SOME APPROXIMATION PROPERTIES AND NUCLEAR OPERATORS IN SPACES OF ANALYTICAL FUNCTIONS

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ABSTRACT. We introduce and investigate a new notion of the approximation property $AP_{[c]}$, where $c = (c_n)$ is an arbitrary positive real sequence, tending to infinity. Also, we study the corresponding notion of $[c]$-nuclear operators in Banach spaces. Some characterization of the $AP_{[c]}$ in terms of tensor products, as well as sufficient conditions for a Banach space to have the $AP_{[c]}$, are given. We give also sufficient conditions for a positive answer to the question: when it follows from the $[c]$-nuclearity of an adjoint operator the nuclearity of the operator itself. Obtained results are applied then to the study of properties of nuclear operators in some spaces of analytical functions. Many examples are given.

1. Introduction and preliminaries

1.1. **Introduction.** A Banach space X has the approximation property AP if the identity operator in X can be approximated, in the topology of compact convergence, by finite rank operators. As was noted by A. Grothendieck [6, Chap I, Lemma 12, p. 112], J. Dieudonné and L. Schwartz showed that every compact subset of a Banach space is contained in the closed convex hull of a sequence, converging to zero (see [4, proof of Theorem 5] or [10, p. 30, Proposition 1.e.2]). Therefore, the notion of the approximation property can be define in the following way: the space X has the AP if for every sequence (x_n) in X with $||x_n|| \to 0$ and every $\varepsilon > 0$ there exists a finite rank operator R in X such that $\sup_n ||x_n - Rx_n|| \leq \varepsilon$. It is natural to replace in this definition the condition "for every sequence (x_n) in X with $||x_n|| \to 0$", e. g., by the condition "$\sum ||x_n||^p < \infty$ for some $p \in (0, \infty)$" and to get corresponding approximation property (say $AP[p]$). It seems that for the first time it was done in the second author's paper [19, Lemma 2.1]. Let us mention that instead of l_p-sequences, we can take in the definition any other sets of zero-sequences.

Once a new notion of approximation property is defined, a natural next step is to study whether some results on the classical approximation property can be extended to the case of this new notion. As examples, we can consider the characterizations of the AP in terms of tensor products, some sufficient conditions for a space to have AP, the statements like "X^* has the $AP \implies X$ has the AP", the connections between the approximation property and the properties

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of so-called nuclear operators, question of whether an operator is nuclear if its adjoint is nuclear etc.

In this paper, we introduce and investigate approximation properties defined (as above) by one-point sets of zero-sequences. Namely, suppose (c_n) is a real positive sequence, tending to ∞. We say that a Banach space X has the approximation property with respect to (c_n) (or the approximation property up to c_n), shortly the $AP_{[c]}$, if for every sequence (x_n) in X with $||x_n|| \leq 1/c_n$ and every $\varepsilon > 0$ there exists a finite rank operator R in X such that $\sup_n ||x_n - Rx_n|| \leq \varepsilon$. Our interest in such properties was inspired by the known fact, that the space H^∞ of bounded analytic functions in the unit disk has the approximation property "up to log" [1, Theorem 9]. As a matter of fact, we first had some new properties of nuclear operators, acting from H^∞ or from the space L_1/H_0^1 (predual to H^∞) (see Sections 5 and 6 below) and then decided to consider, instead of only the sequence $(\log(n+1))$, also other positive sequences (c_n) with $c_n \to \infty$.

These properties $AP_{[c]}$ are closely connected with a new notion of so-called $[c]$-nuclear operators For example, a [log]-nuclear operator between Banach spaces X and Y is an operator $T : X \to Y$, that admits a nuclear representation of type $Tx = \sum_n 1/\log(n+1)\, x'_n(x)\, y_n$ with $\sum_n ||x'_n||\, ||y_n|| < \infty$. One of the main questions in our study of the classes of $[c]$-nuclear operators is to give conditions, under which it follows from $[c]$-nuclearity of an adjoint operator T^* the nuclearity of the operator T itself.

The history of the questions of such a type takes its beginning from a result of A. Grothendieck on the linear operators with nuclear adjoints. He showed in [6] that if a linear operator T maps a Banach space X into a Banach space Y, if T^* is nuclear and if the dual space X^* has the approximation property, then the operator T is nuclear. This result turned up to be sharp (with respect to the space X), in the sense that there exists an operator T in a Banach space X, which is not nuclear but has a nuclear adjoint. This was shown already in 1973 by T. Figiel and W. B. Johnson [5]. Moreover, in their example, the space X possesses the Grothendieck approximation property. Later, E. Oja and O. Reinov [12] gave another sufficient condition for a positive answer to the above question: If T maps X into Y, T^* is nuclear and Y^{***} has the approximation property, then the operator T is nuclear. Again, as was shown in the same paper, the condition is essential: There are a Banach space Z and an operator $T : Z^{**} \to Z$ so that Z^{**} has a basis, T^* is nuclear but T is not nuclear. Of course, here the space Z^{***} (which is, by the way, separable in the example) does not have the approximation property.

It is easy to see that the nuclearity of an adjoint operator $T^* : Y^* \to X^*$ is equivalent to the nuclearity of the operator $\pi_Y T : X \to Y \to Y^{**}$, where π_Y is the natural injection of Y into its second dual Y^{**}. Thus, the above positive results say that, under some approximation conditions posed on the spaces X or Y, the space $N(X,Y)$ of all nuclear operators from X to Y possesses a property of "regularity" (recall that an operator ideal J is regular if it follows from $U : X \to Y$ and $\pi_Y U \in J(X, Y^{**})$ that $U \in J(X,Y)$; see [14]). After the Figiel-Johnson example had appeared, the natural questions about the regularity of such operator ideals as the ideals of p-nuclear or p-integral (in the sense of A. Pietsch) operators were posed in 1970s (e. g., by A. Pełczyński, A. Pietsch, P. Saphar and others; see [17]). In a more general setting, a corresponding question on the regularity of the so-called ideals of (p,q,r)-nuclear operators can be found in the book [14]. Between 1980 and 2017, E. Oja and O. Reinov made several contributions in answering such questions. We refer here only to a nice paper of E. Oja [11], where also some corresponding references can be found.

Roughly speaking, we can divide such problems on the regularity (of in different senses nuclear operators) in three parts: Let A_1 be the set of all projective tensor products of Banach spaces, A_1^+ be the set of all tensor products of Banach spaces, equipped with norms, which are less then the projective tensor norm $||\cdot||_\wedge$ and greater then the operator norm, A_1^- be the set of all subspaces of all projective tensor products of Banach spaces. In a natural way, we can define the corresponding classes $\widetilde{A}_1$, $\widetilde{A}_1^+$ and $\widetilde{A}_1^-$ of operators induced by these sets. In this setting, we can define a notion of regularity (note that it can be defined in more general cases of Banach tensor products or quasi-normed tensor products of Banach spaces): For example, let $X\widetilde{\otimes}Y$ belong to one of the above sets and $J(X^*,Y)$ be the corresponding set of operators. We say that $J(X^*,Y)$ is regular if the conditions

$$U : X^* \to Y,\ \pi_Y U \text{ is generated by an element from } X\widetilde{\otimes}Y^{**}$$

imply that $U \in J(X^*,Y)$ and generated by an element from $X\widetilde{\otimes}Y$.

The cases of the sets A_1 and A_1^+ (for so-called tensor norms; see [23]) were studied carefully by E. Oja in [11]. She showed that if X or Y^{***} has the approximation property, then every space of operators from $\widetilde{A}_1$ and $\widetilde{A}_1^+$ is regular (recall that the corresponding tensor products in [11] are equipped with tensor norms in the sense of [23]). Some results, concerning the case of A_1^- can be found in [20] and [21].

In our paper, we study the case of the sub-set of A_1^- of tensor products, which corresponds to the approximation properties $AP_{[c]}$, mentioned above. We will see that some of the approximation properties that were considered and studied in [20], [21] and [22] (e.g., AP_s, $AP_{1;p,r}$ $AP_{(pq)}$) as well as the corresponding tensor products are, essentially, special cases of our considerations.

Shortly about the content of the paper.

In Subsection 1.2, we present some standard notations concerning Banach spaces, spaces of nuclear operators, tensor products, the approximation property and formulate a classical result of J. Lindenstrauss from his famous paper [9], which we will need below.

In Section 2, we introduce the notion of the approximation property $AP_{[c]}$, where $c := (c_n)$ is a positive real sequence, tending to infinity. We also define the notions of so-called $[c]$-projective tensor products of Banach spaces and corresponding spaces of $[c]$-nuclear operators. The main result here is Theorem 2.3, which gives some characterizations of the $AP_{[c]}$ in terms of tensor products. In the end of the section, we show that if a dual space X^* has the $AP_{[c]}$, then the space X has the $AP_{[c]}$ too.

In Section 3, some sufficient conditions for a Banach space to have the $AP_{[c]}$ are given. As a consequence, we get an essential generalization of some previous facts about the approximation properties in subspaces of quotients of L_p-spaces. In particular, we show that every Banach space has the $AP_{[(\sqrt{n})]}$ (before it was known that every Banach space has the approximation property "up to $o(\sqrt{n})$", that was a generalization of a famous 2/3-result of A. Grothendieck). In the end of the section we present some examples with some assertions. In particular, as a consequence of Example 3.8, we obtain the existence of an asymptotically Hilbertian space without the compact approximation property but with the property $AP_{[\log^{1+\varepsilon}]}$ (see Proposition 3.10).

Section 4 is devoted to a study of operators with $[c]$-nuclear adjoints. We get here an analogue of the main result of the paper [10], showing that the set $N_{[c]}(X^*, Y)$ of all $[c]$-nuclear operators from X^* to Y is almost regular (cf. definition of regularity mentioned above) if either X or Y^{***} has the $AP_{[c]}$. Namely, if X or Y^{***} has the $AP_{[c]}$, $T : X^* \to Y$, $\pi_Y T$ is generated by an element from $X\widetilde{\otimes}Y^{**}$, then $T \in N(X^*, Y)$ (i.e., nuclear) and generated by an element from the projective tensor product $X\widehat{\otimes}Y$ (Theorem 4.1). The main consequence of the theorem is Corollary 4.2 that gives some sufficient conditions for an operator to be nuclear if its adjoint is $[c]$-nuclear.

In Section 5, some examples are given. These examples show, in particular, that the condition "either X or Y^{***} has the $AP_{[c]}$" is essential for the results of the previous section, as well as the conclusion "'T is nuclear" is the best possible (this was known before; see e.g. [21] and Section 6 below). After the examples, we present several results on the nuclear operators in some spaces of analytical functions, for instance: If an operator from L_1/H_0^1 has a $[(\log(n+1))]$-nuclear adjoint, then it is nuclear; if an operator from H^∞ to a Banach space Y is generated by a tensor element from $L_1\widehat{\otimes}Y^{**}$, then it is nuclear as an operator from H^∞ to Y.

Finally, in Section 6, we generalize the notion of the $AP_{[c]}$ to the case where one considers some subset $\mathcal{C}_0$ of the set of all positive real sequences, tending to infinity. We introduce a notion of the approximation property $AP_{\mathcal{C}_0}$ for a Banach space X (the property means that X has the $AP_{[c]}$ for every sequence $c \in \mathcal{C}_0$). Also, we define a corresponding notion of a $\mathcal{C}_0$-nuclear operator. Examples 6.3 and 6.6 show that in some particular cases we get the notions of some approximation properties and the corresponding nuclear operators which were studied, for example, in [22]. We give some generalizations of results from Section 4 and present some applications. For example, we show that if $0 < s < 1$, Z is either any space of the spaces $A, L_1/H_0^1$ or H^∞ or any of its duals and T is an operator from or into Z, then it follows from s-nuclearity of T^* the nuclearity of the operator T itself. The last theorem of the paper (Theorem 6.9) is a direct application of Theorem 4.1 to the case where one of the Banach spaces under consideration if H^∞.

Let us mention that the results of the paper concerning nuclear operators in the spaces of analytical functions were partially presented by the authors at the Voronezh Winter Mathematical School "Modern methods of theory of functions and related problems" (2003, Jan 26–Feb 2, Voronezh, Russia) [8].

1.2. **Preliminaries.** All the spaces under considerations $(X, Y, W \dots)$ are Banach, all linear mappings (operators) are continuous; as usual, $X^*, X^{**}, \dots$ are Banach duals (to X), and $x', x'', \dots$ (or $y', \dots$) are the functionals on $X, X^*, \dots$ (or on $Y, \dots$). If $x \in X, x' \in X^*$ then $\langle x, x' \rangle = \langle x', x \rangle = x'(x)$. $L(X, Y)$ stands for the Banach space of all linear bounded operators from X to Y; $B(X, Y)$ is the Banach space of all continuous bilinear forms on $X \times Y$. Every Banach space is considered as a subspace of its second dual. If needed, by π_Y we denote the natural isometric injection of Y into Y^{**}.

We consider the algebraic tensor product $X \otimes Y$ as the linear subspace of all continuous finite rank operators from X^* to Y. The projective tensor product $X \widehat{\otimes} Y$ of the spaces X and Y is the completion of $X \otimes Y$ with respect to the norm $||z||_\wedge := \inf\{\sum |\lambda_k|\}$, where the infimum is taken over all finite representations of $z \in X \otimes Y$ in the form $z = \sum \lambda_k \, x_k \otimes y_k$ with $||x_k|| = ||y_k|| = 1$. Every element $z \in X \widehat{\otimes} Y$ admits a representation $z = \sum_{k=1}^\infty \lambda_k x_k \otimes y_k$ such that $\sum |\lambda_k| < \infty$ and $||x_k|| = ||y_k|| = 1$. We denote by z^t the transposed tensor element from $Y \widehat{\otimes} X$: $z^t := \sum_{k=1}^\infty \lambda_k y_k \otimes x_k$. If $X = Y^*$, then the functional "trace" on the tensor product $Y^* \widehat{\otimes} Y$ is well defined by the formula trace $z := \sum \lambda_k \, \langle x_k, y_k \rangle$. The Banach dual to $X \widehat{\otimes} Y$ can be identified with the space $L(Y, X^*) = B(X, Y)$ with duality given by "trace": for $z \in X \widehat{\otimes} Y$ and $U \in L(Y, X^*)$ we put $\langle U, z \rangle :=$ trace $U \circ z = \sum \lambda_k \, \langle x_k, U y_k \rangle$.

There is a natural map from the tensor product $X \otimes Y$ to $L(X^*, Y)$, that takes elementary tensors $x \otimes y$ to operators $\langle \pi_X x, \cdot \rangle y$ of rank one. This map is continuous as a map from $(X \otimes Y, || \cdot ||_\wedge)$ to $L(X^*, Y)$ and can be extended to the natural map $j : X \widehat{\otimes} Y \to L(X^*, Y)$. We will denote by $N^w(X^*, Y)$ the Banach space of operators belonging to the image $j(X \widehat{\otimes} Y)$ of this map (one can identify this space with the quotient $X \widehat{\otimes} Y / \operatorname{Ker} j$). If X is dual to a Banach space, say W, then the corresponding map j can be considered as a map from $W^* \widehat{\otimes} Y$ to $L(W, Y)$. We denote by $N(W, Y)$ the image of this map with a natural norm, induced from the quotient $W^* \widehat{\otimes} Y / \operatorname{Ker} j$. The operators from $N(W, Y)$ are called nuclear operators (from W to Y). Thus, in the general case, $N^*(X^*, Y)$ is a subspace of $N(X^*, Y)$. If $z \in X \widehat{\otimes} Y$ (or $z \in W^* \widehat{\otimes} Y$), then we denote by $\widetilde{z}$ the corresponding nuclear operator from X^* to Y (or from W to Y).

A Banach space X has the approximation property (the AP), if for every Y the canonical map j is one-to-one. Equivalently, X has the AP if for every Banach space Y the natural map $Y^* \widehat{\otimes} X \to L(Y, X)$ is one-to-one. The classical definition of the AP for X is: A Banach space X has the AP, if for every compact subset K of X and for any $\varepsilon > 0$ one can find a finite rank operator R in X such that $\sup_{x \in K} ||Rx - x|| \le \varepsilon$. See [4] for further information.

We use standard notations for the classical Banach spaces such as $L_p(\mu)$, $C(K)$, l_p, c_0, l_{pq} etc. By $l^0_{p\infty}$ we denote the minimal kernel $l^{min}_{(p,\infty)}$ of $l_{(p,\infty)}$ (see, e. g., [14, 13.9.3 Remark]: A sequence $c = (c_n)$ belongs to $l^0_{(p,\infty)}$ if and only if $\lim_n n^{1/p} c^*_n = 0$, where (c^*_n) is a non-increasing rearrangement of $|c| := (|c_n|)$. Finally, A and H^∞ are the disk algebra and the space of bounded analytical functions respectively. For information about these spaces that is needed, see [13]. Let us mention only that $A^* = L_1/H^1_0 \oplus L$ and $A^{**} = H^\infty \oplus L^*$, where L is an L_1-space.

We will need below the following fact from [9] (see Proof of Corollary 1 there):

Lemma 1.1. *For every separable Banach space X there exist a separable Banach space Z and a linear homomorphism φ from Z^{**} onto X with the kernel $Z \subset Z^{**}$ so that the subspace $\varphi^*(X^*)$ is complemented in Z^{***} and, moreover, $Z^{***} \cong \varphi^*(X^*) \oplus Z^*$.*

2. Approximation properties $AP_{[c]}$

It is well known that every compact subset of a Banach space is contained in the closed convex hull of a sequence converging to 0 (see, e.g., [6, p. 112 in Ch.I, Lemma 12], or [10, Proposition 1.e.2]. Therefore, the Grothendieck approximation property for a Banach space X can be defined as follows: X has the AP if and only if for every sequence $(x_n)_{n=1}^\infty \subset X$ tending to zero, for any $\varepsilon > 0$ there exists a finite rank (continuous) operator R in X such that for each $n \in \mathbb{N}$ one has $||Rx_n - x_n|| \leq \varepsilon$.

Replacing the set of all c_0-sequences in this definition by one fixed sequence, we get the following main definition. Namely, for a positive real sequence $c := (c_n)$ with $c \to +\infty$, we define a new notion of the approximation property $AP_{[c]}$:

Definition 2.1. A Banach space X has the *approximation property up to* c, the $AP_{[c]}$, if for every $\epsilon > 0$ and any sequence (x_n) in X with $||x_n|| \leq c_n^{-1}$ there exists a finite rank operator R in X such that $||Rx_n - x_n|| \leq \epsilon$ for every n.

Let us denote by $\mathcal{C}$ the set of all such sequences: $\mathcal{C} := \{(c_n) : \ c_n \in \mathbb{R}_+,\, c_n \to +\infty\}$.

Remark 2.2. One can define also the $CAP_{[c]}$, the compact approximation property with respect to c : We need only to change the words "finite rank operator" by "compact operator" in Definition 2.1. See Example 3.8 and Proposition 3.10 below.

Note that X has the classical AP if and only if X has the $AP_{[c]}$ for every $c \in \mathcal{C}$.

For $c \in \mathcal{C}$, let us denote by $X \overset{c}{\otimes} Y$ a subset of the projective tensor product $X\widehat{\otimes}Y$, consisting of all tensors z such that z admits a representation of type

$$z = \sum_{n=1}^\infty \mu_n\, x_n \otimes y_n, \quad \text{where} \quad \sum ||x_n||\, ||y_n|| < \infty,\, |\mu_k| \leq c_k^{-1}\ (k = 1, 2, \dots).$$

Let us note that we can identify (in a sense) $X \overset{c}{\otimes} Y$ with $Y \overset{c}{\otimes} X$ (in a natural way). Also, we can consider $X \overset{c}{\otimes} Y$ as a subset of $X \overset{c}{\otimes} Y^{**}$ (or as a subset of $X^{**} \overset{c}{\otimes} Y$). Indeed, $X \overset{c}{\otimes} Y$ is a subet of $X\widehat{\otimes}Y$, $X\widehat{\otimes}Y$ is a subspace of $X\widehat{\otimes}Y^{**}$ (see [6, Chap. I, Cor. 3, p. 41]) and $X \overset{c}{\otimes} Y^{**}$ is a subset of $X\widehat{\otimes}Y^{**}$. Thus the natural map from $X \overset{c}{\otimes} Y$ to $X \overset{c}{\otimes} Y^{**}$ is one-to-one.

On the other hand, we have a natural mapping from $X\widehat{\otimes}Y$ to $L(X^*, Y)$ (or to $B(X^*, Y^*)$). Therefore, we can consider also a natural map j_c from $X \overset{c}{\otimes} Y$ to $L(X^*, Y)$. The image of this map j_c will be denoted by $N^w_{[c]}(X^*, Y)$. In the particular case where the first space is a dual space, say W^*, we get a canonical mapping $W^*\overset{c}{\otimes}Y \to L(W,Y)$. The image of this map will be denoted by $N_{[c]}(W,Y)$ and the operators from the space $N_{[c]}(W,Y)$ will be called $[c]$-nuclear [8].

Let us present some characterizations of the $AP_{[c]}$ in terms of tensor products.

Theorem 2.3. *For $c \in \mathcal{C}$ and for a Banach space X, the following statements are equivalent:*

1) *X has the $AP_{[c]}$.*

2) *For every Banach space Y the natural mapping from $Y \overset{c}{\otimes} X$ to $B(Y^*, X^*)$ (or to $L(Y^*, X)$) is one-to-one.*

3) *For every Banach space Y the natural mapping $Y^* \overset{c}{\otimes} X \to L(Y,X)$ is one-to-one.*

4) *The natural mapping $X^* \overset{c}{\otimes} X \to L(X,X)$ is one-to-one (or, what is the same, there exists no tensor element $z \in X^* \overset{c}{\otimes} X$ with* trace $z = 1$ *and $\widetilde{z} = 0$, where $\widetilde{z}$ is the associated (with z) operator from X to X).*

Proof. 2) $\implies$ 3) $\implies$ 4) — evident.

4) $\implies$ 3). Suppose that there exists a Banach space Y such that the natural map $Y^* \overset{c}{\otimes} X \to L(Y,X)$ is not one-to-one. Take an element $z \in Y^* \overset{c}{\otimes} X$ which is not zero, but generates a zero operator $\widetilde{z} : Y \to X$. Then we can find an operator $U \in L(X, Y^{**})$ so that trace $U \circ z = 1$. If $z = \sum_{k=1}^\infty \lambda_k\, y_k' \otimes x_k$ is a representation of z in $Y^* \overset{c}{\otimes} X$ $(\sum ||x_n'||\,||y_n|| < \infty,\ |\lambda_k| \le c_k^{-1},\ k = 1, 2, \dots)$, then

$$1 = \text{trace } U \circ z = \sum_{k=1}^\infty \lambda_k \langle U x_k, y_k'\rangle = \sum_{k=1}^\infty \lambda_k\, \langle x_k, U^* y_k'\rangle$$

and $\sum_{k=1}^\infty \lambda_k\, U^* y_k'(x) x_k = 0$ for every $x \in X$. Put $x_k' := \lambda_k U^* y_k'$, $z_0 := \sum_{k=1}^\infty \lambda_k x_k' \otimes x_k \in X^* \overset{c}{\otimes} X$. We have

$$\text{trace } z_0 = 1,\ \ \widetilde{z}_0 \neq 0$$

(by the assumption about X). Consider a 1-dimensional operator $R = x' \otimes x$ in X with the property that trace $R \circ z_0 > 0$. Then

$$0 < \text{trace } R \circ z_0 = \sum_{k=1}^\infty \lambda_k \langle x_k', x\rangle \langle x', x_k\rangle = \sum_{k=1}^\infty \lambda_k\, \langle U^* y_k', x\rangle \langle x', x_k\rangle$$

$$= \langle \sum_{k=1}^\infty \lambda_k\, \langle Ux, y_k'\rangle x_k, x'\rangle = \langle x', \sum_{k=1}^\infty \lambda_k\, U^* y_k'(x) x_k\rangle = 0.$$

1) $\implies$ 4). Let $z \in X^* \overset{c}{\otimes} X$ and trace $z = 1$. Write $z = \sum \lambda_k\, x_k' \otimes x_k$, where the sequences (x_k') and (x_k) are bounded and $(\lambda_k c_k) \in l_1$. Then

$$z = \sum_{k=1}^\infty (\lambda_k c_k\, x_k') \otimes (c_k^{-1}\, x_k).$$

Let $\varepsilon > 0$ be such that $||(\lambda_k c_k)||_{l_1}\ \sup_k ||x_k'|| \cdot \varepsilon < 1/2$. By 1), there exists a finite rank operator $R \in X^* \otimes X$ such that $||R(c_k^{-1} x_k) - c_k^{-1} x_k|| \le \varepsilon$ for each $k \in \mathbb{N}$. It follows that, for this operator R,

$$|\operatorname{trace}(z - R\circ z)| = |\sum_{k=1}^{\infty}\langle \lambda_k c_k x'_k, c_k^{-1}x_k - R(c_k^{-1}x_k)\rangle|$$

$$\leq \sum_{k=1}^{\infty}\lambda_k c_k ||x'_k||\cdot\varepsilon \leq ||(\lambda_k c_k)||_{l_1}\sup_k ||x'_k||\cdot\varepsilon < 1/2.$$

Hence,

$$|\operatorname{trace} R\circ z| \geq 1/2$$

and therefore z generates a non-zero operator $\widetilde{z}$.

3) $\implies$ 2). It follows from 3) that for every Y the natural map $Y^{**}\overset{c}{\otimes} X \to L(Y^*, X)$ is one-to-one. Since $Y\overset{c}{\otimes} X$ is a subset of $Y^{**}\overset{c}{\otimes} X$, we get 2).

4) $\implies$ 1). Suppose that X does not have the $AP_{[c]}$ Then there is a sequence (x_n) such that $||x_n|| \leq c_n^{-1}$ $(n = 1, 2, \dots)$ and there exists an $\varepsilon > 0$ with the property that for any finite rank operator $R\in X^*\otimes X$ the inequality $\sup_n ||Rx_n - x_n|| > \varepsilon$ is valid. Consider the space $C_0(K;X)$ for the compact set $K := \{x_n\}_{n=1}^{\infty}\cup\{0\}$. Every operator U in X can be considered as a continuous function on K with values in X by setting $f_U(k) := U(k)$ for $k\in K$. In particular, for the identity map id in X and for any $R\in X^*\otimes X$ we have

$$||f_{\mathrm{id}} - f_R||_{C_0(K;X)} \geq \varepsilon.$$

The subset $\mathcal{R} := \overline{\{f_R:\ R\in X^*\otimes X\}}^{C_0(K;X)}$ of $C_0(K;X)$ is a closed linear subspace in $C_0(K;X)$. So, there exists an X^*-valued measure $\mu = (x'_k)_{k=1}^{\infty}\in C_0^*(K;X) = l_1(\{x_n\}_{n=1}^{\infty})\cup\{0\};X)$ such that $\mu|_{\mathcal{R}} = 0$ and $\mu(f_{id}) = 1$. In other words, we can find a sequence (x'_k) with $\sum_{k=1}^{\infty}||x'_k|| < \infty$ such that $\sum_{k=1}^{\infty}\langle x'_k, x_k\rangle = 1$ and $\sum_{k=1}^{\infty}\langle x'_k, Rx_k\rangle = 0$ for any $R\in X^*\otimes X$.

Define a tensor element $z\in X^*\widehat{\otimes}X$ by $z := \sum_{k=1}^{\infty}x'_k\otimes x_k$. Since $||x_n|| \leq c_n^{-1}$ for $(n = 1, 2, \dots)$ and $(x'_k)\in l_1(X^*)$, we get that $z\in X^*\overset{c}{\otimes} X$, trace $z = \sum_{k=1}^{\infty}\langle x'_k, x_k\rangle = 1$ and trace $R\circ z = 0$ for every $R\in X^*\otimes X$. This means that the condition 4) is not fulfilled. □

We give here only one of the natural consequences of Theorem 2.3:

Theorem 2.4. *If the dual space X^* has the $AP_{[c]}$, then X has the $AP_{[c]}$ too.*

Proof. We use Theorem 2.3. As it is known [6, Chap. I, Cor. 3, p. 41], the projective tensor product $X^*\widehat{\otimes}X$ is a subspace of the tensor product $X^*\widehat{\otimes}X^{**}$. The tensor product $X^*\overset{c}{\otimes} X$ is a linear subspace of $X^*\widehat{\otimes}X$, as well as $X^*\overset{c}{\otimes} X^{**}$ is a linear subspace of $X^*\widehat{\otimes}X^{**}$. Therefore, the natural map $X^*\overset{c}{\otimes} X\to X^*\overset{c}{\otimes} X^{**}$ is one-to-one. Now if X^* has the $AP_{[c]}$, then the canonical map $X^{**}\overset{c}{\otimes} X^*\to L(X^*, X^*)$ is one-to-one. Since we can identify the tensor product $X^{**}\overset{c}{\otimes} X^*$ with the tensor product $X^*\overset{c}{\otimes} X^{**}$, it follows that the natural map $X^*\overset{c}{\otimes} X\to L(X, X)$ is one-to-one. Thus, if X^* has the $AP_{[c]}$, then X has the $AP_{[c]}$ too. □

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Remark 2.5. The converse, generally, is not true: It is well-known that there are Banach spaces with the AP, whose duals do not have the AP. Hence, if X is one of such spaces, then there exists a sequence $(x'_n) \subset X^*$, tending to zero, so that X^* does not have the $AP_{[c]}$, where $c = (||x'_n||^{-1})$.

3. Sufficient conditions for a Banach space to have the $AP_{[c]}$

We give here some examples which are interesting for our notes. To state them, we formulate and prove the following statement.

Proposition 3.1. *Let $c \in \mathcal{C}$ be a non-decreasing sequence. For a Banach space Y, suppose that*

($\mathcal{C}$) there exist a number sequence (m_n), $m_n \to +\infty$, and a positive constant d such that for every natural number n, for every $\varepsilon > 0$ and for every subspace E of Y with $\dim E \le m_n$ there exists a finite rank operator R in Y so that $||R|| \le dc_{m_n}$ and $||R|_E - id_E||_{L(E,Y)} \le \varepsilon$.

Then $Y \in AP_{[c]}$.

Proof. Suppose that there is an element $z \in Y^* \overset{c}{\otimes} Y$ such that trace $z = a > 0$, but $\tilde{z} = 0$. Consider a representation of z of the kind

$$z = \sum_{k=1}^{\infty} c_k^{-1}\, y'_k \otimes y_k,$$

where $\sum_{k=1}^{\infty} ||y'_k||\, ||y_k|| < +\infty$. Take a decreasing sequence $(b_n) \in c_o$ such that $0 \le b_n \le 1$ for all n and still $\sum_{k=1}^{\infty} b_n^{-1}\, ||y'_k||\, ||y_k|| < +\infty$.

Fix a natural number N, large enough, such that for all $m \ge N$

$$\sum_{k=1}^{m} c_k^{-1}\, \langle y'_k, y_k\rangle \ge a/2 \quad \text{and} \quad d\, b_m \sum_{k=m+1}^{\infty} b_k^{-1} ||y'_k||\, ||y_k|| \le a/8.$$

Fix an $m = m_n, m > N$, put $E := \operatorname{span}\{y_k\}_{k=1}^m$, and apply given conditions ($\mathcal{C}$) to find a corresponding operator $R \in Y^* \otimes Y$ for $n = m$ and $\varepsilon = \dfrac{a/4}{\sum_{k=1}^m c_k^{-1}\, ||y'_k||}$.

By our assumption, trace $R \circ z = 0$. From this, we get

$$0 = \sum_{k=1}^{m} c_k^{-1}\, \langle y'_k, Ry_k\rangle + \sum_{k=m+1}^{\infty} c_k^{-1}\, \langle y'_k, Ry_k\rangle.$$

For the first sum:

$$\sum_{k=1}^{m} c_k^{-1}\, \langle y'_k, Ry_k\rangle \ge \sum_{k=1}^{m} c_k^{-1}\, \langle y'_k, y_k\rangle - \Big|\sum_{k=1}^{m} c_k^{-1}\, \langle y'_k, y_k - Ry_k\rangle\Big|$$

$$\ge a/2 - \sum_{k=1}^{m} c_k^{-1}\, ||y'_k||\, ||y_k - Ry_k|| \ge a/2 - \sum_{k=1}^{m} c_k^{-1}\, ||y'_k|| \sup_{1\le j\le m} ||y_j - Ry_j|| \ge a/2 - a/4 = a/4.$$

For the second sum:

$$\Big|\sum_{k=m+1}^{\infty} c_k^{-1}\langle y'_k, Ry_k\rangle\Big| \le \sum_{k=m+1}^{\infty} c_k^{-1} b_k\, b_k^{-1} ||y'_k||\, ||Ry_k||$$

$$\leq c_m^{-1} b_m \,||R|| \sum_{k=m+1}^{\infty} b_k^{-1} ||y_k'||\,||y_k|| \leq d\, c_m^{-1} b_m\, c_m \sum_{k=m+1}^{\infty} b_k^{-1} ||y_k'||\,||y_k|| =: d_m,$$

where $0 \leq d_m \leq a/8$.

Now, from the last three relations, we obtain: $0 \geq a/4 - d_m$. A contradiction. □

Let us consider some consequences of the proposition 3.1.

Corollary 3.2. *Let $c \in \mathcal{C}$ be a non-decreasing sequence. For a Banach space Y, suppose that there exists a constant $d > 0$ such that for every natural number n and for every n-dimensional subspace E of Y there exists a finite rank operator R in Y so that $||R|| \leq d\, c_n$ and $R|_E = id_E$. Then $Y \in AP_{[c]}$.*

Corollary 3.3. *Let $c \in \mathcal{C}$ be a non-decreasing sequence. For a Banach space Y, suppose that there exists a constant $d > 0$ such that for every natural number n and for every n-dimensional subspace E of Y there exists a finite dimensional subspace F of Y, containing E and dc_n-complemented in Y. Then $Y \in AP_{[c]}$.*

Corollary 3.4. *Let $c \in \mathcal{C}$ be a non-decreasing sequence. For a Banach space Y, suppose that there exists a constant $d > 0$ such that for every natural number n every n-dimensional subspace E of Y is dc_n-complemented in Y. Then $Y \in AP_{[c]}$. Moreover, every subspace of the space Y has the $AP_{[c]}$.*

It is well-known that for each natural number n every n-dimensional subspace E of any Banach space X is $\sqrt{n}$-complemented, i.e. there exists a continuous linear projector P from X onto E with $||P|| \leq \sqrt{n}$ (see [7]). Taking in Corollary 3.4 $c = (\sqrt{n})$, we get

Corollary 3.5. *For any Banach space X, for every $\epsilon > 0$ and any sequence (x_n) in X with $||x_n|| \leq 1/\sqrt{n}$ there exists a finite rank operator R in X such that $||Rx_n - x_n|| \leq \epsilon$ for every n.*

More generally, let X be a subspace of a quotient of an L_p-space ($1 \leq p \leq \infty$). There is a constant $C(p)$ such that if $n \in \mathbb{N}$ and E is an n-dimensional subspace of the space X, then there exists a projector P from X onto E with $||P|| \leq C(p)\, n^{|1/2-1/p|}$ (this follows from [15, Theorem 4.1 and its Corollaries]). Therefore, we get from Corollary 3.4:

Corollary 3.6. *Let $1 \leq p \leq \infty$ and X be a subspace of a quotient of an L_p-space. For every $\epsilon > 0$ and any sequence (x_n) in X with $||x_n|| \leq n^{-|1/2-1/p|}$, there exists a finite rank operator R in X such that $||Rx_n - x_n|| \leq \epsilon$ for every n.*

Remark 3.7. Corollary 3.5 generalizes a theorem of A. Grothendieck about the property $AP_{2/3}$ (see [6], [18] or [20]). Moreover, Corollaries 3.5 and 3.6 are generalizations of the corresponding facts, mentioned in [21, Section 1], where the same conclusions were made for sequences (x_n) with $||x_n|| = o(1/\sqrt{n})$ or with $||x_n|| = o(n^{-|1/2-1/p|})$ respectively. Also, we have the following generalization of the assertion $(***)'$ in that paper (where the Lorentz spaces $l^0_{q,\infty}(X)$ were considered): Given $\alpha \in [0, 1/2]$ and a Banach space X with the property that every finite dimensional subspace F of X is contained in a finite dimensional subspace $E \subset X$, which in turn is $C\,(\dim F)^{\alpha}$-complemented in X, we have

$(***)''$ for every sequence $(x_n) \in l_{q,\infty}(X)$, where $1/q = \alpha$, for any $\varepsilon > 0$ there is a finite rank operator R in X so that $\sup_n ||Rx_n - x_n|| \leq \varepsilon$.

Taking into account Theorem 2.3, we can reformulate this statement and Corollaries 3.5 and 3.6 in terms of tensor products: Given $\alpha \in [0, 1/2]$ and a Banach space X with the property that every finite dimensional subspace F of X is contained in a finite dimensional subspace $E \subset X$, which in turn is $C\,(\dim F)^{\alpha}$-complemented in X, for any Banach space Y the natural mapping $Y \overset{n^{\alpha}}{\otimes} X \to L(Y^*, X)$ is one-to-one. Thus, if $1 \leq p \leq \infty, \alpha = |1/2 - 1/p|$ and X is a subspace of a quotient of an L_p-space, then for any Y the natural mapping $Y \overset{n^{\alpha}}{\otimes} X \to L(Y^*, X)$ is one-to-one. In particular, for any Banach spaces X and Y the natural mapping $Y \overset{\sqrt{n}}{\otimes} X \to L(Y^*, X)$ is one-to-one. Let us mention that the last two statements, as well as $(***)''$, are optimal in the scale of Lorentz sequence spaces (this follows, e.g., from Examples, given in [21]).

Example 3.8. In [16] the second author constructed, for every $\varepsilon > 0$, a separable reflexive space X_ε with the following properties: The space X_ε does not possess the approximation property. There exists a constant $C_\varepsilon > 0$ such that if E is an n-dimensional subspace of X_ε, then 1) $d(E, l_n^2) \leq C_\varepsilon \log^{1+\varepsilon} n$ and 2) E is $C_\varepsilon \log^{1+\varepsilon} n$-complemented in X_ε. Fix $\varepsilon > 0$. By Corollary 3.4, the space X_ε and all of its subspaces have the $AP_{[\log^{1+\varepsilon}]}$, where $\log^{1+\varepsilon} := (\log^{1+\varepsilon}(n+1))$ (but, as was said, does not have the AP). Note that it was used in [16] a construction of the space X_ε from [24]. Therefore, this space can be taken in such a way that a "bad compact set" $K \subset X$ (in the definition of the CAP) possesses the properties described in [24]. Therefore, X_ε does not possess the CAP. Moreover, this space is an asymptotically Hilbertian space as follows from the construction in [16].

Example 3.9. One can find a Banach space W such that W has a Schauder basis and W^* does not have the AP but has, e.g., the $AP_{[\log^2]}$, where $\log^2 = (\log^2(n+1))$. Indeed, let X_ε be a separable reflexive Banach space without the AP, possessing the property $AP_{[\log^{1+\varepsilon}]}$ from Example 3.8. Let Z be a separable space such that Z^{**} has a basis and there exists a linear homomorphism φ from Z^{**} onto X_ε^* so that the subspace $\varphi^*(X_\varepsilon)$ is complemented in Z^{***} and, moreover, $Z^{***} \cong \varphi^*(X_\varepsilon) \oplus Z^*$ (see Lemma 1.1). Put $W := Z^{**}$. This (second dual) space W has a Schauder basis and its dual W^* does not have the AP, but has the $AP_{[\log^{1+\varepsilon}]}$. We can take $\varepsilon = 1$ to get the desired example with $\log^2$..

These examples 3.8 and 3.9 (the space X_ε) seems to be quite interesting, since, as far as we know, this is the first example (mentioned in the literature) of an asymptotically Hilbertian space without the CAP. Let us formulate the result as

Proposition 3.10. *There exists an asymptotically Hilbertian space without the compact approximation property. Moreover, this space can be chosen in such a way that it has the property $AP_{[\log^{1+\varepsilon}]}$, but does not have the CAP.*

Recall that the first example of an asymptotically Hilbertian space without the AP was constructed (by O. Reinov) in 1982 [16], where A. Szankowski's results were used (let us note that in that time there was not yet such notion as "asymptotically Hilbertian space"). Later, in 2000, by applying Per Enflo's example in a version of A. M. Davie [3], P. G. Casazza, C. L. García and W. B. Johnson [2] gave another example of an asymptotically Hilbertian space which fails the approximation property. In [21] O. Reinov got another example by using a construction from [14].

4. Operators with $[c]$-nuclear adjoints

The following theorem is one of the main results of the paper and is an analogue of the main result of the paper [11] (see Introduction above). Let us mention that the condition "either X or Y^{***} has the $AP_{[c]}$" is essential in the theorem, as well as the conclusion "'T is nuclear" is the best possible (see Section 6 below). Notations are as above (before Theorem 2.3).

Theorem 4.1. *Let $c \in \mathcal{C}$, $z \in X \overset{c}{\otimes} Y^{**})$, $T \in L(X^*, Y)$ be such that $\pi_Y T = \widetilde{z} \in N^w_{[c]}(X^*, Y^{**})$. If either $X \in AP_{[c]}$ or $Y^{***} \in AP_{[c]}$, then $T \in N^w(X^*, Y)$. In other words, under these conditions on the spaces involved, from the $[c]$-nuclearity of the conjugate (weak*-to-weak continuous) operator $T^* : Y^* \to X$ it follows that the operator T belongs to the space $N^w(X^*, Y)$* (in particular, is nuclear).

Proof. Suppose there exists a weak*-to-weak continuous operator $T \in L(X^*, Y)$ such that $T \notin N^w(X^*, Y)$, but $\pi_Y T \in N^w_{[c]}(X^*, Y^{**})$. Since either X or Y^{**} has the $AP_{[c]}$ (see Theorem 2.4), $N_{[c]}(Y^*, X) = Y^{**} \overset{c}{\otimes} X \,(= X \overset{c}{\otimes} Y^{**})$. Therefore the operator $\pi_Y T$ can be identified with the tensor element $z \in X \overset{c}{\otimes} Y^{**} \subset X \widehat{\otimes} Y^{**}$; in addition, by the choice of T, $z \notin X \widehat{\otimes} Y$ (the space $X \widehat{\otimes} Y$ is considered as a closed subspace of the space $X \widehat{\otimes} Y^{**}$). Hence there is an operator $U \in L(Y^{**}, X^*) = \big(X \widehat{\otimes} Y^{**}\big)^*$ with the properties that trace $U \circ z =$ trace $(z^t \circ (U^*|_X)) = 1$ and trace $U \circ \pi_Y \circ u = 0$ for each $u \in X \widehat{\otimes} Y$. From the last it follows that, in particular, $U \pi_Y = 0$ and $\pi_Y^* U^*|_X = 0$. In fact, if $x \in X$ and $y \in Y$, then

$$\langle U \pi_Y\, y, x\rangle = \langle y, \pi_Y^* U^*|_X x\rangle = \text{trace } U \circ (x \otimes \pi_Y(y)) = 0.$$

Evidently, the tensor element $U \circ z \in X \overset{c}{\otimes} X^*$ induces the operator $U \pi_Y T$, which is identically equal to zero.

If $X \in AP_{[c]}$ then $X \overset{c}{\otimes} X^* = N^w_{[c]}(X^*, X^*)$ and, therefore, this tensor element is zero what contradicts to the equality trace $U \circ z = 1$.

Let now $Y^{***} \in AP_s$. In this case

$$V := (U^*|_X) \circ T^* \circ \pi_Y^* :\; Y^{***} \to Y^* \to X \to Y^{***}$$

uniquely determines a tensor element z_0 from the $[c]$-projective tensor product $Y^{****} \overset{c}{\otimes}_s Y^{***}$. Let us take any representation $z = \sum \mu_n\, x_n \otimes y''_n$ for z as an element of the space $X \overset{c}{\otimes}_s Y^{**}$. Denoting for the brevity the operator $U^*|_X$ by

U_*, we obtain:

$$Vy''' = U_*\,(T^*\pi_Y^*\,y''') = U_*\,\Big((\sum \mu_n\, y''_n \otimes x_n)\,\pi_Y^*\,y'''\Big)$$

$$= U_*\,\Big(\sum \mu_n\,\langle y''_n, \pi_Y^*\,y'''\rangle\, x_n\Big) = \sum \mu_n\,\langle \pi_Y^{**} y''_n, y'''\rangle\, U_* x_n.$$

So, the operator V (or the element z_0) has in the space $Y^{****} \overset{c}{\otimes}_s Y^{***}$ the representation

$$V = \sum \mu_n\, \pi_Y^{**}(y''_n) \otimes U_*(x_n).$$

Therefore,

$$\text{trace } z_0 = \text{trace } V = \sum \mu_n\,\langle \pi_Y^{**}(y''_n), U_*(x_n)\rangle = \sum \mu_n\,\langle y''_n, \pi_Y^*\,U_* x_n\rangle = \sum 0 = 0$$

(since $\pi_Y^*\,U_* = 0$; see above).

On the other hand,

$$Vy''' = U_*\,(\pi_Y T)^*\,y''' = U_*\circ z^t(y''') = U_*\,\Big(\sum \mu_n\,\langle y''_n, y'''\rangle\, x_n\Big) = \sum \mu_n\,\langle y''_n, y'''\rangle\, U_* x_n,$$

whence $V = \sum \mu_n\, y''_n \otimes U_*(x_n)$. Therefore

$$\text{trace } z_0 = \text{trace } V = \sum \mu_n\,\langle y''_n, U_* x_n\rangle = \sum \langle U y''_n, x_n\rangle = \text{trace } U\circ z = 1.$$

A contradiction. □

Corollary 4.2. *Let $S \in L(X,Y)$ and $S^* \in N_{[c]}(Y^*, X^*)$. If either $X^* \in AP_{[c]}$ or $Y^{***} \in AP_{[c]}$, then $S \in N(X,Y)$.*

Proof. Suppose that the conditions are fulfilled. Let $S^* = \sum \mu_n\, y''_n \otimes x'_n$ be a representation of S^* in $N_{[c]}(Y^*, X^*) = Y^{**} \overset{c}{\otimes} X^*$ (the equality holds by the conditions).

Consider S^{**} as an operator T from X^{**} to Y :

$$Tx'' := \sum \mu_n\,\langle x'_n, x''\rangle y''_n \in \pi_Y(Y) \subset Y^{**},$$

identifying Y with $\pi_Y(Y)$ in a natural way. We are in conditions of Theorem 4.1. By this theorem, $T \in N^w(X^{**}, Y)$, i.e. T admits a nuclear representation $T = \sum \bar{x}'_n \otimes y_n$ (with $\sum ||\bar{x}'_n||\,||y_n|| < \infty$). But $T|_X x = Sx$ for all $x \in X$. □

Corollary 4.3. *Under the conditions of Corollary 4.2, if the space Y has the AP and $X = Y$, then the nuclear trace of S is well-defined and equal to* trace S^*, *i.e. to $\sum \mu_n\,\langle y''_n, x'_n\rangle$ in notation of the proof of Corollary 4.2.*

Proof. Now, we have $Y^*\widehat{\otimes} Y = N(Y,Y)$ and $S = \sum y'_n \otimes y_n$ (with $\sum ||y'_n||\,||y_n|| < \infty$). We can consider the tensor element $\sum y'_n \otimes y_n$ as an element of the tensor product $Y^{**}\widehat{\otimes} Y^*$ with the same projective norm. On the other hand, the tensor element $v := \sum \mu_n\, y''_n \otimes x'_n$ from the proof of Corollary 4.2 represents the operator S^* and this tensor element must belong to a subspace $\pi_Y(Y)\widehat{\otimes} Y^*$ of the space $Y^{**}\widehat{\otimes} Y^*$ (see the proof of Theorem 4.1). This means, in particular, that $\langle id_{Y^*}, v\rangle = \langle id_{Y^*}, \sum y'_n \otimes \pi_Y y_n\rangle$, i.e. $\sum \mu_n\,\langle y''_n, x'_n\rangle = \sum \langle y'_n, y_n\rangle$. □

5. Examples and some applications

Example 5.1. Fix $\varepsilon > 0$ and consider the separable reflexive space X_ε from Example 3.8 above. Recall that this space (as well as its dual) has the property $AP_{[log^{1+\varepsilon}]}$ but does not have the AP. Apply Lemma 1.1 to get a separable Banach space Z_ε with the property that $Z_\varepsilon^{***} \cong \varphi^*(X_\varepsilon^*) \oplus Z_\varepsilon^*$ (φ is the corresponding linear homomorphism from Z_ε^{**} onto X_ε with the kernel Z_ε). Let $u \in X_\varepsilon^* \widehat{\otimes} X_\varepsilon$ with trace $u = 1$ and $\widetilde{u} = 0$, and let $u = \sum x_n' \otimes x_n$ be a representation of the tensor element u in the projective tensor product (where $\sum ||x_n'||\,||x_n|| < \infty$). Taking $z_n'' \in Z_\varepsilon^{**}$ such that $\varphi z_n'' = x_n$ and $||z_n''|| \le 2\,||x_n||$ and putting $z_n''' := \varphi^* x_n'$, consider $\bar{u} := \sum z_n''' \otimes z_n''$, a tensor element of the projective tensor product $Z_\varepsilon^{***} \widehat{\otimes} Z_\varepsilon^{**}$. It is clear that trace $\bar{u} = 1$ and $\widetilde{\bar{u}}(Z_\varepsilon^{**}) \subset \pi_{Z_\varepsilon}(Z_\varepsilon) \subset Z_\varepsilon^{**}$ (since $\varphi \widetilde{\bar{u}} = 0$). On the other hand, $\widetilde{\bar{u}}|_{\pi_{Z_\varepsilon}(Z_\varepsilon)} = 0$. It follows from this that the operator $\widetilde{\bar{u}}$, considered as an operator from Z_ε^{**} to Z_ε, is not nuclear, but is nuclear as an operator from Z_ε^{**} to Z_ε^{**}.

On the other hand, if we take any operator T from Z^{**} to a Banach space Y (respectively, from a Banach space W to Z), for which the operator $\pi_Y T$ (respectively, the operator $\pi_Z T$) has a nuclear representation of the kind

$$\sum_{n=1}^{\infty} \frac{1}{\log^{1+\varepsilon}(n+1)}\, z_n''' \otimes y_n'',$$

where $\sum ||z_n'''||\,||y_n''|| < \infty$ (respectively, of the kind

$$\sum_{n=1}^{\infty} \frac{1}{\log^{1+\varepsilon}(n+1)}\, w_n' \otimes z_n'',$$

where $\sum ||w_n'||\,||z_n''|| < \infty$), then the operator T is nuclear, i.e. has a nuclear representation of the kind $\sum_{n=1}^{\infty} \hat{z}_n''' \otimes y_n$ (respectively, $\sum_{n=1}^{\infty} \hat{w}_n' \otimes z_n$). The last follows from Corollary 4.2.

Example 5.2. Put $X := L_1/H_0^1$ and use Lemma 1.1 to define a separable Banach space Z with the property that $Z^{***} \cong \varphi^*(X^*) \oplus Z^* \cong H^\infty \oplus Z^*$. Since the space Z^* has a basis and H^∞ possesses the property $AP_{[\log(1+n)]}$ [1], the space Z^{***} has the property $AP_{[\log(1+n)]}$ too. Thus, Corollary 4.2 can be applied and we get: For any Banach space W, if $U \in L(W, Z)$ and

$$\pi_Z U w = \sum_{n=1}^{\infty} \frac{1}{\log{(n+1)}} \langle w_n', w \rangle z_n'',\ \ w \in W,$$

where $\sum ||w_n'||\,||z_n''|| < \infty$, then there exist sequences $\{v_n'\} \subset W^*$ and $\{z_n\} \subset Z$ with $\sum ||v_n'||\,||z_n|| < \infty$ so that

$$Uw = \sum_{n=1}^{\infty} \langle v_n', w \rangle z_n,\ \ w \in W.$$

Let us mention that it is unknown whether we can omit "$1/\log(n+1)$" above.

Let us consider the first application of our results to the investigation of properties of nuclear operators in the spaces of analytic functions.

Theorem 5.3. *Let $T \in L(L_1/H_0^1, Y)$. If there exist sequences $(g_n) \subset H^\infty$ and $(y_n'') \subset Y^{*}*$ such that $\sum ||g_n||\,||y_n''|| < \infty$ and $T^*y' = \sum 1/\log(n+1)\,\langle y_n'', y'\rangle g_n$ for all $y' \in Y^*$, then the operator T is nuclear. Moreover, if $Y = L_1/H_0^1$, then the nuclear trace of T is well-defined and equals $\sum 1/\log(n+1)\,\langle y_n'', g_n\rangle$.*

Proof. As we know, the space $H^\infty = (L_1/H_0^1)^*$ has the property $AP_{[(\log(n+1))]}$. Thus, the first part follows from Corollary 4.2. In the case where $Y = L_1/H_0^1$, the trace of T is well-defined since the space L_1/H_0^1 has the AP. The equality follows from Corollary 4.3. □

Proposition 5.4. *Let a linear operator $T : H^\infty \to Y$ be such that there are a sequence of functions $(g_n) \subset L^1$ and a bounded sequence $(y_n'') \subset Y^{**}$, for which $\sum_k \int |g_k|\,dm < \infty$ and*

$$\pi_Y T(f) = \sum_{k=1}^{\infty} \int g_k(t)\,f(t)\,dm(t)\,y_k''.$$

Then the operator T is nuclear as an operator, acting from H^∞ into Y.

Proof. Let $S := T|_A : A \to Y$. Then $\pi_Y S = (\pi_Y T)|_A = \sum g_k^0 \otimes y_k'' \in L_1/H_0^1 \widehat{\otimes} Y^{**}$, where g_k^0 is the image of g_k under the quotient map $L_1 \to L_1/H_0^1$. If $y' \in Y^*$, then

$$S^*y' = \sum_{k=1}^{\infty} \langle y_k'', y'\rangle g_k^0 \in L_1/H_0^1 \quad \text{and} \quad S^{**}|_{H^\infty} = \pi_Y T.$$

Since A^* has the approximation property and S^* is nuclear, the operator S is nuclear too (by Grothendieck). Now, if $S = \sum \psi_n \otimes y_n$ is a nuclear representation of $S : A \to Y$, where $\psi_n \in A^*, y_n \in Y, \sum ||\psi_n||\,||y_n|| < \infty$, then $\pi_Y T = \sum \psi_n|_{H^\infty} \otimes \pi_Y y_n$ (we consider here the elements ψ_n as the functionals on A^{**}). It follows that $T = \sum \psi_n|_{H^\infty} \otimes y_n \in N(H^\infty, Y)$. □

Note that it is unknown whether the conclusion of the proposition is true if we suppose just that the operator $\pi_Y T$ is nuclear. The same can be said about the next consequence of Proposition 5.4.

Corollary 5.5. *Let a linear operator $T : H^\infty \to A$ be such that there are two sequences of functions $(g_n) \subset L^1$ and $(f_n) \subset H^\infty$, for which $\sum_k \int |g_k\,dm < \infty$, $\|f_n\| \le 1$ for each n and*

$$T(f) = \sum_{k=1}^{\infty} \int g_k(t)\,f(t)\,dm(t)\,f_k.$$

Then the operator T is nuclear as an operator, acting from H^∞ into the disk-algebra A.

Proof. As we know, the space H^∞ is a complemented subspace of the second dual A^{**}. Therefore the result follows directly from Proposition 5.4. □

6. Generalizations and further applications

Recall that we denote by $\mathcal{C}$ the set $\mathcal{C} := \{(c_n) :\ c_n \in \mathbb{R}_+,\ c_n \to +\infty\}$.

Let us consider a fixed subset $\mathcal{C}_0$ of this set (for example, we can take the set $\{(c_n) :\ c_n \in \mathbb{R}_+,\ (c_n^{-1}) \in l_2\}$.

Definition 6.1. A Banach space X has the property $AP_{\mathcal{C}_0}$ if it has the property $AP_{[c]}$ for every $c \in \mathcal{C}_0$.

Remark 6.2. Let us denote by $X \overset{\mathcal{C}_0}{\otimes} Y$ the union $\bigcup_{c\in\mathcal{C}_0} X \overset{c}{\otimes} Y$. It is easy to see that Theorem 2.3 is valid also for this kind of tensor products, i. e. Theorem 2.3 holds if we replace $AP_{[c]}$ by $AP_{\mathcal{C}_0}$ and $\overset{c}{\otimes}$ by $\overset{\mathcal{C}_0}{\otimes}$. We omit a formulation of this generalized theorem 2.3.

Example 6.3. Let $0 < p < \infty$ and $\mathcal{C}_0 = l_p^{-1} := \{c = (c_n) :\ c_n \in \mathbb{R}_+,\ (c_n^{-1}) \in l_p\}$. By the main definition 2.1, a space X has the $AP_{l_p^{-1}}$ if for any $c \in l_p^{-1}$, for every $\epsilon > 0$ and any sequence (x_n) in X with $||x_n|| \le c_n^{-1}$ there exists a finite rank operator R in X such that $||Rx_n - x_n|| \le \epsilon$ for every n. Or: X has this property if and only if for every $\epsilon > 0$ and any sequence (x_n) in X with $\sum ||x_n||^p < \infty$ there exists a finite rank operator R in X such that $||Rx_n - x_n|| \le \epsilon$ for every n. We can see that this property $AP_{l_p^{-1}}$ is exactly the property $\widetilde{AP_s} = AP_s$ from [22, Sect. 1-2] (see also [20], [21]), if we take s from the equality $1/s = 1 + 1/q$.

Definition 6.4. An operator $T \in L(X, Y)$ is said to be $\mathcal{C}_0$-nuclear, if there is a sequence $c \in \mathcal{C}_0$ such that $T \in N_{[c]}(X, Y)$.

It is easy to see that Theorem 4.1 can be formulated and proved for the general case of $\mathcal{C}_0$-nuclear operators. But we consider here only an evident generalization of Corollary 4.2:

Theorem 6.5. *Let $S \in L(X, Y)$ and S^* is $\mathcal{C}_0$-nuclear. If either X^* has the property $AP_{\mathcal{C}_0}$ or Y^{***} has the property $AP_{\mathcal{C}_0}$, then $S \in N(X, Y)$.*

Example 6.6. Let $p \in (0, \infty]$, $r \in (0, \infty]$ and consider a tensor product $\widehat{\otimes}_{1;p,r}$ from [22, Sec. 3]. It is defined in the following way: For a couple of Banach spaces X, Y the tensor product $Y \widehat{\otimes}_{1;p,r} X$ consists of those elements z of the projective tensor product $Y \widehat{\otimes} X$ which admit representations of the type

$$z = \sum_{k=1}^{\infty} a_k b_k\, y_k \otimes x_k;\ \ (y_k) \text{ and } (x_k) \text{ are bounded, } (a_k) \in l_1,\ (b_k) \in l_{pr}$$

(recall that in [22] and here one considers $l^0_{p\infty}$ in the case $r = \infty$). If we put $\mathcal{C}_0 := l_{p,r}^{-1} := \{c = (c_n) :\ c_n \in \mathbb{R}_+,\ (c_n^{-1}) \in l_{pr}\}$, then it is clear that $Y \widehat{\otimes}_{1;p,r} X = Y \overset{l_{pr}^{-1}}{\otimes} X$ and the property $AP_{l_{pr}^{-1}}$ is just the property $AP_{1;p,r}$ from [22] (recall that here we consider $l^0_{p\infty}$ in the case $r = \infty$).

Also, let $0 < s < 1$ and $0 < u \le \infty$, or $s = 1$ and $0 < u \le 1$. If $1 + 1/p = 1/s$ and $1 + 1/r = 1/u$, then $AP_{1;p,r} = AP_{(s,u)}$ and $Y \widehat{\otimes}_{1;p,r} X = Y \widehat{\otimes}_{(s,u)}$, where the last tensor product is the "Lorentz tensor product", consisting of those tensor elements from $Y\widehat{\otimes}X$ which admit the representations of type

$$\sum_{k=1}^{\infty} \lambda_k\, y_k \otimes x_k, \text{ where } (y_k) \text{ and } (x_k) \text{ are bounded and } (\lambda_k) \in l_{su}$$

(see [22, Sec 3]). Thus the analogues of Theorems 2.3 and 6.5 are valid for the cases of these tensor products and approximation properties (cf. Remark 6.2). As for the case where $0 < p < \infty$ and $\mathcal{C}_0 = l_{p\infty}^{-1}$, we have: $Y \overset{C_0}{\otimes} X = Y \overset{(n^{1/p})}{\otimes} X$ and $AP_{\mathcal{C}_0} = AP_{[(n^{1/p})]}$ (evidently, $\overset{c_0^{-1}}{\otimes} = \widehat{\otimes}$ and $AP_{c_0^{-1}} = AP$). Recall that every Banach space has the $AP_{[(\sqrt{n})]}$ (see Corollary 3.5). If $p = \infty$, then we get one more new result considering $\mathcal{C}_0 = l_{p\infty}$ in Theorem 6.5 (otherwise we can apply the "generalized" Theorem 2.3 to this case; cf. Remark 6.2).

Note that a partial case of Theorem 6.5 (namely, the case considered in Example 6.3) was studied already in [21]. In this case, the corresponding operators are called s-nuclear (recall that $1/s = 1 + 1/p$). Let us apply Theorem 6.5 to the particular case of the given spaces of analytic functions.

Theorem 6.7. *Let $0 < s < 1$ and W be any of the following Banach spaces: $A, L_1/H_0^1$ or H^∞. Let Z be either the space W or any of its duals (W^* or W^{**} etc.). If Y is a Banach space, $T \in L(Z,Y)$, $U \in L(Y,Z)$, T^* and U^* are s-nuclear, then $T \in N(Z,Y)$ and $U \in N(Y,Z)$.*

Proof. As was shown in [1, Theorem 1], the space H^∞ and all of its duals have the property AP_s for any $s \in (0,1)$. Therefore, the same is true for the spaces A and L_1/H_0^1. Now, the assertion of Theorem 6.7 follows from the considered partial case of Theorem 6.5. □

Remark 6.8. In the case where $s = 1$, the assertion of Theorem 6.7 is known to be valid only if $Z = A$ and $T^* \in N(Y^*, A^*)$ (cf. the proof of Theorem 5.3).

We end the paper with a direct application of Theorem 4.1 to some nuclear bilinear forms on the products of type $H^\infty \times Y$:

Theorem 6.9. *Let $\log := (\log(n+1))$ and $T \in L((H^\infty)^*, X)$ be such that $T^*(X^*) \subset H^\infty$. If there is a tensor element $z \in X^{**} \overset{\log}{\otimes} H^\infty$ which generates the operator T^*, then T is a nuclear operator from $(H^\infty)^*$ to X that can be generated by a tensor element belonging to the projective tensor product $H^\infty \widehat{\otimes} X$.*

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