\frac{1}{p_2}\cdot\frac{p_2}{p}} ||(x_k)||_{w,p_1} =$$

$$||(\alpha_k^{r_1/p})||_p \, ||(x_k)||_{w,p_1} = ||(\alpha_k)||_{w,r_1}^{r_1/p} \, ||(x_k)||_{w,p_1}.$$

By the same reason,

$$||(\alpha_k^{r_1/q}y_k)||_{w,q_2} \le ||(\alpha_k^{r_1/q})||_q \, ||(y_k)||_{w,q_1} = ||(\alpha_k)||_{w,r_1}^{r_1/q} \, ||(y_k)||_{w,q_1}.$$

Hence,

$$||(\alpha_k^{r_1/r})||_{r_2} \, ||(\alpha_k^{r_1/p}x_k)||_{w,p_2} \, ||(\alpha_k^{r_1/q}y_k)||_{w,q_2} \le$$
$$||(\alpha_k)||_{r_1}^{r_1/r} \, ||(\alpha_k)||_{w,r_1}^{r_1/p} \, ||(x_k)||_{w,p_1} \, ||(\alpha_k)||_{w,r_1}^{r_1/q} \, ||(y_k)||_{w,q_1} =$$

$$||(\alpha_k)||_{r_1}\, ||(x_k)||_{w,p_1}\, ||(y_k)||_{w,q_1}.$$

It follows that $\mu_{r_2,p_2,q_2}(z) \le \mu_{r_1,p_1,q_1}(z)$.

Let us recall the following useful fact (see Section 0). If $A \in L(X,W)$, $B \in L(Y,G)$ and $z \in X \widehat{\otimes_{r,p,q}} Y$, then $B \circ z \circ A^* \in W \widehat{\otimes_{r,p,q}} G$ and $\mu_{r,p,q}(B \circ z \circ A^*) \le ||B||\, ||A||\, \mu_{r,p,q}(z)$. Particular cases: $X = W$ and $A = \mathrm{id}_X$ or $Y = G$ and $B = \mathrm{id}_Y$ .

The topological dual to $(X \widehat{\otimes_{r,p,q}} Y, \mu_{r,p,q})$ is the space $\Pi_{\infty,p,q}(X, Y^*)$ of absolutely $(\infty, p, q)$-summing operators from $X$ to $Y^*$ [**5**, Theorem 1.3, p. 57] (recall that $0 < r \le 1$) : If $\tau \in (X \widehat{\otimes_{r,p,q}} Y)^*$ and $x \otimes y \in X \otimes Y$, then the corresponding operator $T$ is defined by $\langle \tau, x \otimes y\rangle = \langle Tx, y\rangle$ [**5**, pp. 56-57]. Recall that, by definition, an operator $T : X \to F$ is absolutely $(\infty, p, q)$-summing if for any finite sequences $(x_k)$ and $(f'_k)$ (from $X$ and $F^*$ respectively) one has

$$\sup_k |\langle Tx_k, f'_k\rangle| \le C\, ||(x_k)||_{w,p} ||(f'_k)||_{w,q}.$$

With a norm $\pi_{\infty,p,q}(T) := \inf C$, the space $\Pi_{\infty,p,q}(X, F)$ is a Banach space and in duality above (for $F = Y^*$) $\pi_{\infty,p,q}(T) = ||\tau||$ (on the right is the norm of the functional $\tau$ in $(X \widehat{\otimes_{r,p,q}} Y)^*$).

Futhermore, taking a sequence in $X \times F^*$, consisting of one nonzero element $(x, f')$, we obtain: If $T \in \Pi_{\infty,p,q}(X, F)$, then $|\langle Tx, f'\rangle| \le \pi_{\infty,p,q}(T)\, ||x||\, ||f'||$; thus, $||T|| \le \pi_{\infty,p,q}(T)$. On the other hand, if $T \in L(X, F)$, then for any finite sequences $(x_k)$ and $(f'_k)$ we have:

$$\sup_k |\langle Tx_k, f'_k\rangle| \le ||T|| \sup_k ||x_k|| \sup_i ||f'_i|| \le ||T||\, ||(x_k)||_{w,p}\, ||(f'_k)||_{w,q}.$$

Therefore, $\Pi_{\infty,p,q}(X, F) = L(X, F)$.

I do not know whether the dual space $\Pi_{\infty,p,q}(X, Y^*)$ separates points of $X \widehat{\otimes_{r,p,q}} Y$. If so, then the natural map $X \widehat{\otimes_{r,p,q}} Y \to X \widehat{\otimes} Y$ is one-to-one. As a matter of fact, it follows from the above considerations, that *the space* $\Pi_{\infty,p,q}(X, Y^*)$ *separates points of* $X \widehat{\otimes_{r,p,q}} Y$ *iff the natural map* $j_{r,p,q} : X \widehat{\otimes_{r,p,q}} Y \to X \widehat{\otimes} Y$ *is one-to-one.*

**Definition 1.1.** We define a tensor product $X \widehat{\otimes}_{r,p,q} Y$ as a linear subspace of the projective tensor product $X \widehat{\otimes} Y$, consisting of all tensor elements $z$, which admit representations of type

$$z = \sum_{k=1}^{\infty} \alpha_k x_k \otimes y_k,\ (\alpha_k) \in l_r,\ (x_k) \in l_{w,p},\ (y_k) \in l_{w,q}$$

and equipped with the quasi-norm $||z||_{\wedge,r,p,q} := \inf ||(\alpha_k)||_r\, ||(x_k)||_{w,p}\, ||(y_k)||_{w,q}$, where the infimum is taken over all representations of $z$ in the above form.

Note that this tensor product is $\beta$-normed (see [**5, 6**]).

*Remark 1.1.* We can define $X \widehat{\otimes}_{r,p,q} Y$ also as a quotient of the space $X \widehat{\otimes_{r,p,q}} Y$ by the kernel of the map $j_{r,p,q}$ (i.e. by the annihilator $L(X, Y^*)_\perp$ of $L(X, Y^*)$ in the space $X \widehat{\otimes_{r,p,q}} Y$). Therefore:

(i) The tensor product $X \widehat{\otimes}_{r,p,q} Y$ is complete, i.e. a quasi-Banach space. This, with the injectivity of the natural map $X \widehat{\otimes}_{r,p,q} Y \to X \widehat{\otimes} Y$ answers a corresponding question of Sten Kaijser ("Why the last map is one-to-one for the "completion" $X \widehat{\otimes}_{r,p,q} Y$?").

(ii) If the dual of $X \widehat{\otimes_{r,p,q}} Y$ separates points of this space, then we can write $X \widehat{\otimes_{r,p,q}} Y = X\widehat{\otimes}_{r,p,q}Y$. In this case "finite nuclear" quasi-norm $\mu_{r,p,q}$ coincides with the tensor quasi-norm $||z||_{\wedge;r,p,q}$ (compare with [**6**, 18.1.10.]).

(iii) The dual space to $X\widehat{\otimes}_{r,p,q}Y$ is still $\Pi_{\infty,p,q}(X,Y^*)$ of absolutely $(\infty,p,q)$-summing operators from $X$ to $Y^*$ with its natural quasi-norm.

It follows from Lemma 1.1 (or, if one wishes, can be proved by the same method)

**Proposition 1.1** Let 1) $0 < r_1 \le r_2 \le 1$, $p_1 \le p_2$ and $q_1 \le q_2$ or 2) $0 < r_1 < r_2 \le 1$, $p_1 \ge p_2$, $q_1 \ge q_2$ and $1/r_2 + 1/p_2 + 1/q_2 \le 1/r_1 + 1/p_1 + 1/q_1$. If $z \in X \otimes Y$, then $||z||_{\wedge;r_2,p_2,q_2} \le ||z||_{\wedge;r_1,p_1,q_1}$. In particular, $||z||_{\wedge;1,\infty,\infty} \le ||z||_{\wedge;r_1,p_1,q_1}$. Consequently, a natural mappings $X\widehat{\otimes}_{r_1,p_1,q_1}Y \to X\widehat{\otimes}_{r_2,p_2,q_2}Y \to X\widehat{\otimes}Y$ are continuos injections of quasi-norms 1.

**Proposition 1.2.** If $X$ or $Y$ has the bounded approximation property, then $\mu_{r,p,q} = || \cdot ||_{\wedge;r,p,q}$ on $X \otimes Y$. Hence, in this case the dual of $X \widehat{\otimes_{r,p,q}} Y$ separates points, $j_{r,p,q}$ is injective and $X \widehat{\otimes_{r,p,q}} Y = X\widehat{\otimes}_{r,p,q}Y$ (and equals to the corresponding space of $(r,p,q)$-nuclear operators; see below Corollary 2.1).

*Proof.* It is enough to show that the map $j_{r,p,q}$ is injective. Since $X \widehat{\otimes_{r,p,q}} Y = Y \widehat{\otimes_{r,q,p}} X$, it is enough to consider the case, where $Y \in C$-MAP, $C \in [1,\infty)$. Let $z \in X \otimes Y$ and let $\widetilde{z} : X^* \to Y$ be an operator, associated with $z$ (note that this is one-to-one correspondence). There exists a finite rank operator $R : Y \to Y$ such that $||R|| \le C+1$ and $R\widetilde{z} (:= (\widetilde{\mathrm{id}_X \otimes R})(z)) = \widetilde{z}$ (see [**6**, 10.2.5, p. 131]). Fix $\delta > 0$ and choose a representation for $z$, $z = \sum_{k=1}^\infty \alpha_k\, x_k \otimes y_k$, with $||(\alpha_k)||_r\, ||(x_k)||_{w,p}\, ||(y_k)||_{w,q} \le ||z||_{\wedge;r,p,q}\,(1+\delta)$. Let $E := R(Y) \subset Y$, $M := \dim E$ and $\varepsilon = \varepsilon(M) \in (0,\delta]$ (to be chosen later). Then $\widetilde{z} = \sum_{k=1}^\infty \alpha_k\, \langle x_k, \cdot\rangle \varphi_k$, where $\varphi_k = Ry_k \in E$. Let $N$ be such that $||(\alpha_k)_N^\infty||_r\, ||(x_k)_N^\infty||_{w,p}\, ||\varphi_N^\infty||_{w,q} \le \varepsilon\, ||z||^\beta_{\wedge;r,p,q}$.

Now, since $E$ is finite dimensional, $\mathrm{id}_E$ admit a representation in $E^* \widehat{\otimes_{r,p,q}} E$ which give us an estimation from above for $\mu_{r,p,q}(\mathrm{id}_E)$ by a constant $C = C(M)$ depending only on $M$. Indeed, take an isomorphism $A : E \to l_2^M$ with $||A|| = 1, ||A^{-1}|| \le \sqrt{M}$ (see e.g. [**7**, Corollary 3.9]). Since $\mathrm{id}_{l_2^M} = \sum_{k=1}^M e'_k \otimes e_k$, $\mu_{r,p,q}(\mathrm{id}_{l_2^M}) \le M^{1/\beta}$. Therefore, $\mathrm{id}_E = A^{-1}\, \mathrm{id}_{l_2^M}\, A = \sum_{k=1}^M A^* e'_k \otimes A^{-1} e_k$ and $\mu_{r,p,q}(\mathrm{id}_E) \le M^{1/\beta+1/2}$. So, for any $v \in X \otimes E$, considering $\mathrm{id}_E \circ v$ $(= \mathrm{id}_E\, \widetilde{v})$ we obtain an inequality $\mu_{r,p,q}(\mathrm{id}_E \circ v) \le M^{1/\beta+1/2}\, ||\widetilde{v}||$. Since $||\widetilde{v}|| \le ||v||_{\wedge;r,p,q}$ we get

$$\mu_{r,p,q}(\mathrm{id}_E \circ v) \le C(M)\, ||v||_{\wedge;r,p,q}.$$

Hence, for our $z$ we get

$$\mu^\beta_{r,p,q}(z) \le \mu^\beta_{r,p,q}\Big(\sum_{k=1}^N \alpha_k\, x_k \otimes \varphi_k\Big) + \mu^\beta_{r,p,q}\Big(\sum_{k=N+1}^\infty \alpha_k\, x_k \otimes \varphi_k\Big) \le$$

$$||R||^\beta\, \mu^\beta_{r,p,q}\Big(\sum_{k=1}^N \alpha_k\, x_k \otimes y_k\Big) + C(M)^\beta\, ||\sum_{k=N+1}^\infty \alpha_k\, x_k \otimes \varphi_k||^\beta_{\wedge;r,p,q} \le$$

$$||R||^\beta\, (||(\alpha_k)||_r\, ||(x_k)||_{w,p}\, ||(y_k)||_{w,q})^\beta + C(M)^\beta\, (||(\alpha_k)_N^\infty||_r\, ||(x_k)_N^\infty||_{w,p}\, ||\varphi_N^\infty||_{w,q})^\beta \le$$

$$(1+\delta)^\beta\, ||R||^\beta\, ||z||^\beta_{\wedge;r,p,q} + \varepsilon^\beta C(M)^\beta\, ||z||^\beta_{\wedge;r,p,q}.$$

Taking $\varepsilon < \delta$ to have $\varepsilon^\beta C(M)^\beta < \delta$, we obtain $\mu^\beta_{r,p,q}(z) \le [(1+\delta)^\beta\, (C+1)^\beta + \delta]\, ||z||^\beta_{\wedge;r,p,q}$.

This inequality means, in particular, that the quasi-norms $\mu_{r,p,q}$ and $||\cdot||_{\wedge;r,p,q}$ are equivalent on the space $X\otimes Y$, thus giving the same completions. Since the space $X\widehat{\otimes}_{r,p,q}Y$ is complete (Remark (i) above), we obtain that the natural quotient map $\widehat{X\otimes_{r,p,q}Y} \to X\widehat{\otimes}_{r,p,q}Y$ is injective (recall that the last space is a subspace of the projective product $X\widehat{\otimes}Y$).

*Remark 1.2.* For an "operator" situation, see Corollary 2.1 below and (for $1\le p,q,\le\infty$) [**6**, pp. 249-251].

## Approximation properties

We begin with the main definition.

**Definition 2.1.** A Banach space $X$ has the approximation property $AP_{r,p,q}$ if for every Banach space $Y$ the canonical mapping $Y\widehat{\otimes}_{r,p,q}X\to L(Y^*,X)$ is one to one.

**Proposition 2.1.** The following conditions are equivalent:
1) $X$ has the $AP_{r,p,q}$.
2) For every Banach space $W$ the natural map $W^*\widehat{\otimes}_{r,p,q}X\to L(W,X)$ is one-to-one.
3) The natural map $X^*\widehat{\otimes}_{r,p,q}X\to L(X):=L(X,X)$ is one-to-one.

*Proof.* Implications 1) $\implies$ 2) $\implies$ 3) are evident.

3) $\implies$ 1). Suppose that the canonical map $X^*\widehat{\otimes}_{r,p,q}X\to L(X)$ is one-to-one, but there exists a Banach space $Y$ such that the natural map $Y\widehat{\otimes}_{r,p,q}X\to L(Y^*,X)$ is not injective. Let $z\in Y\widehat{\otimes}_{r,p,q}X$ be such that $z\ne 0$ and the associated operator $\widetilde{z}$ is a zero operator. Then we can find an operator $V$ from $L(Y,X^*)$ (the dual space to the projective tensor product $Y\widehat{\otimes}X$) so that trace $V\circ z^t=1$, where, as usual, $z^t$ is the transposed tensor element, $z^t\in X\widehat{\otimes}Y$. Since $V\circ z^t\in X\widehat{\otimes}X^*$ and trace $V\circ z^t=1$, the tensor element $(V\circ z^t)^t$ (which, evidently, belongs to $X^*\widehat{\otimes}_{r,p,q}X$) is not zero. Contradiction.

*Remark 2.1.* One can introduce also (in a similar way) some new notions of the approximation properties by using the Lapresté tensor products. We do not consider these properties here because we do not know how to work with the tensor products if their Banach duals do not separate points.

The following assertion is an analogue of [**4**, Prop. 6.2]. Its proof is contained in the corresponding proof of Proposition 6.2 from [**4**]. But since a situation now is a little bit different from the one there (quasi-norms are here not "selfadjoint"), we present a proof here.

**Proposition 2.2.** If $X^*$ has the $AP_{r,p,q}$, then $X$ has the $AP_{r,q,p}$.

*Proof.* We will use Proposition 1. As it is known [**1**], the projective tensor product $X^*\widehat{\otimes}X$ is a Banach subspace of the tensor product $X^*\widehat{\otimes}X^{**}$. The tensor product $X^*\widehat{\otimes}_{r,q,p}X$ is a linear subspace of $X^*\widehat{\otimes}X$, as well as $X^*\widehat{\otimes}_{r,q,p}X^{**}$ is a linear subspace of $X^*\widehat{\otimes}X^{**}$. Therefore, the natural map $X^*\widehat{\otimes}_{r,q,p}X\to X^*\widehat{\otimes}_{r,q,p}X^{**}$ is one-to-one. Now if $X^*$ has the $AP_{r.p.q}$, then the canonical map $X^{**}\widehat{\otimes}_{r,p,q}X^*\to L(X^*,X^*)$ is one-to-one. Since we can identify the tensor product $X^{**}\widehat{\otimes}_{r,p,q}X^*$ with the tensor product $X^*\widehat{\otimes}_{r,q,p}X^{**}$, it follows that the natural map $X^*\widehat{\otimes}_{r,q,p}X\to L(X,X)$ is one-to-one. Thus, if $X^*$ has the $AP_{r,p,q}$, then $X$ has the $AP_{r,q,p}$.

*Remark 2.2.* The inverse statement is not true. Some examples are given in [**4**, Remark 6.1].

*Remark 2.3.* (i) From the proof it follows that: *For any $X$ and $Y$ the natural map $X\widehat{\otimes}_{r,q,p}Y \to X\widehat{\otimes}_{r,q,p}Y^{**}$ is one-to-one.*

(ii) On the other hand: *For any $X$ and $Y$ the natural map $X\,\widehat{\otimes_{r,q,p}}\,Y \to X\,\widehat{\otimes_{r,q,p}}\,Y^{**}$* is an isometric embedding. To prove this, it is enough to apply Principle of Local Reflexivity (see e.g. [**6**, E.3.1]) as it was done in [**6**, 18.1.12] for the case $1 \le p, q \le \infty$.

Recall that a linear map $T : X \to Y$ is called $(r, p, q)$-nuclear if it has a representation $T = \sum_{k=1}^{\infty} \alpha_k \langle x'_k, \cdot\rangle y_k$, where $(\alpha_k) \in l_r$, $(x'_k) \in l_{w,p}(X^*)$ and $(y_k) \in l_{w,q}(Y)$. Every such a map is continuous. The space $N_{r,p,q}(X, Y)$ of all $(r, p, q)$-nuclear operators from $X$ to $Y$ can be considered as a quotient of the tensor product $X^*\widehat{\otimes}_{r,p,q}Y$ (as well as a quotient of $X^*\,\widehat{\otimes_{r,p,q}}\,Y$) by the kernel of the natural map $X^*\widehat{\otimes}_{r,p,q}Y \to L(X, Y)$. We equip this space with the induced quasi-norm ($\beta$-norm) denoted by $\nu_{r,p,q}$. If the corresponding quotient map has a trivial kernel, then we write $N_{r,p,q}(X, Y) = X^*\widehat{\otimes}_{r,p,q}Y$ (respectively, $N_{r,p,q}(X, Y) = X^*\,\widehat{\otimes_{r,p,q}}\,Y$). Thus, $X$ has the $AP_{r,p,q}$ iff for every Banach space $Y$ the equality $N_{r,p,q}(Y, X) = Y^*\widehat{\otimes}_{r,p,q}X$ holds.

*Remark 2.4.* 1) If $t \in (0, +\infty]$, then $N_{t,p,q}(X, Y)$ (the space of $(t, p, q)$-nuclear operators) can be defined by the analogues way: an operator $T : X \to Y$ is $(t, p, q)$-nuclear, if it can be written in the form $T = \sum_{k=1}^{\infty} \alpha_k \langle x'_k, \cdot\rangle y_k$, where $(\alpha_k) \in l_t$, $(x'_k) \in l_{w,p}(X^*)$ and $(y_k) \in l_{w,q}$. The quasi-norm $||\cdot||_{N_{t,p,q}}$ is defined for $T$ as $\inf ||(\alpha_k)||_t||(x'_k)||_{w,p}||(y_k)||_{w,q}$, where infimum is taken over all appropriate representations of $T$. If $1/t + 1/p + 1/q = 1$, then it is a norm.

2) In notation, we follow J.-T. Lapresté [**5**], only changing a triple $(p, r, s)$ there to $(r, p, q)$ here (see also [**6**] ; nota bene: A. Pietsch [**6**, 18.1] uses different notations for this class of operators and considers the cases where $1 \le p, q \le \infty$.)

It follows from Proposition 1.1:

**Proposition 2.3** Let 1) $0 < r_1 \le r_2 \le 1$, $p_1 \le p_2$ and $q_1 \le q_2$ or 2) $0 < r_1 < r_2 \le 1$, $p_1 \ge p_2$, $q_1 \ge q_2$ and $1/r_2 + 1/p_2 + 1/q_2 \le 1/r_1 + 1/p_1 + 1/q_1$. If $X$ has the $AP_{r_2,p_2,q_2}$, then $X$ has the $AP_{r_1,p_2,q_3}$. In particular, the $AP$ of A. Grothendieck implies any $AP_{r,p,q}$.

**Corollary 2.1.** (i) If $X$ has the bounded approximation property, then for all $r, p, q$ and $Y$ the equalities $N_{r,p,q}(Y, X) = Y^*\widehat{\otimes}_{r,p,q}X = Y^*\,\widehat{\otimes_{r,p,q}}\,X$ hold (with the same quasi-norms). (ii) If $Y^*$ has the bounded approximation property, then for all $r, p, q$ and $X$ the equalities $N_{r,p,q}(Y, X) = Y^*\widehat{\otimes}_{r,p,q}X = Y^*\,\widehat{\otimes_{r,p,q}}\,X$ hold (with the same quasi-norms).

*Proof.* Apply Propositions 1.2 and 2.3.

**Lemma 2.1.** The tensor product $\cdot\,\widehat{\otimes_{r,p,2}}\,\cdot$ is injective, i.e. if $i : Y \to W$ is an isometric imbedding and $z \in X\,\widehat{\otimes_{r,p,2}}\,Y$, then $\mu_{r,p,2}(z) = \mu_{r,p,2}(i \circ z)$.

*Proof.* It is clear that $\mu_{r,p,2}(z) \ge \mu_{r,p,2}(i\circ z)$. Let $\varepsilon > 0$ and $\sum_{k=1}^{N} \alpha_k x_k \otimes \varphi_k$ be a finite representation of $i\circ z$ in $X\otimes W$ such that $||(\alpha_k)||_r||(x_k)||_{w,p}||(\varphi_k)||_{w,q} \le (1+\varepsilon)\mu_{r,p,2}(i\circ z)$. Define an operator $S \in L(l_2^N, W)$ and a tensor element $z_0 \in X\,\widehat{\otimes_{r,p,2}}\,l_2^N$ by $S = \sum e'_k \otimes \varphi_k$ and $z_0 = \sum \alpha_k x_k \otimes e_k$. Let $E := \overline{\widetilde{z_0}(X^*)} \subset l_2^N$ and $P : l_2^N \to l_2^N$ be an orthogonal projector from $l_2^n$ onto $E$. Then $S \circ z_0 = i \circ z$, $\widetilde{SP \circ z_0} = \widetilde{i \circ z}$ and $SP(l_2^N) \subset i(Y)$. It follows that $z = (i^{-1}|_{i(Y)}SP) \circ z_0 = \sum \alpha_k x_k \otimes i^{-1}|_{i(Y)}SPe_k$ (as the elements of $X \otimes Y$) and

$$\mu_{r,p,2}(z) \le ||S||\,||(\alpha_k)||_r||(x_k)||_{w,p} = ||(\alpha_k)||_r||(x_k)||_{w,p}||(\varphi_k)||_{w,q} \le (1+\varepsilon)\mu_{r,p,2}(i\circ z).$$

Therefore, the natural map $\mathrm{id}_X \otimes i : X \otimes Y \to X \otimes W$ is an isometric imbedding and it extends to the isometry $\mathrm{id}_X\,\overline{\otimes}i : X\,\widehat{\otimes_{r,p,2}}\,Y \to X\,\widehat{\otimes_{r,p,2}}\,W$.

*Remark 2.5.*. It follows from Lemma 2.1 and from definitions of $\widehat{\otimes}_{r,p,2}$ (and $N_{r,p,2}$) that $\widehat{\otimes}_{r,p,2}$ (and $N_{r,p,2}$) are injective (consider the quotient maps $\widehat{\otimes_{r,p,2}} \to \widehat{\otimes}_{r,p,2} \to N_{r,p,2}$). On the other hand, the injectivity of $\widehat{\otimes}_{r,p,2}$ (and $N_{r,p,2}$) can be proved in the same way as above by consideration the infinite representations of $z, \widetilde{z}$ (instead of finite ones) in the given proof. Also, we see from the proof that if $\sum_{k=1}^{\infty} \alpha_k x_k \otimes \varphi_k$ is a representation of $i \circ z$ in $X \widehat{\otimes_{r,p,2}} W$ (see [**5**, Proposition 1.3, p. 52], then the corresponding representation for $z$ in $X \widehat{\otimes_{r,p,2}} Y$ can be taken of the type $\sum_{k=1}^{\infty} \alpha_k x_k \otimes y_k$ with $||(y_k)||_{w,2} \leq ||(\varphi_k)||_{w,2}$. The same is true for $\widehat{\otimes}_{r,p,2}$ and $N_{r,p,2}$.

The first part of the following fact is partially known (cf. [**6**, 18.1.15-18.1.16] for $1 \leq p, q \leq \infty$).

**Proposition 2.4.** For any Banach spaces $X, Y$ the equalities

$$N_{r,p,2}(Y,X) = Y^* \widehat{\otimes}_{r,p,2} X = Y^* \widehat{\otimes_{r,p,2}} X \text{ and } N_{r,2,q}(Y,X) = Y^* \widehat{\otimes}_{r,2,q} X = Y^* \widehat{\otimes_{r,2,q}} X$$

hold (with the same quasi-norms). In particular, every Banach space has the $AP_{r,p,2}$ and the $AP_{r,2,p}$.

*Proof.* As is known, the operator ideal $N_{r,p,2}$ is injective, (see [**6**, 18.1.8] for the case $1 \leq p, q \leq \infty$); apply (factorization) Theorem 2.5 [**5**] in other cases). I.e., if $X \subset G$, $T \in L(Y,X)$ and $T \in N_{r,p,2}(Y,G)$, then $T \in N_{r,p,2}(Y,X)$ (with the same quasi-norm). Also, the tensor product $\cdot \widehat{\otimes_{r,p,2}} \cdot$ is injective too (Lemma 2.1)Now, let $z \in Y^* \widehat{\otimes_{r,p,2}} X$ and $i : X \to L_\infty$ be an isometric embedding of $X$ into an $L_\infty$-space. Since $L_\infty$ has the MAP, $\nu_{r,p,2}(\widetilde{i \circ z}) = \mu_{r,p,2}(i \circ z)$ (see Corollary 2.1). Hence, $\mu_{r,p,2}(z) = \mu_{r,p,2}(i \circ z) = \nu_{r,p,2}(\widetilde{i \circ z}) = \nu_{r,p,2}(i\widetilde{z}) \leq \nu_{r,p,2}(\widetilde{z}) \leq \mu_{r,p,2}(z)$.

To get the last two equalities it is enough to apply the surjectivity of the operator ideal $N_{r,2,q}$ and Corollary 2.1 (second part), by using the same idea as above, or just to apply Lemma 2.1 and Remark 2.5 (second part): Take $z \in Y^* \widehat{\otimes_{r,2,q}} X$ and a quotient map $Q : L_1 \to Y$. Considering $(z \circ Q)^t$ as an element of $X \widehat{\otimes_{r,q,2}} L_\infty$, we get:

$$\nu_{r,q,2}(\widetilde{Q^* \circ z^t}) = \mu_{r,q,2}(Q^* \circ z^t) = \mu_{r,q,2}(z^t) = \mu_{r,2,q}(z).$$

But $\nu_{r,q,2}(\widetilde{Q^* \circ z^t}) \leq \nu_{r,2,q}(\widetilde{z \circ Q}) \leq \nu_{r,2,q}(z)$; thus, $\mu_{r,2,q}(z) \leq \nu_{r,2,q}(z)$.

,

*Remark 2.6.* The fact that every $X$ has the $AP_{1,2,\infty}$ is essentially contained in [**6**, 27.4.10, Proposition]. It is strange, but it seems that a corresponding fact for $AP_{1,\infty,2}$ appears here for the first time. Note that this fact follows also from the preceding by virtue of Proposition 2.2: if every $X$ has the $AP_{1,2,\infty}$, then $X^*$ possesses this property, and by Proposition 2.2 $X$ has the $AP_{1,\infty,2}$.

Many of the above approximation properties were considered earlier, e.g. in the papers [**2, 3, 4**] etc:

(i) For $p = q = \infty$, we get the $AP_r$ from [**3, 4**].

(ii) For $p = \infty$, we get the $AP_{[r,q]}$ from [**2, 4**].

(iii) For $q = \infty$, we get the $AP^{[r,p]}$ from [**2, 4**].

Following notations from [**4**] (see also [**2**]), we denote

$N_{r,\infty,\infty}$ by $N_r$,

$N_{r,\infty,q}$ by $N_{[r,q]}$,

$N_{r,p,\infty}$ by $N^{[r,p]}$,

$\widehat{\otimes}_{r,\infty,\infty}$ by $\widehat{\otimes}_r$,
$\widehat{\otimes}_{r,\infty,q}$ by $\widehat{\otimes}_{[r,q]}$,
$\widehat{\otimes}_{r,p,\infty,}$ by $\widehat{\otimes}^{[r,p]}$.
The corresponding notations are used also for the $AP_{r,p,q}$ (see above (i)–(iii)).

Almost all the information about Banach spaces without (or with) the properties $AP_r$, $AP_{[r,q]}$ and $AP^{[r,p]}$ which is known to us by now, can be found in [**2, 3, 4**]. Other results in this direction are the subject of the forthcoming paper of the author.

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