Around Grothendieck's theorem on operators with nuclear adjoins

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Nuclear operators

Definition

$$T : X \to Y$$

is nuclear, if it's admits a factorization

$$T : X \xrightarrow{A} l_\infty \xrightarrow{\Delta} l_1 \xrightarrow{B} Y,$$

where $\Delta(a_k) = (\delta_k a_k),\ (\delta_k) \in l_1.$

Question

Let $T : X \to Y$ *and* T^* *be nuclear:*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "A"] & Y \arrow[r, "\pi_Y"] & Y^{**} \\ l_\infty \arrow[r, "\Delta"] & l_1 \arrow[r, "B"] & Y \arrow[u, "\pi_Y"] \end{tikzcd} \qquad X^* \xleftarrow{A^*} l_1 \xleftarrow{\Delta} l_\infty \xleftarrow{B^*} Y^*$$

$$\Downarrow \xRightarrow{\exists ?} \quad X \xrightarrow{A_0} l_\infty \xrightarrow{\Delta_0} l_1 \xrightarrow{B_0} Y$$

AP and Grothendieck's theorem

Definition

X has the approximation property (the AP), if

$$\forall\ \varepsilon > 0,\ (x_n) \in c_0(X)$$

$$\exists\ R : X \to X, \quad \textit{of finite rank}:$$

$$\sup_n ||Rx_n - x_n|| \leq \varepsilon.$$

Theorem

Let $T \in L(X, Y)$ be such that T^* is nuclear.
If X^* has the AP, then T is nuclear from X to Y.

A. Grothendieck, Produits tensoriels topologiques et espases nucléaires, Mem. Amer. Math. Soc., Volume 16, 1955, 196 + 140.

Simple Example

EXAMPLE.

$$\begin{tikzcd} C[0,1] \arrow[r, "U"] \arrow[dr, bend right, "\pi U"'] & Y \arrow[d, "\pi"] \\ & Y^{**} \end{tikzcd}$$

$$\pi U(f) = \int_0^1 f(t)\,\bar{g}(t)\,dt$$

$$\bar{g} \in L_1([0,1], Y^{**})$$

$$\begin{tikzcd} C[0,1] \arrow[r, "U"] \arrow[d, "A"] \arrow[dr, bend right, "\pi U"] & Y \arrow[d, "\pi"] \\ l_\infty \arrow[d, "\Delta"] & Y^{**} \\ l_1 \arrow[ur, bend right, "B"'] & \end{tikzcd}$$

$$\pi U(f) \in N(C[0,1], Y^{**})$$

$$\Delta(a_k) = (\delta_k a_k)$$

Since $C^*[0,1] \in AP$, we have:

$$U \in N(C[0,1], Y).$$

Figiel-Johnson Theorem 1973

Theorem

There exists a Banach space X : X has the AP,

$$\exists\, T : X \to X,\ \ T^* \ \text{ is nuclear, but } T \ \text{ is not.}$$

T. Figiel, W.B. Johnson, The approximation property does not imply the bounded approximation property, Proc. Amer. Math. Soc., Volume 41 (1973), 197–200.

Oja-Reinov 1987

Theorem

Let $T \in L(X, Y)$ be such that T^* is nuclear.
If Y^{***} has the AP, then T is nuclear from X to Y.

Theorem

There exists a Banach space X : X, X^*, X^{**} have the AP,

$$\exists\, T : X^{**} \to X, \;\; T^* \;\text{ is nuclear, but } T \;\text{ is not.}$$

E. Oja, O.I. Reinov, Un contre-exemple à une affirmation de A.Grothendieck, C. R. Acad. Sc. Paris. — Serie I, Volume 305 (1987), 121–122.

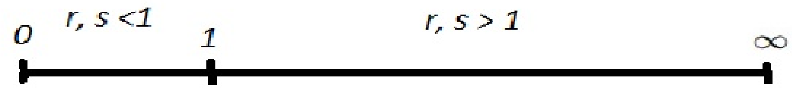
Up to now, we considered the factorizations through

$$l_\infty \xrightarrow{\Delta} l_1.$$

General questions were posed in different time about the factorizations through diagonal operators of type

$$l_r \xrightarrow{\Delta} l_s$$

[e. g., by A. Pełczyński F. Pietsch, P. Saphar, A. Hinrichs etc]. Different methods were used by the author to get some sharp results for $1 \leq r, s \leq \infty$ and for $0 < r, s \leq 1$.



$$l_r \xrightarrow{\Delta} l_s$$

Diagrams

Is it true that if we have

$$\begin{tikzcd}
X \arrow[rrr, "T"] \arrow[dr, bend right, "B"'] & & & Y \arrow[d, "\pi_Y"] \\
& l_r \arrow[r, "\Delta"] & l_s \arrow[r, "A"] & Y^{**}
\end{tikzcd}$$

then one has a factorization

$$\begin{tikzcd}
X \arrow[drrr, bend left, "T"] \arrow[dr, dashed, "(\exists ?)"] \arrow[ddr, bend right, "B"'] & & & \\
& l_p \arrow[r, "\Delta_1"] & l_q \arrow[r, "(\exists ?)"] & Y \arrow[d, "\pi_Y"] \\
& l_r \arrow[r, "\Delta"] & l_s \arrow[r, "A"] & Y^{**}
\end{tikzcd}$$

Particular cases: $r = p = \infty$, $s = q \geq 1$ (so-called q-nuclear operators).

Example of normed tensor products

We need some information about topological tensor products.

Example

Let $p \geq 1$. $\mathcal{F}(X,Y) = \{$all finite rank operators $R : X \to Y.\}$
Put $n_p(R) := \inf\{||A||\,||\Delta_n||\,||B||\}$,
where inf is taken over all factorizations of R of type

$$R:\ X \xrightarrow{B} l_\infty^n \xrightarrow{\Delta_n} l_p^n \xrightarrow{A} Y,\ \ n \in \mathbb{N}.$$

We consider $\mathcal{F}(X,Y)$ as a tensor product $X^* \otimes Y$:

$$x'(\cdot)\,y \in \mathcal{F}(X,Y) \ \text{ means } \ x' \otimes y \in X^* \otimes Y.$$

Let $X^* \widehat{\otimes}_p Y$ be a completion of $(X^* \otimes Y, n_p)$.
Then $X^* \widehat{\otimes}_p Y$ generates a Banach space $N_p(X,Y)$ of p-nuclear operators T :

$$T:\ X \to l_\infty \xrightarrow{\Delta} l_p \to Y.$$

If $p = 1$, $0 < s < 1$ and $\Delta \sim (\delta_k) \in l_s$, then we get s-nuclear operators.
Sharp versions of the Grothendieck's theorem were obtained by the author for these situations: $0 < s \leq 1$, $1 \leq p < \infty$.
We have two different cases here:
1) $1 \leq p < \infty$ — a case of "tensor norms".
2) $0 < s < 1$ — a case of "tensor quasi-norms"
("triangle inequality"with a constant: $||a + b|| \leq C\,(||a|| + ||b||)$).

Рис.: Norms/Quasi-norms illustrated

Projective/injective tensor norms

As was said above, $X^* \otimes Y$ can be considered as $\mathcal{F}(X, Y)$:
$x' \otimes y : X \to Y$, $(x' \otimes y)x = \langle x', x\rangle y$.
Projective norm $||\cdot||_\wedge (= n_1)$ can be defined by

$$(X^* \otimes Y, ||\cdot||_\wedge)^* = L(Y, X^{**}).$$

with duality

$$\langle U, x' \otimes y\rangle = \langle Uy, x'\rangle =: \text{trace } U \circ (x' \otimes y), \ \ U \in L(Y, X^{**}).$$

Completion: $X^* \widehat{\otimes} Y$ (projective tensor product).
Injective norm is the usual operator norm $||\cdot||$:

$$(X^* \otimes Y, ||\cdot||) \subset (L(X, Y), ||\cdot||) = L(X, Y).$$

Completion: $X \widehat{\widehat{\otimes}} Y$ (injective tensor product).

Tensor norms and α-nuclear operators

Tensor norm α on $X^* \otimes Y$ (roughly):

$$|| \cdot || \leq \alpha(\cdot) \leq || \cdot ||_\wedge$$

(plus some additional properties).

$$X^* \widehat{\otimes}_\alpha Y \; - \quad \text{completion of} \;\; (X^* \otimes Y, \alpha).$$

One has a natural map:

$$j_\alpha : X^* \widehat{\otimes}_\alpha Y \to L(X, Y).$$

The image is

$$N_\alpha(X, Y) := j_\alpha(X^* \widehat{\otimes}_\alpha Y) - \alpha\text{-nuclear operators.}$$

Eve Oja's Theorem 2012

Theorem

Let $T \in L(X, Y)$ and $\pi_Y T \in N_\alpha(X, Y^{**})$. If X^* or Y^{***} has the AP, then $T \in N_\alpha(X, Y)$.

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[dr, bend right, "N_\alpha"'] & Y \arrow[d, "\pi_Y"] \\ & Y^{**} \end{tikzcd} \qquad\qquad \begin{tikzcd} X^* & Y^* \arrow[l, "T^* \in N^\alpha"'] \end{tikzcd}$$

Eve Oja, Grothendieck's nuclear operator theorem revisited with an application to pn-ull sequences, J, Func. Anal. 263 (2012) 2876-2892

Remark

1) E. Oja has proved a more general theorem.
2) The author showed that the conditions about the AP ("AP w.r. to the projective tensor norm") can be changed by a weaker conditions ("AP w.r. to a tensor norm").

Tensor quasi-norms

Description

Projective tensor quasi-norm (on all tensor products $Z \otimes W$):
Take a quasi-norm β *on* $X^* \otimes Y$ *such that* $\beta(\cdot) \geq n_1(\cdot) = ||\cdot||_\wedge$,
consider a completion $X^* \widehat{\otimes}_\beta Y$. *It is supposed that a natural map*

$$X^* \widehat{\otimes}_\beta Y \to X^* \widehat{\otimes} Y.$$

is one-to-one
(plus some additional properties).
β^t *is a transposed quasi-norm:* $\beta^t(z \otimes w) := \beta(w \otimes z)$

Example

Let $0 < s \leq 1$.

$$X^* \widehat{\otimes}_s Y := \{z \in X^* \widehat{\otimes} Y : \ z = \sum x'_k \otimes y_k, \sum ||x'_k||^s \, ||y_k||^s < \infty\}.$$

Main questions

One has a natural map:

$$j_\beta : X^* \widehat{\otimes}_\beta Y \to L(X, Y).$$

The image is

$$N_\beta(X, Y) := j_\beta(X^* \widehat{\otimes}_\beta Y) - \beta\text{-nuclear operators.}$$

In Example: s-nuclear operators. A question for these operators:

Question

Let $T : X \to Y$ and T^* be s-nuclear.
Is T s-nuclear too?

A. Hinrichs, A. Pietsch, p-nuclear operators in the sense of Grothendieck, Math. Nachr., Volume 283, No. 2 (2010), 232–261.

Corresponding question in general case:

Question

Let $T : X \to Y$ and T^* be β^t-nuclear:

$$
\begin{tikzcd}
X \arrow[r, "T"] \arrow[dr, bend right, "N_\beta"'] & Y \arrow[d, "\pi_Y"] \\
 & Y^{**}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X^* & Y^* \arrow[l, "T^* \in N_{\beta^t}"']
\end{tikzcd}
$$

$$\Downarrow \overset{??}{\Longrightarrow} \quad X \xrightarrow{T} Y \text{ is } \beta\text{-nuclear?}$$

Generally, the answer is NO (already for Hinrichs–Pietsch question).

For a formulation of the main result we need one more definition.
Recall that
Z has the AP iff for every W the natural map

$$W^*\widehat{\otimes}Z \to L(W,Z)$$

is one-to-one (A. Grothendieck).

Definition

Z has the AP_β if for every W the natural map

$$W^*\widehat{\otimes}_\beta Z \to L(W,Z)$$

is one-to-one.

In Example above (s-nuclear operators), this property is the AP_s.
The *AP* implies all the *AP*'s!

Result

We have the following sharp result:

Theorem

Let $T \in L(X, Y)$ and $\pi_Y T \in N_\beta(X, Y^{**})$ (or, what is the same, $T^* \in N_\beta(Y^*, X^*)$). If X^* or Y^{***} has the AP_{β^t}, then $T \in N_1(X, Y)$ (i.e. Grothendieck nuclear).

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[dr, bend right, "N_\beta"'] & Y \arrow[d, "\pi_Y"] \\ & Y^{**} \end{tikzcd} \qquad\qquad X^* \xleftarrow[T^* \in N_{\beta^t}]{} Y^*$$

$$X^* \quad \text{or} \quad Y^{***} \in AP_{\beta^t}$$

$$\Downarrow \implies X \xrightarrow{T} Y \text{ is } \textit{nuclear}.$$

Comment to the result

As was said, the Hinrichs–Pietsch question (on s-nuclearity) has a negative answer:

$$\exists\, X, Y, T : X \to Y : \forall\, s \in (2/3, 1]\;\; T^* \in N_s(Y^*, X^*)$$

$$\text{but} \quad T \notin N_1(X, Y).$$

O. I. Reinov, On linear operators with s-nuclear adjoints, $0 < s \leq 1$, J. Math. Anal. Appl., Volume 415 (2014) 816-824.

By the way, IT IS OPEN for a long time:

Question

Let $T : X \to Y$ and $T^* \in N_{2/3}(Y^*, X^*)$.
Is it true that $T \in N_{2/3}(X, Y)$?

Applications

Let us recall that if A and H^∞ denote the disk algebra and the space of bounded analytical functions respectively then $A^* = L_1/H_0^1 \oplus L$ and $A^{**} = H^\infty \oplus L^*$, where L is an L_1-space. It is known that H^∞ and all its duals have the AP_s for all $s \in (0,1)$.

Bourgain J., Reinov O.I. *On the approximation properties for the space H^∞*, Math. Nachr. – 122 (1985). - P. 19-27.

Thus, our first application is

Theorem

Let $0 < s < 1$ and $A^{(n)}$ be either the space A or any of its duals (A^* or A^{**} etc.). If Y is any Banach space, $T \in L(A^{(n)}, Y)$, $U \in L(Y, A)$, T^* and U^* are s-nuclear, then $T \in N(A^{(n)}, Y)$ and $U \in N(Y, A)$.

Applications

In particular, we have:

$$\forall\, s \in (0,1),\ \forall\, Y:$$

$$\begin{tikzcd} L_1/H_0^1 \arrow[r, "T"] \arrow[dr, bend right, "N_s"'] & Y \arrow[d, hook, "\pi_Y"] \\ & Y^{**} \end{tikzcd} \implies L_1/H_0^1 \xrightarrow{T} Y \text{ is } \textit{nuclear}.$$

$$\begin{tikzcd} Y \arrow[r, "U"] \arrow[dr, bend right, "N_s"'] & A \arrow[d, hook, "\pi_A"] \\ & H^\infty \end{tikzcd} \implies Y \xrightarrow{U} A \text{ is } \textit{nuclear}.$$

Concluding Remarks and Applications

Recalling: X has the AP iff $\forall\ \varepsilon > 0$, $\forall\,(x_n) \in c_0(X)$
$\exists\ R : X \to X$ of finite rank : $\sup_n ||Rx_n - x_n|| \leq \varepsilon$.
Or: the natural map $X^* \widehat{\otimes} X \to N(X,X)$ is one-to-one.

If $s \in (0,1)$ and $1/s = 1 + 1/p$, then $X \in AP_s$ iff
$\forall\ \varepsilon > 0$, $\forall\,(x_n) \in l_p(X)$
$\exists\ R : X \to X$ of finite rank : $\sup_n ||Rx_n - x_n|| \leq \varepsilon$.
Or: the natural map $X^* \widehat{\otimes}_s X \to N(X,X)$ is one-to-one.
We used the last definition when we considered s-nuclear operators defined on L_1/H_0^1.

BUT: $H^\infty = (L_1/H_0^1)^*$ has the AP "up to log", i.e.
$\forall\ \varepsilon > 0$, $\forall\,(x_n) \subset X$: $||x_n|| \leq 1/\log(n+1)$
$\exists\ R : X \to X$ of finite rank : $\sup_n ||Rx_n - x_n|| \leq \varepsilon$.

Bourgain J., Reinov O.I. *On the approximation properties for the space H^∞*, Math. Nachr. – 122 (1985). - P. 19-27.

One can introduce a definition of the corresponding property $AP_{[\log]}$ and to get, in particular:

Theorem

Let $T \in L(L_1/H_0^1, Y)$. If there exist sequences $(g_n) \subset H^\infty$ and $(y_n'') \subset Y^{**}$ such that $\sum ||g_n||\,||y_n''|| < \infty$ and

$$T^* y' = \sum \frac{1}{\log(n+1)} \langle y_n'', y' \rangle g_n \quad \text{for all } y' \in Y^*,$$

then the operator T is nuclear. Moreover, if $Y = L_1/H_0^1$, then the nuclear trace of T is well-defined and equals $\sum 1/\log(n+1)\,\langle y_n'', g_n \rangle$.

Thank you for your attention!