On Z_d-symmetry of spectra of some nuclear operators

Oleg Reinov

Abstract. It was shown by M. I. Zelikin (2007) that the spectrum of a nuclear operator in a Hilbert space is central-symmetric iff the traces of all odd powers of the operator equal zero. B. Mityagin (2016) generalized Zelikin's criterium to the case of compact operators (in Banach spaces) some of which powers are nuclear, considering even a notion of so-called $\mathbb{Z}_d$-symmetry of spectra introduced by him. We study α-nuclear operators generated by the tensor elements of so-called α-projective tensor products of Banach spaces, introduced in the paper (α is a quasi-norm). We give exact generalizations of Zelikin's theorem to the cases of $\mathbb{Z}_d$-symmetry of spectra of α-nuclear operators (in particular, for s-nuclear and for (r,p)-nuclear operators). We show that the results are optimal.

1. Introduction

It is well known that every nuclear (= trace class) operator on a Hilbert space has the absolutely summable sequence of eigenvalues [**21**]. Moreover, the famous Lidskiĭ theorem [**11**] says that for such an operator its trace is equal to the sum of all its eigenvalues (written in according to their algebraic multiplicities).

It is clear that if the spectrum of such an operator is central-symmetric, then its trace equals zero. Moreover, since every power of a nuclear operator T is nuclear too and has a central-symmetric spectrum if T has, we see that, for such T, trace $T^k = 0$ for every odd natural number k.

M.I. Zelikin has noticed that for an finite dimensional spaces the converse is also true (see [**23**, Theorem 1]), and then he proved the corresponding theorem for any nuclear operator in a separable Hilbert space ([**23**, Theorem 2]). At the same time, his proof was rather complicated. We are going to present, in particular, a more simple proof below.

Recall that the spectrum of a compact operator is central-symmetric, if together with any eigenvalue $\lambda \neq 0$ it has the eigenvalue $-\lambda$ of the same multiplicity. Thus,

AMS Subject Classification 2010: 47B06 Riesz operators; eigenvalue distributions; approximation numbers, s-numbers, Kolmogorov numbers, entropy numbers, etc. of operators. 47B10 Operators belonging to operator ideals (nuclear, p-summing, in the Schatten-von Neumann classes, etc.). 46B28 Spaces of operators; tensor products; approximation properties.

Key words: nuclear operator; tensor product; approximation property; eigenvalue.

M.I. Zelikin has proved: The spectrum of a nuclear operator A acting on a separable Hilbert space is central-symmetric iff trace $A^{2n-1} = 0$, $\forall\, n \in \mathbf{N}$.

Let us mention that this theorem can not be extended to the case of general Banach spaces: it follows from Grothendieck-Enflo-Davie results [**7, 4, 3**] that there exists a nuclear operator T in the space l_1 of absolutely summable sequences such that $T^2 = 0$ but trace $T = 1$ (the operator can be chosen even in such a way, that it is s-nuclear for every $s \in (2/3, 1]$; see Definition 1 below and [**13**, 10.4.5]).

A right generalization of Zelikin's theorem was found by B. Mityagin [**12**]. He introduced a notion of so-called Z_d-symmetry of the spectra of compact operators in Banach spaces and gave a criterium for the spectra of an operator (some of which power is nuclear) to be Z_d-symmetric. For $d = 2$, this gives a generalization of the criterium of M.I. Zelikin. We will use this notion of the Z_d-symmetry to formulate and to prove an *exact* generalization of Zelikin's theorem for the case of subspaces of quotients of L_p-spaces (thus getting, in a simpler way, Zelikin's result putting $p = 2$ and $d = 2$). However, we will have to consider so-called s-nuclear operators instead of nuclear ones in Zelikin's theorem. To formulate our main result, let us recall the definitions of s-nuclearity of operators and of Z_d-symmetry of spectra.

Definition 1.1 (A. Grothendieck). *An operator $T : X \to Y$ is s-nuclear $(0 < s \le 1)$, if*

$$\exists\ (x'_k) \subset X^*, (y_k) \subset Y :\ \sum_{k=1}^{\infty} ||x'_k||^s\, ||y_k||^s < \infty,$$

$$T(x) = \sum_{k=1}^{\infty} x'_k(x) y_k,\ \ \forall\ x \in X.$$

For $s = 1$, they say that T is nuclear.

Let us note that A. Grothendieck in [**7**] called such operators "applications de puissance p.éme sommable".

Definition 1.2 (B. Mityagin). *Let T be an operator in X, all non-zero spectral values of which are eigenvalues of finite multiplicity and have no limit point except possibly zero. For a fixed $d = 2, 3, \ldots$ and for the operator T, the spectrum of T is called $\mathbb{Z}_d$-symmetric, if $0 \ne \lambda \in \mathrm{sp}\,(T)$ implies $t\lambda \in \mathrm{sp}\,(T)$ for every $t \in \sqrt[d]{1}$ and of the same multiplicity.*

Our generalization of the Zelikin's theorem is:

Theorem 1.1. *Let Y be a subspace of a quotient (or a quotient of a subspace) of some $L_p(\mu)$-space, $1 \le p \le \infty$ and $1/r = 1 + |1/2 - 1/p|$. If $T : Y \to Y$ is r-nuclear, then* trace T *is well-defined. For a fixed $d = 2, 3, \ldots,$ the spectrum of T is $\mathbb{Z}_d$-symmetric iff*

$$\text{trace } T^{kd+j} = 0 \ \textit{ for all }\ k = 0, 1, 2, \ldots \ \textit{ and }\ j = 1, 2, \ldots, d-1.$$

In particular, if trace $T \ne 0$, *then* $T^2 \ne 0$.

Note that if $d = 2$, we obtain an exact generalization of Zelikin's theorem on the central symmetry.

Also, we present some sharp (optimal in r, p) generalizations of Zelikin's theorem to the case of so-called (r, p)-nuclear and dually (r, p)-nuclear operators (see Theorem 4.2).

Theorems 1.1 and 4.2 are optimal with respect to p and r :

THEOREM 1.2. *Let* $p \in [1, \infty], p \neq 2$, $1/r = 1 + |1/2 - 1/p|$. *There exists a nuclear operator* V *in* l_p *(in* c_0 *for* $p = \infty$*) such that*

1) V *is* s*-nuclear for each* $s \in (r, 1]$;
2) V *is not* r*-nuclear;*
3) $\operatorname{trace} V = 1$ *and* $V^2 = 0$.

Note that for $p = \infty$ or $p = 1$ we have $r = 2/3$ and for $p = 2$ we have $r = 1$. The proofs will be given in Sections 4.5, 4.6

Let us note that some of the implications of our results on $\mathbb{Z}$-symmetry of spectra are the consequences of Mityagin's theorem. But it seems that our proofs are shorter. Besides, our aim was to obtain the *exact* generalizations of Zelikin's theorem in an independent way.

2. Content

In Section 3, we present the general notations concerning Banach spaces, spaces of operators, tensor products, vector-valued sequence spaces.

In first Subsection of Section 4, we give a definition of projective tensor quasi-norms α and introduce the α-projective tensor products of Banach spaces. We show that these tensor products are continuously imbedded in the projective products of A. Grothendieck. For complete α-projective tensor products, we define α-nuclear operators in a natural way (as elements of corresponding factor spaces). Also in a natural way, we define a notion of the approximation property AP_α, give a simple characterization of Banach spaces with this property and present some main examples.

In second Subsection of Section 4, we consider some properties of the α-projective tensor products of spectral type l_1 (so that all α-nuclear operators have absolutely summable sequences of their eigenvalues). In particular, we are interested in the question of when the trace formulas are true. In the end, examples are given.

In third Subsection of Section 4, we introduce so-called α-extension and α-lifting properties for a projective tensor quasi-norms α. W are interested here in connection between the AP_α, trace formulas and the statements of type "trace $T = 1 \implies T^2 \neq 0$".

In forth Subsection of Section 4, we proof one of the main our theorem on $\mathbb{Z}_d$-symmetry of spectra of α-nuclear operators. We apply the results to some concrete quasi-normed tensor products, getting a generalization of Zelikin's theorem to the case of (r, p, q)-nuclear operators in general Banach spaces.

In fifth Subsection of Section 4, a proof of Theorem 1.1 is given.

Finally, in sixth (last) Subsection of Section 4, we show that the main results of the previous subsections are sharp. Maybe, it it worthwhile to mention a new result on the asymptotically Hilbertian spaces (the last theorem in the paper).

3. Notation and preliminaries

Throughout, we denote by $X, Y, E, F, W \dots$ Banach spaces over a field $\mathbb{K}$ (which is either $\mathbb{R}$ or $\mathbb{C}$); $X^*, Y^*, \dots$ are Banach dual to $X, Y, \dots$. By $x, y, x', \dots$ (maybe with indices) we denote elements of $X, Y, \dots, Y^* \dots$ respectively. $\pi_Y : Y \to Y^{**}$ is a natural isometric imbedding. By a subspace of a Banach space we mean a closed linear subspace.

Notations $l_p, (0 < p \leq \infty, n = 1, 2, \dots)$, c_0 are standard; e_k $(k = 1, 2, \dots)$ is the k-th unit vector in l_p or c_0 (when we consider the unit vectors as the linear functionals, we use notation e'_k). We use id_X for the identity map in X.

It is denoted by $F(X, Y)$ a vector space of all linear continuous finite rank mappings from X to Y. By $X \otimes Y$ we denote the algebraic tensor product of the spaces X and Y. $X \otimes Y$ can be considered as a subspace of the vector space $F(X^*, Y)$ (namely, as a vector space of all linear weak*-to-weak continuous finite rank operators). We can identify also the tensor product (in a natural way) with a corresponding subspace of $F(Y^*, X)$. If $X = W^*$, then $W^* \otimes Y$ is identified with $F(W, Y)$. If $z \in X \otimes Y$, then $\widetilde{z}$ is the corresponding finite rank operator. If $z \in X^* \otimes X$ and e.g. $z = \sum_{k=1}^n x'_k \otimes x_k$, then trace $z := \sum_{k=1}^n \langle x'_k, x_k \rangle$ does not depend on representation of z in $X^* \otimes X$. $L(X, Y)$ is a Banach space of all linear continuous mappings ("operators") from X to Y equipped with the usual operator norm.

If $A \in L(X, W)$ and $B \in L(Y, G)$, then a linear map $A \otimes B : X \otimes Y \to W \otimes G$ is defined by $A \otimes B(x \otimes y) := Ax \otimes By$ (and then extended by linearity). Since $\widetilde{A \otimes B}(z) = B\widetilde{z}A^*$ for $z \in X \otimes Y$, we will use notation $B \circ z \circ A^* \in W \otimes G$ for $A \otimes B(z)$. In the case where X is a dual space, say F^*, and $T \in L(W, F)$ (so, $A = T^* : F^* \to W^*$), one considers a composition $B\widetilde{z}T$; in this case $T^* \otimes B$ maps $F^* \otimes Y$ into $W^* \otimes Y$ and we use notation $B \circ z \circ T$ for $T^* \otimes B(z)$.

A projective tensor product $X \widehat{\otimes} Y$ of Banach spaces X and Y is defined as a completion of $X \otimes Y$ with respect to the norm $|| \cdot ||_\wedge$: if $z \in X \otimes Y$, then $||z||_\wedge := \inf \sum_{k=1}^n ||x_k|| \, ||y_k||$, where infimum is taken over all representation of z as $\sum_{k=1}^n x_k \otimes y_k$. We can try to consider $X \widehat{\otimes} Y$ also as operators $X^* \to Y$ or $Y^* \to X$, but this correspondence is, in general, not one-to-one. However, the natural map $(X \otimes Y, || \cdot ||_\wedge) \to L(X^*, Y)$ is continuous and can be extended to the completion $X \widehat{\otimes} Y$; for a tensor element $z \in X \widehat{\otimes} Y$, we still denote by $\widetilde{z}$ the corresponding operator. Note that $X \widehat{\otimes} Y = Y \widehat{\otimes} X$ in a sense. If $z \in X \widehat{\otimes} Y, \varepsilon > 0$, then one can represent z as $z = \sum_{k=1}^\infty x_k \otimes y_k$ with $\sum_{k=1}^\infty ||x_k|| \, ||y_k|| < ||z||_\wedge + \varepsilon$. For $z \in X^* \widehat{\otimes} X$ with a "projective representation" $z = \sum_{k=1}^\infty x'_k \otimes x_k$, trace $z := \sum_{k=1}^\infty \langle x_k, y_k \rangle$ does not depend of representation of z. The Banach dual $(X \widehat{\otimes} Y)^*$ equals $L(Y, X^*)$ (with duality $\langle T, z \rangle = \text{trace } T \circ z$.)

One more notation: If $\mathfrak{A}$ is an operator ideal **[13]** then we often use the notation $\mathfrak{A}(X)$ for the space $\mathfrak{A}(X, X)$.

Finally,

$$l_p(X) := \{(x_i) \subset X :\ ||(x_i)||_p := \big(\sum ||x_i||^p\big)^{1/p} < \infty\},$$
$$l_\infty(X) := \{(x_i) \subset X :\ ||(x_i)||_\infty := \sup_i ||x_i|| < \infty\},$$
$$l_p^w(X) := \{(x_i) \subset X :\ ||(x_i)||_{w,p} := \sup_{||x'||\le 1} \big(\sum |\langle x', x_i\rangle|^p\big)^{1/p} < \infty\},$$
$$l_\infty^w(X) := \{(x_i) \subset X :\ ||(x_i)||_{w,\infty} := \sup_i ||x_i|| < \infty\}.$$

Note that if $p \le q$, then $||\cdot||_q \le ||\cdot||_p$ and $||\cdot||_{w,q} \le ||\cdot||_{w,p}$. If $0 < p \le \infty$, then p' is a conjugate exponent: $1/p + 1/p' = 1$ if $p \ge 1$ and $p' = \infty$ if $p \in (0,1]$.

4. Quasi-normed tensor products and approximation properties

4.1. Projective quasi-norms and approximation properties. Let α be a function on a vector space E, $\alpha : E \to \widehat{\mathbb{R}}$. We say that α is a *quasi-norm* on E if 1) $\alpha(E) \subset [0, +\infty]$ and $\alpha(x) = 0$ implies $x = 0$; 2) there exists a constant $C > 0$ such that $\alpha(x+y) \le C\,[\alpha(x) + \alpha(y)]$ for $x, y \in E$; 3) $\alpha(ax) = |a|\,\alpha(x)$ for $a \in \mathbb{K}, x \in E$.

Definition 4.1. (i) *Given a pair (E, α), where α is a quasi-norm on a vector space E, a quasi-normed space associated with the pair (E, α) is the quasi-normed vector space*

$$E_\alpha := \{x \in E :\ \alpha(x) < \infty\}.$$

(ii) *The quasi-normed space E_α is complete (= a quasi-Banach space), if every Cauchy sequence in E_α α-converges to an element of E_α.*

Note that E_α is a quasi-normed vector space in the sense of [**9**, p. 159] and we may generate the corresponding topology (see [**9**, p. 159-160], [**1**, p. 445]).

Remark 4.1. 1) It may be $E_\alpha = E$.

2) It is well known [**1**, p. 445] that if E_α is a quasi-normed space, then there are a number $\beta \in (0,1]$ and a β-norm $||\cdot||$ on E_α which is equivalent to the quasi-norm α. Recall that a β-norm on a vector space F is a quasi-norm $||\cdot|| : F \to \mathbb{R}$ such that for all $x, y \in F$ one has the following β-triangle inequality: $||x+y||^\beta \le ||x||^\beta + ||y||^\beta$.

Now, let α be a quasi-norm on a projective tensor product $X\widehat{\otimes}Y$ such that $\alpha(x \otimes y) = ||x||\,||y||$ for $x \in X, y \in Y$. The associated quasi-normed tensor product (which will be denoted by $X\widehat{\otimes}_\alpha Y$ and called "α-projective tensor product") is the α-closure of $X \otimes Y$ in $(X\widehat{\otimes}Y)_\alpha$ (in the concrete cases we will use some specific notations). Thus,

$$X\widehat{\otimes}_\alpha Y := \{u \in X\widehat{\otimes}Y :\ \alpha(u) < \infty \ \text{ and } \ \exists\, (u_n) \subset X \otimes Y :\ \alpha(u - u_n) \underset{n\to\infty}{\to} 0\}.$$

More generally:

Definition 4.2. (i) *Let $\widehat{\otimes}$ denotes the class of all tensor elements of the projective tensor products of arbitrary Banach spaces. A projective tensor quasi-norm α is a map from $\widehat{\otimes}$ to $\widehat{\mathbb{R}}$ such that α is a quasi-norm on each component $X\widehat{\otimes}Y$ with the properties:*

(Q_1) *$\alpha(x \otimes y) = ||x||\,||y||$ for $x \in X, y \in Y$.*

(Q_2) *There exists a constant $C > 0$ such that $\alpha(u_1 + u_2) \leq C\,[\alpha(u_1) + \alpha(u_2)]$ for all X, Y and $u_1, u_2 \in X\widehat{\otimes}Y$.*

(Q_3) *If $u \in X\widehat{\otimes}Y$, $A \in L(X,E)$ and $B \in L(Y,F)$, then $\alpha(A \otimes B(u)) \leq ||A||\,\alpha(u)\,||B||$.*

(ii) *A projective tensor quasi-norm α ia said to be complete, if every α-projective tensor product $X\widehat{\otimes}_\alpha Y$ is complete, that is quasi-Banach.*

For every projective tensor quasi-norm α there exist $\beta \in (0,1]$ and an equivalent β-norm $||\cdot||_\beta$ on $\widehat{\otimes}$ so that $X\widehat{\otimes}_\alpha Y = X\widehat{\otimes}_{||\cdot||_\beta} Y$ (i.e. there exists a quasi-norm $||\cdot||_\beta$ with β-triangle inequality such that for some positive constants C_1, C_2 and for all projective tensor elements u the inequalities $C_1\alpha(u) \leq ||u||_\beta \leq C_2\alpha(u)$ hold). Thus, we may assume, if needed, that a priori α is a β-norm.

We are not going to consider here in detail the properties of just introduced objects. But we need below the fact that the inclusions $X\widehat{\otimes}_\alpha Y \hookrightarrow X\widehat{\otimes}Y$ are continuous for all Banach spaces X, Y (in the main Example 4.1 below this will be automatically fulfilled).

Proposition 4.1. *Let α be a complete projective tensor norm. The natural injections $X\widehat{\otimes}_\alpha Y \to X\widehat{\otimes}Y$ are continuous for all Banach spaces X and Y. Moreover, there is a constant $d = d(\alpha)$ such that for all X, Y and $u \in X\widehat{\otimes}_\alpha Y$ we have: $||u||_\wedge \leq d\,\alpha(u)$.*

Proof. Suppose the last assertion is not true and there exist the sequences $(X_n), (Y_n)$ and (u_n) with $u_n \in X_n\widehat{\otimes}_\alpha Y_n$ so that $\alpha(u_n) \leq 1/(2C)^n$ and $||u_n||_\wedge \geq n$. Put $X := \big(\sum X_n\big)_{l_2}$ and $Y := \big(\sum Y_n\big)_{l_2}$. Let $i_n : X_n \to X$ and $j_n : Y_n \to Y$ be the natural injections. Consider the sequence $(z_N) := \big(\sum_{k=1}^N (i_k \otimes j_k)(u_k)\big)$. For any natural numbers K and m, we have:

$$\alpha\Big(\sum_{k=K+1}^{K+m} (i_k \otimes j_k)(u_k)\Big) \leq \sum_{k=1}^{m} C^k\,\alpha\big((i_{K+k} \otimes j_{K+k})(u_{K+k})\big) \leq \sum_{k=1}^{\infty} \frac{C^k}{(2C)^{K+k}} \leq \frac{1}{(2C)^K}.$$

Hence, (z_N) is a Cauchy sequence in $X\widehat{\otimes}_\alpha Y$ and, by the completeness of α, converges to an element $u := \sum_{k=1}^\infty (i_k \otimes j_k)(u_k) \in X\widehat{\otimes}_\alpha Y$. On the other hand, if $P_n : X \to X_n$ and $Q_n : Y \to Y_n$ are the natural "projections", then $||u||_\wedge \geq ||(P_n \otimes Q_n)(u)||_\wedge = ||u_n||_\wedge \geq n$. $\square$

Since $X\widehat{\otimes}_\alpha Y$ ia a linear subspace of $X\widehat{\otimes}Y$, the space $L(Y, X^*)$ separates points of $X\widehat{\otimes}_\alpha Y$. If $u \in X\widehat{\otimes}_\alpha Y$, then $u = 0$ iff trace $U \circ u = 0$ for every $U \in L(Y, X^*)$. In particular, the dual space $(X\widehat{\otimes}_\alpha Y)^*$ separates points of $X\widehat{\otimes}_\alpha Y$.

It is clear that every tensor element $u \in X\widehat{\otimes}_\alpha Y$ generates a nuclear operator $\widetilde{u} : X^* \to Y$. If X is a dual space, say E^*, then we get a canonical mapping $j_\alpha : E^*\widehat{\otimes}_\alpha Y \to L(E, Y)$. The image of j_α is denoted here by $N_\alpha(E, Y)$, and we equip it with an "α-nuclear" quasi-norm ν_α : This is a quasi-norm induced from $E^*\widehat{\otimes}_\alpha Y$ via the quotient map $E^*\widehat{\otimes}_\alpha Y \to N_\alpha(E, Y)$. If the projective tensor quasi-norm α is complete, then $N_\alpha(E, Y)$ is a quasi-Banach space.

Definition 4.3. *Let α be a complete projective tensor quasi-norm. We say that a Banach space X has the approximation property AP_α, if for every Banach space E the canonical map $E^*\widehat{\otimes}_\alpha X \to N_\alpha(E,X)$ is one-to-one (in other words, if $E^*\widehat{\otimes}_\alpha X = N_\alpha(E,X)$).*

Note that if $\alpha = ||\cdot||_\wedge$, then we get the classical approximation property AP of A. Grothendieck [**7**]. It must be clear that the AP implies the AP_α, for any projective tensor quasi-norm.

We will need below the following

Lemma 4.1. *A Banach space X has the AP_α iff the canonical map $X^*\widehat{\otimes}_\alpha X \to L(X)$ is one-to-one.*

Proof. It is enough to repeat (word for word with same notations) the proof of [**20**, Proposition 6.1]. $\square$

Example 4.1. Let $0 < r, s \le 1$, $0 < p, q \le \infty$ and $1/r + 1/p + 1/q = 1/\beta \ge 1$. We define a tensor product $X\widehat{\otimes}_{r,p,q}Y$ as a linear subspace of the projective tensor product $X\widehat{\otimes}Y$, consisting of all tensor elements z which admit representations of type

$$(1) \qquad z = \sum_{k=1}^{\infty} \alpha_k x_k \otimes y_k,\ (\alpha_k) \in l_r,\ (x_k) \in l_{w,p}(X),\ (y_k) \in l_{w,q}(Y);$$

we equip it with the quasi-norm $||z||_{r,p,q} := \inf ||(\alpha_k)||_r\, ||(x_k)||_{w,p}\, ||(y_k)||_{w,q}$, where the infimum is taken over all representations of z in the above form (1). Note that this tensor product is β-normed (cf. [**10**], where it is considered a "finite-sums-representation" version of the above tensor product). It is quasi-Banach (for the completeness, see the author's preprint "Approximation properties associated with quasi-normed operator ideals of (r,p,q)-nuclear operators"[1]). The corresponding quasi-normed operator ideal $N_{r,p,q}$ is the quasi-Banach ideal of (r,p,q)-nuclear operators (cf. [**13, 10**]). In particular cases where one or two of the exponents p, q are ∞, we will use the notations close to those from [**18, 20**] (here we change p', q' to p, q) : We denote $N_{r,\infty,\infty}$ by N_r, $N_{r,\infty,q}$ by $N_{[r,q]}$, $N_{r,p,\infty}$ by $N^{[r,p]}$, $\widehat{\otimes}_{r,\infty,\infty}$ by $\widehat{\otimes}_r$, $\widehat{\otimes}_{r,\infty,q}$ by $\widehat{\otimes}_{[r,q]}$, $\widehat{\otimes}_{r,p,\infty}$, by $\widehat{\otimes}^{[r,p]}$.

The corresponding notations are used also for the $AP_{r,p,q}$:

(i) For $p = q = \infty$, we get the AP_r from [**20**].

(ii) For $p = \infty$, we get the $AP_{[r,q]}$ from [**18, 20**].

(iii) For $q = \infty$, we get the $AP^{[r,p]}$ from [**18, 20**].

We will need some known facts concerning the approximation properties from Example 4.1. Let us collect them in

Lemma 4.2. 1) [**16**, Corollary 10] *Let $s \in (0,1]$, $p \in [1,\infty]$ and $1/s = 1 + |1/p - 1/2|$. If a Banach space Y is isomorphic to a subspace of a quotient (or to a quotient of a subspace) of an L_p-space then it has the property AP_s.*

---

[1] http://www.mathsoc.spb.ru/preprint/2017/index.html#08

2) [**18**, Corollary 4.1], [**20**, Theorem 7.1] *Let $1/r - 1/p = 1/2$. Every Banach space has the properties $AP_{[r,p']}$ and $AP^{[r,p']}$.*

A proof of the assertion 2) can be found below (see Example 4.3). See also [**20**] for some other results in this direction.

Remark 4.2. As a matter of fact, a proof of the assertion that every Banach space has the $AP^{[1,2]}$ is contained implicitly in [**13**]. It was obtained also there that this assertion (after applying some results of Complex Analysis) implies the Grothendieck-Lidskiǐ type trace formulas for operators from $N^{[1,2]}$ [**13**, 27.4.11] (and this implies the Lidskiǐ trace formula for trace-class operators in Hilbert spaces and the Grothendieck trace formula for $N_{2/3}$ as well). On the other hand, there is a very simple way to get these results on $AP^{[1,2]}$ and $N^{[1,2]}$ from the Lidskiǐ theorem (see the proofs of [**20**, Theorems 7.1-7.3] for $p = 2$).

4.2. Spectral type l_1. Let T be an operator in X, all non-zero spectral values of which are eigenvalues of finite multiplicity and have no limit point except possibly zero. Put $\lambda(T) = \{\lambda \in \text{eigenvalues}\,(T) \setminus \{0\}\}$ (the eigenvalues of T are taken in according to their multiplicities). We say that an operator $T \in L(X,X)$ is *of spectral type* l_1, if the sequence of all eigenvalues $\lambda(T) := (\lambda_k(T))$ is absolutely summable. In this case, we can define the *spectral trace of* T : $\operatorname{sp}\operatorname{tr}(T) := \sum \lambda_k(T)$. We say that a subspace $L_1(X,X) \subset L(X,X)$ is *of spectral type* l_1, if every operator $T \in L_1(X,X)$ is of spectral type l_1. Recall that an operator ideal $\mathfrak{A}$ is of spectral type l_1, if every its component $\mathfrak{A}(X,X)$ is of spectral type l_1.

Definition 4.4. *Let α be a projective tensor quasi-norm. The tensor product $X\widehat{\otimes}_\alpha X$ is of spectral type l_1, if the space $N_\alpha(X,Y)$ is of spectral type l_1. The projective tensor quasi-norm α (or the tensor product $\widehat{\otimes}_\alpha$) is of spectral type l_1, if the corresponding operator ideal N_α is of spectral type l_1.*

Example 4.2. $N_1(H)$ $(= N_{[1,2]}(H) = N^{[1,2]}(H) = S_1(H)$, trace class operators in a Hilbert space) is of spectral type l_1 [**21**]. $\widehat{\otimes}_{2/3}$ and $N_1 \circ N_1$ are of spectral type l_1 [**7**]. $N^{[1,2]}$ is of spectral type l_1 (see [**13**, see 27.4.9, end of the proof]). $N_{[1,2]}$ is of spectral type l_1 (see [**20**, Theorem 7.2 for $p = 2$]; it follows also from the previous assertion). More general, if $1/r - 1/p = 1/2$, then $\widehat{\otimes}_{[r,p]} = N_{[r,p]}$, $\widehat{\otimes}^{[r,p]} = N^{[r,p]}$ and they are of spectral type l_1 (see [**20**, Theorems 7.1-7.3]).

Let us note that in all cases in Example 4.2 the trace formula for corresponding operators (say, T) is valid: $\operatorname{trace} T = \operatorname{sp}\operatorname{tr} T$. A general result in this direction is

Proposition 4.2. *Let α be a complete projective tensor quasi-norm of spectral type l_1. For every Banach space X with the AP_α and every $T \in N_\alpha(X)$, one has:* $\operatorname{trace} T = \operatorname{sp}\operatorname{tr} T$. *Conversely, if for a Banach space X and for every $z \in X^*\widehat{\otimes}_\alpha X$ the equality* $\operatorname{trace} z = \operatorname{sp}\operatorname{tr} \widetilde{z}$ *holds, then X possesses the AP_α.*

PROOF. Let X has the AP_α. Since the ideal N_α is quasi-Banach and of spectral type l_1, by White's theorem [**22**, Theorem 2.2] the spectral trace is linear and continuous on N_α. On the other hand, by Proposition 4.1 the usual (nuclear) trace is continuous on $X^*\widehat{\otimes}_\alpha X$, which can be identified with $N_\alpha(X)$ by assumption about X. Since the tensor product $X^* \otimes X$ is dense in $X^*\widehat{\otimes}_\alpha X$, we obtain that trace $T = \mathrm{sp\,tr}\; T$.

Now, suppose that X does not have the AP_α. By Lemma 4.1, there exists an element $z \in X^*\widehat{\otimes}_\alpha X$ such that trace $z = 1$ and $\widetilde{z} = 0$. By assumptions, $\mathrm{sp\,tr}\;\widetilde{z} =$ trace $z = 1$. Contradiction. $\square$

EXAMPLE 4.3. Let $0 < r \le 1$, $1 \le p \le 2$, $1/r = 1/2 + 1/p$.

1) If $T \in N_{[r,p']}(X)$ (see Example 4.1), then T admits a factorization

$$T = BA : X \overset{A}{\to} l_p \overset{B}{\to} X,\;\; A \in N_r(X, l_p),\; B \in L(l_p, X).$$

The complete systems of eigenvalues of $T = BA$ and AB are the same. But $AB \in N_r(l_p, l_p)$. Therefore, AB is of spectral type l_1, as any r-nuclear operator in l_p [**8**, Theorem 7]. It follows from this that $N_{[r,p']}$ is of spectral type l_1. It is easy to see that if $z \in X^*\widehat{\otimes}_{[r,p']}X$ such that $\widetilde{z} = T$, then trace $z =$ trace AB (recall that l_p has the AP). But trace $AB = \mathrm{sp\,tr}\; AB$ (it was shown, e.g., in [**19, 20**] and follows also from Proposition 4.2). Hence, for each $z \in X^*\widehat{\otimes}_{[r,p']}X$ we have: trace $z = \mathrm{sp\,tr}\;\widetilde{z}$. By the second part of Proposition 4.2, every Banach space has the property $AP_{[r,p']}$ ($= AP_{r,\infty,p'}$, see Example 4.1; thus, we gave a proof of Lemma 4.2, 2) for the case of $AP_{[r,p']}$).

2) If $T \in N^{[r,p']}(X)$ (see Example 4.1), then T admits a factorization

$$T = BA : X \overset{A}{\to} l_p \overset{B}{\to} X,\;\; A \in L(X, l_p),\; B \in N_r(l_p, X).$$

As in 1), we see that for each $z \in X^*\widehat{\otimes}^{[r,p']}X$ we have: trace $z = \mathrm{sp\,tr}\;\widetilde{z}$. Furthermore, by the second part of Proposition 4.2, every Banach space has the property $AP^{[r,p]}$ ($= AP^{r,\infty,p'}$, see Example 4.1; thus, we have a proof of Lemma 4.2, 2) for the case of $AP^{[r,p']}$).

4.3. α-extension property and α-lifting property. We give now two definitions, which will be of use below. Let us note that these definitions can be generalized in many different ways.

DEFINITION 4.5. *Let α be a complete projective tensor quasi-norm. A Banach space X has the α-extension property, if for any subspace $X_0 \subset X$ and every tensor element $z_0 \in X_0^*\widehat{\otimes}_\alpha X_0$ there exists an extension $z \in X^*\widehat{\otimes}_\alpha X_0$ (so that $z \circ i = z_0$ and* trace $i \circ z =$ trace z_0*, where $i : X_0 \to X$ is the natural injection). A Banach space X has the α-lifting property, if for every subspace X_0 and every tensor element $z_0 \in (X/X_0)^*\widehat{\otimes}_\alpha X/X_0$ there exists a lifting $z \in (X/X_0)^*\widehat{\otimes}_\alpha X$ (so that $Q \circ z = z_0$, where Q is a quotient map from X onto X/X_0, and* trace $z \circ Q =$ trace z_0).

EXAMPLE 4.4. For instance, every Banach space has the $||\cdot||_{r,\infty,q}$-extension property and $||\cdot||_{r,p,\infty}$-lifting property (see Example 4.1). For the tensor products $(\widehat{\otimes}_s, ||\cdot||_{s,\infty,\infty})$, $s \in (0,1]$, all Banach spaces have both the $||\cdot||_{s,\infty,\infty}$-extension and $||\cdot||_{s,\infty,\infty}$-lifting properties. This follows from Hahn-Banach theorem and from definition of Banach quotients.

PROPOSITION 4.3. *Let α be a complete projective tensor quasi-norm and X have the α-extension property. Suppose that X possesses the AP_α, but there exists a subspace $X_0 \subset X$ without the AP_α. There exists an operator $S \in N_\alpha(X)$ such that* trace $S = 1$ *and* $S^2 = 0$.

PROOF. Take $z_0 \in X_0 \widehat{\otimes}_\alpha X_0$ with trace $z_0 = 1$ and $\widetilde{z}_0 = 0$ (we use Lemma 4.1). By assumption, there exists $z \in X^* \widehat{\otimes}_\alpha X_0$ such that $z_0 = z \circ i$ and trace $i \circ z = 1$, where $i : X_0 \to X$ is an inclusion. Here is a diagram for the operators:

$$X_0 \xrightarrow{i} X \xrightarrow{\widetilde{z}} X_0 \xrightarrow{i} X. \tag{2}$$

Now, X has the AP_α. Therefore, we can identify the operator $S := \widetilde{i \circ z}$ with the tensor element $i \circ z$. It is clear that trace $S = 1$ and $S^2 = 0$. $\square$

The following proposition is a strengthening of Proposition 4.2 in an important case.

PROPOSITION 4.4. *Let α be a complete projective tensor quasi-norm of spectral type l_1 and X have the α-extension property. If for every $z \in X^* \widehat{\otimes}_\alpha X$ the equality* trace $z = \operatorname{sp\,tr} \widetilde{z}$ *holds, then every subspace X_0 of X possesses the AP_α. Consequently, for every $T \in N_\alpha(X_0)$, one has:* trace $T = \operatorname{sp\,tr} T$.

PROOF. Firstly, note that by Proposition 4.2 X has the AP_α. Let X_0 be a subspace of X, $i : X_0 \to X$ be an inclusion map and $z_0 \in X_0^* \widehat{\otimes}_\alpha X_0$ with trace $z_0 = 1$. Take an extension $z \in X^* \widehat{\otimes}_\alpha X_0$ (as in Definition 4.5; hence, $\widetilde{z}|_{X_0} = \widetilde{z}_0$ and trace $i \circ z =$ trace z_0) and consider the operators $\widetilde{i \circ z} : X \to X$ and $\widetilde{z \circ i} : X_0 \to X_0$ (see the diagram (2)). By the principle of related operators [**13**, 27.3.3], $\operatorname{sp\,tr} \widetilde{i \circ z} = \operatorname{sp\,tr} \widetilde{z \circ i}$. By assumption, $\operatorname{sp\,tr} \widetilde{i \circ z} =$ trace $i \circ z$ Now, since X has the AP_α, it follows from the equality trace $i \circ z =$ trace z_0 that

$$1 = \text{trace } z_0 = \operatorname{sp\,tr} \widetilde{i \circ z} = \operatorname{sp\,tr} \widetilde{z \circ i} = \operatorname{sp\,tr} \widetilde{z}_0.$$

Therefore, $\widetilde{z}_0 \neq 0$. By Lemma 4.1, X_0 has the AP_α. The last statement follows from the first part of Proposition 4.2. $\square$

The following propositions are in a sense dual the previous ones.

PROPOSITION 4.5. *Let α be a complete projective tensor quasi-norm and X have the α-lifting property. Suppose that X possesses the AP_α, but there exists a factor space X/X_0 ($X_0 \subset X$) without the AP_α. There exists an operator $S \in N_\alpha(X)$ such that* trace $S = 1$ *and* $S^2 = 0$.

PROOF. Take $z_0 \in X/X_0 \widehat{\otimes}_\alpha X/X_0$ with trace $z_0 = 1$ and $\widetilde{z}_0 = 0$ By assumption, there exists $z \in (X/X_0)^* \widehat{\otimes}_\alpha X$ such that $Q \circ z = z_0$, where Q is a factor map from X onto X/X_0, and trace $z \circ Q =$ trace $z_0 = 1$. Here is a diagram for the operators:

$$X \xrightarrow{Q} X/X_0 \xrightarrow{\widetilde{z}} X \xrightarrow{Q} X/X_0 \xrightarrow{\widetilde{z}} X. \tag{3}$$

Now, X has the AP_α. Therefore, we can identify the operator $S := \widetilde{z \circ Q}$ with the tensor element $z \circ Q$. It is clear that trace $S = 1$ and $S^2 = 0$. $\square$

Proposition 4.6. *Let α be a complete projective tensor quasi-norm of spectral type l_1 and X have the α-lifting property. If for every $z \in X^*\widehat{\otimes}_\alpha X$ the equality* trace $z = $ sp tr $\widetilde{z}$ *holds, then every quotient X/X_0 of X possesses the AP_α. Consequently, for every $T \in N_\alpha(X/X_0)$, one has:* trace $T =$ sp tr T.

Proof. By Proposition 4.2, X has the AP_α. Let X_0 be a subspace of X, $Q: X \to X/X_0$ be a factor map and $z_0 \in (X/X_0)^*\widehat{\otimes}_\alpha X/X_0$ with trace $z_0 = 1$. Take a lifting $z \in (X/X_0)^*\widehat{\otimes}_\alpha X$ (as in Definition 4.5; hence, $Q \circ z = z_0$, and trace $z \circ Q =$ trace z_0) and consider the operators $\widetilde{z \circ Q} : X \to X$ and $\widetilde{Q \circ z} : X/X_0 \to X/X_0$ (see the diagram (3)). By the principle of related operators [**13**, 27.3.3], sp tr $\widetilde{z \circ Q} =$ sp tr $\widetilde{Q \circ z}$. By assumption, sp tr $\widetilde{z \circ Q} =$ trace $z \circ Q$. Now, since X has the AP_α, it follows from the equality trace $z \circ Q =$ trace z_0 that

$$1 = \text{trace } z_0 = \text{sp tr } \widetilde{z \circ Q} = \text{sp tr } \widetilde{Q \circ z} = \text{sp tr } \widetilde{z}_0.$$

Therefore, $\widetilde{z}_0 \neq 0$. By Lemma 4.1, X_0 has the AP_α. The last statement follows from the first part of Proposition 4.2. $\square$

An immediate consequence of Propositions 4.4 and 4.6 is

Proposition 4.7. *Let α be a complete projective tensor quasi-norm of spectral type l_1 such that every Banach space has both the α-extension property and the α-lifting property. If for every $z \in X^*\widehat{\otimes}_\alpha X$ the equality* trace $z =$ sp tr $\widetilde{z}$ *holds, then every quotient of any subspace of X (= every subspace of any quotient of X) possesses the AP_α. Consequently, for $X_0 \subset X_1 \subset X$, $Y = X_1/X_0$ (or for $X_0 \subset X$, $Y \subset X/X_0$) and for every $T \in N_\alpha(Y)$ one has:* trace $T =$ sp tr T.

Proof. Apply in different orders Propositions 4.4 and 4.6. $\square$

4.4. Applications. $\mathbb{Z}_d$-symmetry for $N_{[r,p]}$ and $N^{[r,p]}$. One of our main result (in context of the $\mathbb{Z}_d$-symmetry of the spectra of nuclear operators) is

Theorem 4.1. *Let α be a complete projective tensor quasi-norm of spectral type l_1 and let a Banach space X have the AP_α. For a fixed $d = 2, 3, \dots$, the spectrum of an operator $T \in N_\alpha(X)$ is $\mathbb{Z}_d$-symmetric if and only if*

$$\text{trace } T^{kd+j} = 0 \ \textit{ for all } \ k = 0, 1, 2, \dots \ \textit{ and } \ j = 1, 2, \dots, d-1.$$

In particular, if trace $T \neq 0$, *then* $T^2 \neq 0$.

Proof. Let the spectrum of an operator $T \in N_\alpha(X)$ is $\mathbb{Z}_d$-symmetric. The traces trace T^n $(n \in \mathbb{N})$ are well defined since $T^n \in N_\alpha(X)$ and X has the AP_α. Take an integer $l := kd+j$ with $0 < j < d$. The eigenvalue sequences of T and T^l can be arranged in such a way that $\{\lambda_n(T)^l\} = \{\lambda_n(T^l)\}$ (see [**14**, 3.2.24, p. 147]). Since the spectrum of T^l is absolutely summable, trace $T^l = \sum_{\lambda \in \text{sp}\,(T^l)} \lambda$, $\sum_{t \in \sqrt[d]{1}} t = 0$ and we may assume that $\{\lambda_m(T^l)\} = \{\lambda_m(T)^l\}$, we get that trace $T^{kd+j} = 0$.

To prove the converse, we need some information from Fredholm Theory. Let u be an element of the projective tensor product $Y^*\widehat{\otimes} Y$, where Y is an arbitrary

Banach space. Recall that the Fredholm determinant $\det(1-wu)$ of u (see [**7**, Chap II, §1, $n°4$, p. 13], [**6**], [**13**] or [**14**] is an entire function

$$\det(1-wu) = 1 - w \text{ trace } u + \dots + (-1)^n w^n \alpha_n(u) + \dots,$$

all zeros of which are exactly (according to their multiplicities) the inverses of nonzero eigenvalues of the operator $\widetilde{u}$, associated with the tensor element u. By [**7**, Chap II, §1, $n°4$, Corollaire 2, pp. 17-18], this entire function is of the form

$$\det(1-wu) = e^{-w \,\mathrm{trace}\, u} \prod_{i=1}^{\infty} (1-ww_i)\, e^{ww_i},$$

where $\{w_i = \lambda_i(\widetilde{u})\}$ is a complete sequence of all eigenvalues of the operator $\widetilde{u}$. Hence, there exists a $\delta > 0$ such that for all $w, |w| \le \delta$, we have

$$\det(1-wu) = \exp\Big(\sum_{n=1}^{\infty} c_n\, w^n \,\mathrm{trace}\, u^n\Big) \tag{4}$$

(see [**6**, p. 350]; cf. [**5**, Theorem I.3.3, p. 10]).

Now, let trace $T^{kd+r} = 0$ for all $k = 0, 1, 2, \dots$ and $r = 1, 2, \dots, d-1$. By (4), $\det(1-wT) = \exp\big(\sum_{m=1}^{\infty} c_{md}\, w^{md} \,\mathrm{trace}\, T^{md}\big)$ in a neighborhood V of zero. Hence, for the analytic function $f(w) := \det(1-wT)$, we have: there exists a $\delta > 0$ such that for all $w, |w| \le \delta$, $f(tw) = f(w)$ for every $t \in \sqrt[d]{1}$. By the uniqueness theorem, the complete system of eigenvalues of T is $\mathbb{Z}_d$-symmetric. □

Applying Theorem 4.1 to the tensor products $\widehat{\otimes}_{[r,p']}$ and $\widehat{\otimes}^{[r,p']}$ and using Example 4.3, we get the following generalizations of Zelikin's theorem:

THEOREM 4.2. *Let* $0 < r \le 1$, $1 \le p \le 2$, $1/r = 1/2 + 1/p$ *and* $d = 2, 3, \dots$. *For any Banach space* X *and every operator* $T \in N_{[r,p']}(X)$ *(or* $T \in N^{[r,p']}(Z)$*) we have: The spectrum of an operator* $T \in N_\alpha(X)$ *is* $\mathbb{Z}_d$*-symmetric if and only if*

$$\mathrm{trace}\, T^{kd+j} = 0 \ \textit{ for all } \ k = 0, 1, 2, \dots \ \textit{ and } \ j = 1, 2, \dots, d-1.$$

In particular, if trace $T \neq 0$, *then* $T^2 \neq 0$.

We obtain Zelikin's theorem, if we put $d = 2$, $r = 1, p = 2$ and $X = H$ (a Hilbert space), since $N_1(H) = S_1(H) = N_{[1,2]}(H) = N^{[1,2]}(H)$.

4.5. Proof of Theorem 1.1. Here it is

PROOF. Let $T \in N_r(Y)$. Under the conditions of the theorem we have: every quotient of every subspace of an L_p-space has the AP_r, $\lambda(T) \in l_1$ and the trace of T is well defined and equals the sum of the eigenvalues of T (written in according to their multiplicities; see, e.e., [**16, 20**]).

Supposing that the spectrum of T is $\mathbb{Z}_d$-symmetric, we can proceed as in the beginning of the proof of Theorem 4.1 to obtain that trace $T^{kd+j} = 0$ for all $k = 0, 1, 2, \dots$ and $j = 1, 2, \dots, d-1$.

To proof the converse, we repeat word for word the second part of the proof of Theorem 4.1. □

4.6. Sharpness of main results. We need the following auxialary result:

LEMMA 4.3. *If* $r \in [2/3, 1), q \in (2, \infty]$ *and* $1/r = 3/2 - 1/q$, *then there exist a subspace* $Y_q \subset l_q$ (c_0 *for* $q = \infty$) *and a tensor element* $w_q \in Y_q^* \widehat{\otimes} Y_q$ *so that* $w_q \in Y_q^* \widehat{\otimes}^{[s,q]} Y_q$ *for every* $s > r$, trace $w_q = 1$ *and* $\widetilde{w}_q = 0$. *Moreover,* w_q *can be chosen in such a way that* $w_q = \sum_{k=1}^\infty e'_k|_{Y_q} \otimes y_k$, *where* (e'_k) *is a sequence of the linear functionals on* l_q *generated by the unit vectors from* $l_{q'}$ *and* (y_k) *is in* $l_s(Y_q)$ *for all* $s > r$.

PROOF. Let us look at the proof from [**17**, Example 2] and take the space Y_q and the tensor element w_q from that proof. We have: Y_q is isometrically imbedded into l_q, $w_q = \sum_{k=1}^\infty e'_k|_{Y_q} \otimes y_k$, where (e'_k) and (y_k) are as above. □

The following two theorems show that Theorem 4.2 is optimal.

THEOREM 4.3. *Let* $r \in [2/3, 1)$, $q \in (2, \infty]$, $1/r = 3/2 - 1/q$. *There exists a nuclear operator* V *in* l_q (*in* c_0 *for* $q = \infty$) *such that*

1) $V \in N^{[s,q]}(l_q)$ *for each* $s \in (r, 1]$;
2) V *is neither in* $N^{[r,q]}(l_q)$ *nor* r*-nuclear;*
3) trace $V = 1$ *and* $V^2 = 0$.

PROOF. Take a pair (Y_q, w_q) from Lemma 4.3 and let $i : Y_q \to l_q$ be an isometric imbedding. Define $v \in l_q^* \widehat{\otimes} l_q$ by $v = \sum_{k=1}^\infty e'_k \otimes i y_k$ and put $V := \widetilde{v}$. This operator possesses the properties 1)–3) (we have to mention only that $N^{[r,q]}(l_q) \subset N^r(l_q)$ and that if $T \in N_r(l_q)$ with trace $z = 1$, then $T^2 \neq 0$ by Theorem 1.1). □

THEOREM 4.4. *Let* $r \in [2/3, 1)$, $p \in [1, 2)$, $1/r = 1/2 + 1/p$. *There exists a nuclear operator* U *in* l_p *such that*

1) $U \in N_{[s,p']}(l_p)$ *for each* $s \in (r, 1]$;
2) U *is neither in* $N_{[r,p']}(l_p)$ *nor* r*-nuclear;*
3) trace $V = 1$ *and* $V^2 = 0$.

PROOF. Consider $U := V^*$, where V is from the previous theorem. □

Now, *Theorem 1.2 follows from the above theorems, since, e.g.,* $N^{[s,q]} \subset N^s$.

One more auxiliary fact:

LEMMA 4.4. *Let* $r \in (2/3, 1], q \in [2, \infty), 1/r = 3/2 - 1/q$. *One can find the number sequences* (q_k) *and* (n_k) *with* $q_n > q, q_n \to q$ *and* $k_n \to \infty$ *for which the following statement is true: There exist a Banach space* Y_0 *and a tensor element* $w \in Y_0^* \widehat{\otimes}_r Y_0$ *so that* $Y_0 \subset Y := \left(\sum_k l_{q_k}^{n_k}\right)_{l_q}$, $w \neq 0, \tilde{w} = 0$, *the space* Y_0 (*as well as* Y_0^*) *has the* AP_s *for every* $s < r$ (*but does not have the* $AP_{r,\bar{q}}$ *for any* $\bar{q} \in (q, \infty]$). *Moreover,* w *can be chosen in such a way that* $w = \sum_{k=1}^\infty \sum_{m=1}^{n_k} e'_{mk}|_{Y_0} \otimes y_{mk}$, *where* (e'_{mk}) *is a weakly* $\bar{q}$*-summable* ($\forall\, \bar{q} > q$) *sequence of the linear functionals on* Y *generated by the unit vectors from* Y^* *and* (y_{mk}) *is in* $l_r(Y_0) \setminus \cup_{s<r} l_s(Y_0)$.

PROOF. It is enough to take the space Y_0 and the tensor element w from the proof of [**17**, Example 1] and put $n_k := 3 \cdot 2^k$ in that proof. After this we get exactly the desired Banach space and tensor element. We have also: $Y_0 \subset Y \subset l_{\bar{q}}$ for every $\bar{q} > q$. Hence, the sequence $(e'_{mk}|_{Y_0})$ is weakly $\bar{q}$-summable ($\forall\, \bar{q} > q$). □

THEOREM 4.5. *Let $r \in (2/3, 1], q \in [2, \infty), 1/r = 3/2 - 1/q$. One can find the number sequences (q_k) and (n_k) with $q_k > q, q_k \to q$ and $n_k \to \infty$ for which the following statement is true:*

There exists a nuclear operator U in $Y := \left(\sum_k l_{q_k}^{n_k}\right)_{l_q}$ such that

1) *$U \in N^{[r,\bar{q}]}(Y)$ for each $\bar{q} > q$.*
2) *U is not in $N^{[r,q]}(Y)$.*
3) *trace $U = 1$ and $U^2 = 0$.*

PROOF. Take a pair (Y_0, w) from Lemma 4.4 and let $j : Y_0 \to Y$ be an injection map. Define $u \in Y^* \widehat{\otimes} Y$ by $u = \sum_{k=1}^{\infty} \sum_{m=1}^{n_k} e'_{mk} \otimes j y_{mk}$ and put $U := \widetilde{u}$. This operator possesses the properties 1)–3) (we have to mention only that if $T \in N_{[r,q]}(Y)$ with trace $z = 1$, then $T^2 \neq 0$ by Theorem 4.2). □

THEOREM 4.6. *Let $r \in (2/3, 1]$, $p \in (1, 2]$, $1/r = 1/2 + 1/p$. One can find the number sequences (p_k) and (n_k) with $p_k < p, p_k \to p$ and $n_k \to \infty$ for which the following statement is true:*

There exists a nuclear operator V in $E := \left(\sum_k l_{p_k}^{n_k}\right)_{l_p}$ such that

1) *$V \in N_{[r,\bar{q}]}(E)$ for each $\bar{q} > q$.*
2) *V is not in $N_{[r,q]}(E)$.*
3) *trace $V = 1$ and $V^2 = 0$.*

PROOF. Consider $V := U^*$, where U is from the previous theorem. □

Let us emphasize an important particular case of Theorems 4.5 and 4.6, namely, the case of so-called "asymptotically Hilbertian spaces" (see, e.g., [**2**] for a definition):

THEOREM 4.7. *There exist an asymptotically Hilbertian space $Y_2 := \left(\sum_k l_{q_k}^{n_k}\right)_{l_2}$ ($q_k \to 2$ and $n_k \to \infty$) and a nuclear operator U in this space so that*

1) *$U \in N^{[1,2+\varepsilon]}(Y_2)$ for each $\varepsilon > 0$.*
2) *U is not in $N^{[1,2]}(Y_2)$.*
3) *trace $U = 1$ and $U^2 = 0$.*

The corresponding statements hold for the adjoint operator U^*.

As we know, the last theorem is the best strengthening of related results from [**2**], [**15**] and [**17**].

References

[1] Yoav Benyamini, Joram Lindenstrauss, Geometric Nonlinear Functional Analysis: 1, American Mathematical Society Colloquium Publications 48, 2000

[2] P. G. Casazza, C. L. García, W. B. Johnson, An example of an asymptotically Hilbertian space which fails the approximation property, Proceedings of the American Mathematical Society, Vol. 129, 10, 2001, 3017-3024.

[3] Davie A.M., The approximation problem for Banach spaces, Bull. London Math. Soc., Vol 5, 1973, 261–266

[4] Enflo P., *A counterexample to the approximation property in Banach spaces*, Acta Math., **130** (1973), 309–317

[5] I. Gohberg, S. Goldberg, N. Krupnik, *Traces and Determinants of Linear Operators*, Birkhäuser Verlag, Basel-Boston-Berlin, 2000.

[6] A. Grothendieck, *La théorie de Fredholm*, Bull. Soc. Math. France, **84** (1956), 319–384.
[7] A. Grothendieck: *Produits tensoriels topologiques et éspaces nucléaires*, Mem. Amer. Math. Soc., **16** (1955).
[8] H. König: *On the eigenvalue spectrum of certain operator ideals*, Coll. Math. **44** (1981), 15–28.
[9] Köthe, G., Topological vector spaces I, Berlin-Heidelberg-New York, 1969.
[10] Lapreste, J. T.: *Opérateurs sommants et factorisations à travers les espaces L_p*, Studia Math. 57 (1976), 47-83
[11] V. B. Lidskiĭ, *Nonselfadjoint operators having a trace*, Dokl. Akad. Nauk SSSR, **125** (1959), 485–487.
[12] B. S. Mityagin, *A criterion for the $\mathbb{Z}_d$-symmetry of the spectrum of a compact operator*, J. Operator Theory, **76**:1 (2016), 57–65.
[13] A. Pietsch, Operator ideals, North-Holland, 1978.
[14] A. Pietsch, *Eigenvalues and s-Numbers*, Cambridge studies in advanced mathematics, **13**, Cambridge University Press, 1987.
[15] O. I. Reinov, Banach spaces without approximation property, Functional Analysis and Its Applications, vol 16, issue 4, 1982, 315-317.
[16] O. I. Reinov, Q. Latif: *Grothendieck-Lidskiĭ theorem for subspaces of quotients of L_p-spaces*, Banach Center Publications 102 (2014), 189-195.
[17] O. I. Reinov, *On linear operators with s-nuclear adjoints,* $0 < s \leq 1$, J. Math. Anal. Appl. **415** (2014), 816-824.
[18] O. I. Reinov, Q. Latif, *Distribution of eigenvalues of nuclear operators and Grothendieck-Lidski type formulas*, Journal of Mathematical Sciences, Springer, Vol. 193, No. 2, 2013, 312-329.
[19] O. Reinov, Q. Latif: *Grothendieck-Lidskiĭ theorem for subspaces of L_p-spaces*, Math. Nachr. **286**, No. 2-3 (2013), 279–282.
[20] Oleg Reinov, *Some Remarks on Approximation Properties with Applications*, Ordered Structures and Applications, Trends in Mathematics, 371-394, 2016
[21] H. Weyl, *Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. Nat. Acad. Sci. USA, **35** (1949), 408–411.
[22] M.C. White: *Analytic multivalued functions and spectral trace*, Math. Ann. **304** (1996), 665-683.
[23] M. I. Zelikin, A criterion for the symmetry of a spectrum", Dokl. Akad. Nauk 418 (2008), no. 6, 737-740

St. Petersburg State University, Saint Petersburg, RUSSIA.
E-mail address: `orein51@mail.ru`