Препринт Санкт-Петербургского математического общества
Поступил 26.11.2023
Доступен на сайте
http://www.mathsoc.spb.ru/preprint/

Boundary Point Principle for divergence type parabolic equations with principal coefficients discontinuous in time [preliminary version]

Darya E. Apushkinskaya, Alexander I. Nazarov

1 Introduction

1.1 Notation and conventions

Throughout the paper we use the following notation:
$x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$ is a point in $\mathbb{R}^n$;
$(x;t) = (x', x_n; t) = (x_1, \dots, x_n; t)$ is a point in $\mathbb{R}^{n+1}$;
$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, $\qquad \mathbb{R}^{n+1}_+ = \{(x;t) \in \mathbb{R}^{n+1} : x_n > 0\}$;
$|x|, |x'|$ are the Euclidean norms in corresponding spaces;
$B_r(x^0)$ is the open ball in $\mathbb{R}^n$ with center x^0 and radius r;
$B_r = B_r(0)$;
$Q_r(x^0;t^0) = B_r(x^0) \times (t^0 - r^2; t^0)$; $\quad Q_r = Q_r(0;0)$;

D_i denotes the operator of (weak) differentiation with respect to x_i;
$D = (D', D_n) = (D_1, \dots, D_{n-1}, D_n); \quad \partial_t = \dfrac{\partial}{\partial t}$.
We adopt the convention that the indices i and j run from 1 to n. We also adopt the convention regarding summation with respect to repeated indices.

We use standard notation for the functional spaces. For a bounded domain $\mathcal{E} \subset \mathbb{R}^{n+1}$ we understand $\mathcal{C}^{1,0}_{x,t}(\overline{\mathcal{E}})$ as the space of $u \in \mathcal{C}(\overline{\mathcal{E}})$ such that $Du \in \mathcal{C}(\overline{\mathcal{E}})$.

Definition 1. We say that a continuous function $\sigma : [0,1] \to \mathbb{R}_+$ belongs to the class $\mathcal{D}$ if

- σ is increasing, and $\sigma(0) = 0$;
- $\sigma(\tau)/\tau$ is summable and decreasing.

Remark 1. It should be noted that our assumption about the monotonicity of $\sigma(\tau)/\tau$ is not restrictive, and moreover, without loss of generality σ can be assumed continuously differentiable on $(0;1]$ (see [AN16, Remark 1.2] and [AN19, Remark 1] for details).

For $\sigma \in \mathcal{D}$ we define the function $\mathcal{J}_\sigma$ as

$$\mathcal{J}_\sigma(s) := \int_0^s \frac{\sigma(\tau)}{\tau}\, d\tau.$$

Definition 2. Let $\mathcal{E}$ be a bounded domain in $\mathbb{R}^n$. We say that a function $\zeta : \mathcal{E} \to \mathbb{R}$ belongs to the class $\mathcal{C}^{0,\mathcal{D}}(\mathcal{E})$, if

- $|\zeta(x) - \zeta(y)| \leqslant \sigma(|x-y|)$ for all $x, y \in \overline{\mathcal{E}}$, and σ belongs to the class $\mathcal{D}$.

Similarly, suppose that $\mathcal{E}$ is a bounded domain in $\mathbb{R}^{n+1}$. A function $\zeta : \mathcal{E} \to \mathbb{R}$ is said to belong to the class $\mathcal{C}^{0,\mathcal{D}}_x(\mathcal{E})$, if

- $|\zeta(x;t) - \zeta(y;t)| \leqslant \sigma(|x-y|)$ for all $(x;t), (y;t) \in \overline{\mathcal{E}}$, and σ belongs to the class $\mathcal{D}$.

We use the letters C and N (with or without indices) to denote various constants. To indicate that, say, C depends on some parameters, we list them in parentheses: $C(\dots)$.

2 Statement of the problem

Let Q be a bounded domain in $\mathbb{R}^{n+1}$ with topological boundary ∂Q. We define the parabolic boundary $\partial' Q$ as the set of all points $(x^0;t^0) \in \partial Q$ such that for any $\varepsilon > 0$, we have $Q_\varepsilon(x^0;t^0) \setminus \overline{Q} \neq \emptyset$. By $d_p(x;t)$ we denote the parabolic distance between $(x;t)$ and $\partial' Q$ which is defined as follows:

$$d_p(x;t) := \sup\{\rho > 0 : Q_\rho(x;t) \cap \partial' Q = \emptyset\}.$$

Next, we define the lateral surface $\partial'' Q$ as the set of all points $(x^0;t^0) \in \partial' Q$ such that $Q_\varepsilon(x^0;t^0) \cap Q \neq \emptyset$ for any $\varepsilon > 0$.

We suppose that Q satisfies the *parabolic interior* $\mathcal{C}^{1,\mathcal{D}}$*-paraboloid* condition. It means that in a local coordinate system $\partial'' Q$ is given by the equation $x_n = F(x';t)$, where F is a $\mathcal{C}^1$-function such that $F(0;0) = 0$ and the inequality

$$F(x';t) \leqslant \sqrt{|x'|^2 - t} \cdot \sigma(\sqrt{|x'|^2 - t}) \qquad \text{for} \quad t \leqslant 0 \tag{1}$$

holds in some neighborhood of the origin. Here σ is a $\mathcal{C}^1$-function belonging to the class $\mathcal{D}$ (see Remark 1).

Let an operator $\mathcal{M}$ be defined by the formula

$$\mathcal{M}u := \partial_t u - D_i(a^{ij}(x;t)D_j u) + b^i(x;t)D_i u = 0. \tag{2}$$

Suppose that the coefficients of $\mathcal{M}$ satisfy the following conditions:

$$\begin{gathered} \nu \mathcal{I}_n \leq (a^{ij}(x;t)) \leq \nu^{-1}\mathcal{I}_n, \\ a^{ij} \in \mathcal{C}^{0,\mathcal{D}}_x(Q) \qquad \text{for all} \quad i,j = 1,\dots,n, \end{gathered} \tag{3}$$

and

$$\omega_p^-(r) \to 0 \quad \text{and} \quad \omega_p^+(r) \to 0 \qquad \text{as} \quad r \to 0, \tag{4}$$

where

$$\omega_p^-(r) := \sup_{(x;t)\in Q} \int\limits_{Q_r(x;t)\cap Q} \frac{|\mathbf{b}(y;s)|}{(t-s)^{(n+1)/2}} \exp\left(-\gamma\, \frac{|x-y|^2}{t-s}\right) \times$$
$$\times \frac{d_p(y;s)}{d_p(y;s) + \sqrt{|x-y|^2 + t - s}}\, dyds;$$

$$\omega_p^+(r) := \sup_{(x;t)\in Q} \int\limits_{Q_r(x;t+r^2)\cap Q} \frac{|\mathbf{b}(y;s)|}{(s-t)^{(n+1)/2}} \exp\left(-\gamma\,\frac{|x-y|^2}{s-t}\right)\times$$
$$\times\frac{d_p(y;s)}{d_p(y;s)+\sqrt{|x-y|^2+s-t}}\,dyds.$$

Here ν is a positive constant, $\mathcal{I}_n$ is identity $(n\times n)$-matrix, $\mathbf{b}(y;s) = (b^1(y;s),\dots,b^n(y;s))$, and γ is a positive constant to be determined later, depending only on n, ν and on the moduli of continuity of the coefficients a^{ij} in spatial variables.

Remark 2. Notice that in any strict interior subdomain of $\overline{Q}\setminus\partial' Q$ condition (4) means that $\mathbf{b}$ is an element of the parabolic Kato class $\mathbf{K}_n$, see [CKP12]. Indeed, in this case (4) can be rewritten as follows:

$$\sup_{(x;t)\in Q} \int\limits_{(t-r^2,t+r^2)\times B_r(x)} \frac{|\mathbf{b}(y;s)|}{|s-t|^{(n+1)/2}}\cdot\exp\left(-\gamma\,\frac{|x-y|^2}{|s-t|}\right) dyds \to 0 \quad (5)$$

as $r\to 0$.

This condition differs from Definition 3.1 [CKP12] only in that the integration in [CKP12] is over $(t-r^2,t+r^2)\times\mathbb{R}^n$. However, using the covering of $\mathbb{R}^n\setminus B_r(x)$ by the balls of radius $r/3$ one can check that corresponding suprema converge to zero simultaneously.

In the whole domain Q our condition (4) is weaker then $\mathbf{b}\in\mathbf{K}_n$.

To formulate our main result we need the following notion.

Definition 3. For a point $(x;t)\in\overline{Q}$ we define its *dependence set* as the set of all points $(y;s)\in\overline{Q}$ admitting a vector-valued map $\mathfrak{F}:[0,1]\mapsto\mathbb{R}^{n+1}$ such that the last coordinate function $\mathfrak{F}_{n+1}$ is strictly increasing and

$$\mathfrak{F}(0) = (y;s); \quad \mathfrak{F}(1) = (x;t); \quad \mathfrak{F}((0,1))\subset Q.$$

If Q is a right cylinder with generatrix parallel to the t-axis, then for any $(x;t)\in\overline{Q}$ the dependence set is $\overline{Q}\cap\{s<t\}$.

Theorem 2.1. *Let Q be a bounded domain in $\mathbb{R}^{n+1}$, let $\partial''Q$ satisfy the interior parabolic $\mathcal{C}^{1,\mathcal{D}}$-paraboloid condition, let $\mathcal{M}$ be defined by (2), and let assumptions (3)-(4) be satisfied.*

In addition, assume that a function $u \in \mathcal{C}^{1,0}_{x;t}(\overline{Q})$ satisfies, in the weak sense, the inequality

$$\mathcal{M}u \geqslant 0 \qquad in \quad Q.$$

Then, if u attends its minimum at a point $(x^0;t^0) \in \partial''Q$, and u is nonconstant on the dependence set of $(x^0;t^0)$, we have

$$\frac{\partial u}{\partial \mathbf{n}}(x^0;t^0) < 0.$$

Here $\frac{\partial}{\partial \mathbf{n}}$ denotes the derivative with respect to the spatial exterior normal on $\partial''Q \cap \{t = t^0\}$.

Remark 3. Notice that we do not care of the behavior of u after t^0. Thus, without loss of generality we suppose $Q = Q \cap \{t < t_0\}$. Moreover, we may assume that $(x^0;t^0) = (0;0)$, and $\partial''Q$ is locally a paraboloid

$$x_n = \mathcal{P}(x';t) := \sqrt{|x'|^2 - t} \cdot \sigma(\sqrt{|x'|^2 - t}),$$

where $\sigma \in \mathcal{D}$ is smooth.

3 Estimates of solutions

We begin with flattening the boundary of the paraboloid by the coordinate transform

$$\tilde{x}' = x'; \quad \tilde{t} = t; \quad \tilde{x}_n = x_n - \mathcal{P}(x';t). \tag{6}$$

Proposition 3.1 ([AN19, Lemma 3.2]). *Assumptions (3) and (4) on a^{ij} and $\mathbf{b}$ remain valid under transform (6).*

Thus, we may consider $\partial''Q$ locally as a flat boundary $x_n = 0$ and assume, without loss of generality, that $Q_R \cap \mathbb{R}^{n+1}_+ \subset Q$.

Next, we take for $0 < \rho \leqslant R/2$ the cylinder $\mathcal{A}_\rho = Q_\rho(x^\rho; 0)$ (here $x^\rho = (0, \ldots, 0, \rho)$). Define the auxiliary function z as the solution of the initial-boundary value problem

$$\begin{cases} \mathcal{M}_0 z := \partial_t z - D_i(a^{ij}(x;t) D_j z) = 0 & \text{in} \quad \mathcal{A}_\rho, \\ z = 0 & \text{on} \quad \partial'' \mathcal{A}_\rho, \\ z(x; -\rho^2) = \varphi(\frac{x - x^\rho}{\rho}) & \text{for} \quad x \in B_\rho(x^\rho), \end{cases} \tag{7}$$

where φ is a smooth cut-off function such that

$$\varphi(x) = 1 \quad \text{for} \quad |x| < 1/2; \qquad \varphi(x) = 0 \quad \text{for} \quad |x| > 3/4.$$

The existence of (unique) weak solution z follows from the general parabolic theory.

Theorem 3.2. *The function z belongs to $\mathcal{C}^{1,0}_{x,t}(\overline{\mathcal{A}}_\rho)$ for sufficiently small ρ. Moreover, there exists a positive constant $\widetilde{\rho}_0 \leqslant R/2$ depending only on n, ν and σ, such that the inequality*

$$|Dz(x;t)| \leqslant \frac{C_1(n,\nu)}{\rho}, \qquad (x;t) \in \overline{\mathcal{A}}_\rho, \tag{8}$$

holds true for all $\rho \leqslant \widetilde{\rho}_0$.

Proof. We partially follow the line of the proof of [AN19, Theorem 3.3]. Let x^* be an arbitrary point in $\overline{B}_\rho(x^\rho)$. We introduce the auxiliary function ψ_{x^*} as the solution of the problem

$$\begin{cases} \mathcal{M}_0^{x^*} \psi_{x^*} = 0 & \text{in} \quad \mathcal{A}_\rho, \\ \psi_{x^*} = 0 & \text{on} \quad \partial'' \mathcal{A}_\rho, \\ \psi_{x^*}(x; -\rho^2) = \varphi(\frac{x - x^\rho}{\rho}) & \text{for} \quad x \in B_\rho(x^\rho), \end{cases} \tag{9}$$

where $\mathcal{M}_0^{x^*} := \partial_t - D_i a^{ij}(x^*; t) D_j$ is the operator with coefficients frozen at the point x^* (and thus constant in spatial variables). It is well known (see [KN09, Section 5]) that $\psi_{x^*} \in W^{2,1}_p(\mathcal{A}_\rho)$ for any $1 < p < \infty$. By homogeneity argument,

$$|D\psi_{x^*}(y;s)| \leqslant \frac{N_1(n,\nu)}{\rho}, \qquad (y;s) \in \overline{\mathcal{A}}_\rho. \tag{10}$$

Setting $w^{(1)} = z - \psi_{x^*}$ we observe that $w^{(1)}$ vanishes on $\partial'\mathcal{A}_\rho$. Hence, $w^{(1)}$ can be represented in the cylinder $\mathcal{A}_\rho$ as

$$\begin{aligned} w^{(1)}(x;t) &= \int\limits_{\mathcal{A}_\rho\cap\{s\leqslant t\}} \Gamma_\rho^{x^*}(x,y;t,s)\mathcal{M}_0^{x^*} w^{(1)}(y;s)\,dyds,\\ &\overset{(\star)}{=} \int\limits_{\mathcal{A}_\rho\cap\{s\leqslant t\}} \Gamma_\rho^{x^*}(x,y;t,s)\left(\mathcal{M}_0^{x^*} z(y) - \mathcal{M}_0 z(y)\right)\,dy, \end{aligned}$$

where $\Gamma_\rho^{x^*}$ stands for the Green function of the operator $\mathcal{M}_0^{x^*}$ in $\mathcal{A}_\rho$. The equality $(\star)$ follows from the relation $\mathcal{M}_0^{x^*}\psi_{x^*} = \mathcal{M}_0 z = 0$.

Applying integration by parts we get another version of the representation formula:

$$\begin{aligned} w^{(1)}(x;t) = \int\limits_{\mathcal{A}_\rho\cap\{s\leqslant t\}} & D_{y_i}\Gamma_\rho^{x^*}(x,y;t,s)\left(a^{ij}(x^*;s) - a^{ij}(y;s)\right)\times\\ &\times D_j z(y;s)\,dyds. \end{aligned}$$

Differentiating both sides with respect to x_k, $k = 1,\dots,n$, we get the system of equations

$$\begin{aligned} D_k z(x;t) - \int\limits_{\mathcal{A}_\rho\cap\{s\leqslant t\}} & D_{x_k}D_{y_i}\Gamma_\rho^{x^*}(x,y;t,s)\times\\ &\times\left(a^{ij}(x^*;s) - a^{ij}(y;s)\right) D_j z(y;s)\,dyds = D_k\psi_{x^*}(x;t). \end{aligned} \tag{11}$$

Now we put $x^* = x$ and get the relation

$$\left(\mathbb{I} - \mathbb{T}_1\right) Dz = \mathbf{\Psi}, \tag{12}$$

where

$$\mathbf{\Psi} = D\psi_{x^*}(x;t)\Big|_{x^*=x}$$

while $\mathbb{T}_1$ denotes the matrix integral operator whose kernel is matrix T_1 with entries

$$\begin{aligned} T_1^{kj}(x,y;t,s) = D_{x_k}D_{y_i}&\Gamma_\rho^{x^*}(x,y;t,s)\Big|_{x^*=x}\times\\ &\times\left(a^{ij}(x;s) - a^{ij}(y;s)\right)\chi_{\{s\le t\}}. \end{aligned}$$

It is easy to see that $\mathbf{\Psi} \in \mathcal{C}(\overline{\mathcal{A}}_\rho)$. Therefore, the statement of Theorem follows from the next assertion.

Lemma 3.3. *The operator* $\mathbb{T}_1$ *is bounded in* $\mathcal{C}(\overline{\mathcal{A}}_\rho)$, *and*

$$\|\mathbb{T}_1\|_{\mathcal{C}\to\mathcal{C}} \leqslant C_2\, \mathcal{J}_\sigma(2\rho),$$

where C_2 *depends only on* n *and* ν.

Proof. The following estimate for the Green function $\Gamma_\rho^{x^*}(x,y;t,s)$ is obtained similarly to [KN09, Theorem 3.6]:

$$|D_x D_y \Gamma_\rho^{x^*}(x,y;t,s)| \leqslant \frac{N_2}{(t-s)^{(n+2)/2}} \exp\Big(-N_3\,\frac{|x-y|^2}{t-s}\Big), \tag{13}$$

where N_2 and N_3 are completely determined by n and ν.

Combination of (13) with condition (3) gives for $r \leqslant 2\rho$ and any $(x;t) \in \overline{\mathcal{A}}_\rho$

$$\int\limits_{Q_r(x;t)\cap\mathcal{A}_\rho} |T_1(x,y;t,s)|\, dyds$$

$$\leqslant \int\limits_{t-r^2}^{t} \int\limits_{B_r(x)} \frac{N_2\,\sigma(|x-y|)}{(t-s)^{(n+2)/2}} \exp\Big(-N_3\,\frac{|x-y|^2}{t-s}\Big)\, dyds.$$

Change of variables $\varrho = |x-y|/\sqrt{t-s}$, $\tau = |x-y|$ gives

$$\begin{aligned} &\int\limits_{Q_r(x;t)\cap\mathcal{A}_\rho} |T_1(x,y;t,s)|\, dyds \\ &\leqslant \int\limits_0^\infty \int\limits_0^r N_4 \exp\left(-N_3\varrho^2\right)\varrho^{n-1}\frac{\sigma(\tau)}{\tau}\, d\tau d\varrho \leqslant C_2\mathcal{J}_\sigma(r) \end{aligned} \tag{14}$$

(N_4 and C_2 depend only on n and ν).

For a vector function $\mathbf{f} \in \mathcal{C}(\overline{\mathcal{A}}_\rho)$ and for all $(x;t) \in \overline{\mathcal{A}}_\rho$ we have

$$|\mathbb{T}_1\mathbf{f}(x;t)| \leqslant \|\mathbf{f}\|_{\mathcal{C}(\overline{\mathcal{A}}_\rho)} \cdot \int\limits_{\mathcal{A}_\rho} |T_1(x,y;t,s)|\, dyds \leqslant C_2\, \mathcal{J}_\sigma(2\rho) \cdot \|\mathbf{f}\|_{\mathcal{C}(\overline{\mathcal{A}}_\rho)}.$$

It remains to show that $\mathbb{T}_1\mathbf{f} \in \mathcal{C}(\overline{\mathcal{A}}_\rho)$. We take arbitrary points $(x;t), (\tilde{x};\tilde{t}) \in \overline{\mathcal{A}}_\rho$ and assume without loss of generality that $t < \tilde{t}$. Then for any small $\delta > 0$ we have

$$(\mathbb{T}_1\mathbf{f})(x;t) - (\mathbb{T}_1\mathbf{f})(\tilde{x};\tilde{t}) = J_1 + J_2$$
$$:= \Big(\int\limits_{\mathcal{A}_\rho \cap Q_\delta(\tilde{x};\tilde{t})} + \int\limits_{\mathcal{A}_\rho \setminus Q_\delta(\tilde{x};\tilde{t})} \Big) \big(T_1(x,y;t,s) - T_1(\tilde{x},y;\tilde{t},s)\big)\mathbf{f}(y;s)\,dyds.$$

If $(x;t) \in \overline{Q}_{\delta/2}(\tilde{x};\tilde{t})$ then (14) gives

$$|J_1| \leqslant \|\mathbf{f}\|_{\mathcal{C}(\overline{\mathcal{A}}_\rho)} \cdot \int\limits_{\mathcal{A}_\rho \cap Q_\delta(\tilde{x};\tilde{t})} \big(|T_1(x,y;t,s)| + |T_1(\tilde{x},y;\tilde{t},s)|\big)\,dyds$$
$$\leqslant 2C_2\,\mathcal{J}_\sigma(3\delta/2) \cdot \|\mathbf{f}\|_{\mathcal{C}(\overline{\mathcal{A}}_\rho)}.$$

Thus, given ε we can choose δ such that $|J_1| \leqslant \varepsilon$.

Next, $D_xD_y\Gamma_\rho^{x^*}(x,y;t,s)$ is continuous w.r.t. $(x;t)$ and w.r.t. x^* for $(x;t) \neq (y;s)$. Therefore, $T_1(x,y;t,s)$ is continuous w.r.t. $(x;t)$ for $(x;t) \neq (y;s)$. Thus, it is equicontinuous on the compact set

$$\{(x,y;t,s) :\ (x;t) \in \overline{Q}_{\delta/2}(\tilde{x};\tilde{t}) \cap \overline{\mathcal{A}}_\rho,\ (y;s) \in \overline{\mathcal{A}}_\rho \setminus Q_\delta(\tilde{x};\tilde{t})\}.$$

Therefore, for chosen δ we obtain, as $(x;t) \to (\tilde{x};\tilde{t})$,

$$|J_2| \leqslant \|\mathbf{f}\|_{\mathcal{C}(\overline{\mathcal{A}}_\rho)} \cdot \mathrm{meas}(\overline{\mathcal{A}}_\rho) \max_{(y;s)\in\mathcal{A}_\rho\setminus Q_\delta(\tilde{x};\tilde{t})} |T_1(x,y;t,s) - T_1(\tilde{x},y;\tilde{t},s)| \to 0,$$

and the Lemma follows. □

We continue the proof of Theorem 3.2. Choose the value of ρ_0 so small that $\mathcal{J}_\sigma(2\rho_0) \leqslant (2C_2)^{-1}$, where C_2 is the constant from Lemma 3.3. Then by the Banach theorem the operator $\mathbb{I} - \mathbb{T}_1$ in (12) is invertible. This gives $z \in \mathcal{C}^{1,0}_{x,t}(\overline{\mathcal{A}}_\rho)$. Moreover, Lemma 3.3 and inequality (10) provide (8). The proof is complete. □

For $\rho \leqslant \rho_0$ we introduce the Green function $\Gamma_{0,\rho}(x,y;t,s)$ of the operator $\mathcal{M}_0$ in the cylinder $\mathcal{A}_\rho$. The following statement is an analogue of Theorem 2.6 [CKP12] for the operators with $\mathcal{C}^{0,\mathcal{D}}_x$-coefficients.

Theorem 3.4. *The function* $D_x\Gamma_{0,\rho}(x,y;t,s)$ *is continuous for* $(x;t)\neq(y;s)$*, and the estimate*

$$|D_x\Gamma_{0,\rho}(x,y;t,s)|\leqslant N_4\min\Big\{\frac{1}{(t-s)^{(n+1)/2}};\frac{\operatorname{dist}\{y,\partial B_\rho(x^\rho)\}}{(t-s)^{(n+2)/2}}\Big\}\times \\ \times\exp\Big(-N_5\,\frac{|x-y|^2}{t-s}\Big) \tag{15}$$

holds for any $(x;t),(y;s)\in\mathcal{A}_\rho$*,* $s<t$*. Here* N_4 *and* N_5 *are the constants depending only on* n*,* ν*, and* σ*.*

Proof. Under construction. □

Further, we introduce the barrier function v defined as the weak solution of the initial-boundary value problem

$$\begin{cases} \mathcal{M}v= 0 & \text{in} \quad \mathcal{A}_\rho,\\ v= 0 & \text{on} \quad \partial''\mathcal{A}_\rho,\\ v(x;-\rho^2)= \varphi(\frac{x-x^\rho}{\rho}) & \text{for} \quad x\in B_\rho(x^\rho), \end{cases} \tag{16}$$

where φ is the same as in (7).

Theorem 3.5. *Let* $\mathbf{b}$ *satisfy the first relation in (4) with* $\gamma=N_5(n,\nu,\sigma)$ *(here* N_5 *is the constant in (15)). Then there exists a positive* $\widehat{\rho}_0\leqslant\rho_0$ *such that for all* $\rho\leqslant\widehat{\rho}_0$ *the problem (16) admits a unique solution* $v\in\mathcal{C}^{1,0}_{x,t}(\overline{\mathcal{A}}_\rho)$*. Moreover, the inequality*

$$|Dv(x;t)-Dz(x;t)|\leqslant C_3\,\frac{\omega_p^-(2\rho)}{\rho} \tag{17}$$

holds true for any $(x;t)\in\mathcal{A}_\rho$*. Here* $C_3=C_3(n,\nu,\sigma)>0$*,* $\widehat{\rho}_0$ *is completely defined by* n*,* ν*,* σ*, and* ω*, while* $z\in\mathcal{C}^{1,0}_{x,t}(\overline{\mathcal{A}}_\rho)$ *is defined in (7).*

Proof. We follow the line of proof of Theorem 3.5 in [AN19]. Consider in $\mathcal{A}_\rho$ the auxiliary function $w^{(2)}=v-z$. We observe that $w^{(2)}$ vanishes on $\partial'\mathcal{A}_\rho$, and

$$\mathcal{M}_0w^{(2)}=-b^i\left(D_iw^{(2)}+D_iz\right)\quad\text{in}\quad\mathcal{A}_\rho.$$

Hence, $w^{(2)}$ can be represented in A_ρ via the Green function $\Gamma_{0,\rho}(x,y;t,s)$ as

$$w^{(2)}(x;t) = -\int\limits_{\mathcal{A}_\rho\cap\{s\leqslant t\}} \Gamma_{0,\rho}(x,y;t,s)b^i(y;s)\times \\ \times\left(D_i w^{(2)}(y;s) + D_i z(y;s)\right)\,dyds.$$

Differentiation with respect to x_k gives

$$D_k w^{(2)}(x;t) = -\int\limits_{\mathcal{A}_\rho\cap\{s\leqslant t\}} D_{x_k}\Gamma_{0,\rho}(x,y;t,s)b^i(y;s)\times \\ \times\left(D_i w^{(2)}(y;s) + D_i z(y;s)\right)\,dyds.$$

Therefore, we get the relation

$$\left(\mathbb{I} + \mathbb{T}_2\right) Dw^{(2)} = -\mathbb{T}_2 Dz, \tag{18}$$

where $\mathbb{T}_2$ denotes the matrix operator whose (k,i) entries are integral operators with kernels $D_{x_k}\Gamma_{0,\rho}(x,y;t,s)b^i(y;s)\chi_{\{s\le t\}}$.

The statement of Theorem follows from the next assertion.

Lemma 3.6. *The operator $\mathbb{T}_2$ is bounded in $\mathcal{C}(\overline{\mathcal{A}}_\rho)$, and*

$$\|\mathbb{T}_2\|_{\mathcal{C}\to\mathcal{C}} \leqslant C_4\,\omega_p^-(2\rho),$$

where C_4 depends only on n, ν, and σ.

Proof. Recall that $\rho \le R/2$ and $Q_R \cap \mathbb{R}^{n+1}_+ \subset Q$. Thus,

$$\operatorname{dist}\{y, \partial B_\rho(x^\rho)\} \leqslant d_p(y;s)$$

for any $(y;s) \in \mathcal{A}_\rho$, and the combination of estimate (15) with the first relation in (4) gives for $r \leqslant 2\rho$ and $(x;t) \in \overline{\mathcal{A}}_\rho$

$$\int\limits_{Q_r(x;t)\cap\mathcal{A}_\rho} |D_x\Gamma_{0,\rho}(x,y;t,s)|\,|\mathbf{b}(y;s)|\,dyds \leqslant N_6(n)N_4\,\omega_p^-(r) \tag{19}$$

(here N_4 is the constant in (15)).

For arbitrary vector function $\mathbf{f} \in \mathcal{C}(\overline{\mathcal{A}}_\rho)$ and for all $(x;t) \in \overline{\mathcal{A}}_\rho$ we have

$$
\begin{aligned}
|\mathbb{T}_2\mathbf{f}(x;t)| &\leqslant \|\mathbf{f}\|_{\mathcal{C}(\overline{\mathcal{A}}_\rho)} \cdot \int\limits_{\mathcal{A}_\rho} |D_x\Gamma_{0,\rho}(x,y;t,s)|\,|\mathbf{b}(y;s)|\,dyds \\
&\leqslant N_6 N_4\,\omega_p^-(2\rho) \cdot \|\mathbf{f}\|_{\mathcal{C}(\overline{\mathcal{A}}_\rho)}.
\end{aligned}
$$

It remains to show that $\mathbb{T}_2\mathbf{f} \in \mathcal{C}(\overline{\mathcal{A}}_\rho)$. We take arbitrary points $(x;t), (\tilde{x};\tilde{t}) \in \overline{\mathcal{A}}_\rho$ and assume without loss of generality that $t < \tilde{t}$. Then for any small $\delta > 0$ we have

$$
\begin{aligned}
&(\mathbb{T}_2\mathbf{f})(x;t) - (\mathbb{T}_2\mathbf{f})(\tilde{x};\tilde{t}) = \widetilde{J}_1 + \widetilde{J}_2 \\
&:= \Big(\int\limits_{\mathcal{A}_\rho \cap Q_\delta(\tilde{x};\tilde{t})} + \int\limits_{\mathcal{A}_\rho \setminus Q_\delta(\tilde{x};\tilde{t})} \Big) \big(D_x\Gamma_{0,\rho}(x,y;t,s) - D_x\Gamma_{0,\rho}(\tilde{x},y;\tilde{t},s)\big) \times \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times [\mathbf{b}(y;s) \cdot \mathbf{f}(y;s)]\,dyds.
\end{aligned}
$$

If $(x;t) \in \overline{Q}_{\delta/2}(\tilde{x};\tilde{t})$ then (14) gives

$$
\begin{aligned}
|\widetilde{J}_1| \leqslant \|\mathbf{f}\|_{\mathcal{C}(\overline{\mathcal{A}}_\rho)} \cdot &\int\limits_{\mathcal{A}_\rho \cap Q_\delta(\tilde{x};\tilde{t})} \big(|D_x\Gamma_{0,\rho}(x,y;t,s)| + |D_x\Gamma_{0,\rho}(\tilde{x},y;\tilde{t},s)|\big) \times \\
&\times |\mathbf{b}(y;s)|\,dyds \leqslant 2N_6N_4\,\omega(3\delta/2) \cdot \|\mathbf{f}\|_{\mathcal{C}(\overline{\mathcal{A}}_\rho)}.
\end{aligned}
$$

Thus, given ε we can choose δ such that $|\widetilde{J}_1| \leqslant \varepsilon$.

On the other hand, $D_x\Gamma_{0,\rho}(x,y;t,s)$ is continuous for $(x;t) \neq (y;s)$. Thus, it is equicontinuous on the compact set

$$
\{(x,y;t,s):\ (x;t) \in \overline{Q}_{\delta/2}(\tilde{x};\tilde{t}) \cap \overline{\mathcal{A}}_\rho,\ (y;s) \in \overline{\mathcal{A}}_\rho \setminus Q_\delta(\tilde{x};\tilde{t})\}.
$$

Therefore, for chosen δ we obtain, as $(x;t) \to (\tilde{x};\tilde{t})$,

$$
\begin{aligned}
|\widetilde{J}_2| \leqslant \|\mathbf{f}\|_{\mathcal{C}(\overline{\mathcal{A}}_\rho)} \cdot &\max_{\mathcal{A}_\rho \setminus Q_\delta(\tilde{x};\tilde{t})} |D_x\Gamma_{0,\rho}(x,y;t,s) - D_x\Gamma_{0,\rho}(\tilde{x},y;\tilde{t},s)| \times \\
&\times \int\limits_{A_\rho} |\mathbf{b}(y;s)|\,dyds \to 0,
\end{aligned}
$$

and the Lemma follows. □

We continue the proof of Theorem 3.5. Choose the value of $\widehat{\rho}_0$ so small that $\omega_p^-(2\widehat{\rho}_0) \leqslant (2C_4)^{-1}$, where C_4 is the constant from Lemma 3.6. Then by the Banach theorem the operator $\mathbb{I}+\mathbb{T}_2$ in (18) is invertible. This gives the existence and uniqueness of $w^{(2)} \in \mathcal{C}^{1,0}_{x,t}(\overline{\mathcal{A}}_\rho)$, and thus, the unique solvability of the problem (16). Moreover, Lemma 3.6 and inequality (8) provide (17). The proof is complete. □

To prove Theorem 2.1 we need the following maximum principle.

Lemma 3.7. *Let $\mathcal{M}$ be defined by (2), and let assumptions (3)-(4) be satisfied in a domain $\mathcal{E} \subset \mathbb{R}^{n+1}$. Let a function $w \in \mathcal{C}^{1,0}_{x,t}(\mathcal{E})$ satisfy $\mathcal{M}w \geqslant 0$ in $\mathcal{E}$. If w attains its minimum in a point $(x^0;t^0) \in \overline{\mathcal{E}} \setminus \partial'\mathcal{E}$ then $w = const$ on the closure of the dependence set of $(x^0;t^0)$.*

Proof The Harnack inequality for parabolic divergence-type operators was established in the paper [Zha96] under the following assumptions: the principal coefficients a^{ij} are Hölder continuous, and the lower-order coefficients b^i satisfy (5) with arbitrary $\gamma > 0$ (and integration over $(t-r^2, t+r^2) \times \mathbb{R}^n$ that is inessential, see Remark 2). However, it is mentioned in [Zha96] that the assumption of the Hölder continuity of principal coefficients is needed only for the pointwise gradient estimate of the Green function for the operator $\mathcal{M}_0$, see (15).

By Theorem 3.4 this estimate holds for operators with $\mathcal{C}^{0,\mathcal{D}}_x$ coefficients. Further, in fact only (5) with a certain γ occuring in the estimate of $D\Gamma_{0,\rho}$ is used in [Zha96]. The latter coincides with the assumption $\mathbf{b} \in \mathbf{K}_n$.

Since our assumption (4) implies $\mathbf{b} \in \mathbf{K}_n$ in any strict interior subdomain of $\overline{Q} \setminus \partial' Q$ (see Remark 2), the strong maximum principle holds for the operator $\mathcal{M}$. □

Remark 4. Similarly to Lemma 3.7 in [AN19], this Lemma is the only point where we need the second relation in (4). If we could prove at least weak maximum principle for the operator $\mathcal{M}$ using only the quantity ω_p^-, we did not need ω_p^+ at all. Unfortunately, we cannot do it, and the question whether the second relation in (4) is necessary for the Boundary Point Principle remains open.

Proof of Theorem 2.1.

We begin with the function ψ_0 and recall that $\psi_0 \in W_p^{2,1}(\mathcal{A}_\rho)$ for any $1 < p < \infty$. Therefore, the equation (9) can be rewritten in the non-divergence form:

$$\begin{cases} \partial_t\psi_0 - a^{ij}(0;t)D_iD_j\psi_0 = 0 & \text{in} \quad \mathcal{A}_\rho, \\ \psi_0 = 0 & \text{on} \quad \partial''\mathcal{A}_\rho, \\ \psi_0(x;-\rho^2) = \varphi(\frac{x-x^\rho}{\rho}) & \text{for} \quad x \in B_\rho(x^\rho). \end{cases}$$

So, it is well known[1] (see [Výb57] and [Fri58]) that the boundary point principle holds for ψ_0. By rescaling $\mathcal{A}_\rho$ into $\mathcal{A}_1$ we get the estimate

$$D_n\psi_0(0;0) \geqslant \frac{N_7(n,\nu)}{\rho}.$$

Next, the relation (11), Lemma 3.3, and inequality (8) imply for sufficiently small ρ

$$\begin{aligned} D_nz(0;0) &\geqslant D_n\psi_0(0;0) - \|\mathbb{T}_1 Dz\|_{\mathcal{C}(\overline{\mathcal{A}}_\rho)} \\ &\geqslant \frac{N_7}{\rho} - C_1C_2\frac{\mathcal{J}_\sigma(2\rho)}{\rho} \geqslant \frac{N_7}{2\rho}. \end{aligned}$$

The relation (17) gives for sufficiently small ρ

$$\begin{aligned} D_nv(0;0) &\geqslant D_nz(0;0) - |Dv(0;0) - Dz(0;0)| \\ &\geqslant \frac{N_7}{2\rho} - C_3\,\frac{\omega_p^-(2\rho)}{\rho} \geqslant \frac{N_7}{4\rho}. \end{aligned}$$

We fix such a ρ. Since u is nonconstant on the dependence set of $(0;0)$, Lemma 3.7 ensures

$$u(x;-\rho^2) - u(0;0) > 0 \qquad \text{for} \quad x \in B_{3\rho/4}(x^\rho).$$

Therefore, we have for sufficiently small ε

$$\mathcal{M}(u - u(0;0) - \varepsilon v) \geqslant 0 \quad \text{in} \ \mathcal{A}_\rho; \qquad u - u(0;0) - \varepsilon v \geqslant 0 \quad \text{on} \ \partial'\mathcal{A}_\rho.$$

[1] In [Výb57] and [Fri58], classical solutions were considered; however, the proof works also for strong solutions belonging to $W_{n+1}^{2,1}(\mathcal{A}_\rho)$ by the Aleksandrov–Bakelman–Krylov maximum principle ([Kry76], see also [Naz05]).

By Lemma 3.7 the estimate $u - u(0;0) \geqslant \varepsilon v$ holds true in $\mathcal{A}_\rho$, with equality at the origin. This gives

$$\frac{\partial u}{\partial \mathbf{n}}(0;0) = -D_n u(0;0) \leqslant -\varepsilon D_n v(0;0),$$

which completes the proof. □

Acknowledgments

The research of the authors is supported by the Russian Science Foundation grant 22-21-00027 fulfilled at St. Petersburg Department of the Steklov Mathematical Institute.

Список литературы

[AN16] D.E. Apushkinskaya and A.I. Nazarov, *A counterexample to the Hopf-Oleinik lemma (elliptic case)*, Anal. PDE **9** (2016), no. 2, 439–458.

[AN19] ______, *On the Boundary Point Principle for divergence-type equations*, Rendiconti Lincei – Matematica E Applicazioni **30** (2019), no. 4, 677–699.

[CKP12] S. Cho, P. Kim, and H. Park, *Two-sided estimates on Dirichlet heat kernels for time-dependent parabolic operators with singular drifts in $C^{1,\alpha}$-domains*, J. Differential Equations **252** (2012), no. 2, 1101–1145.

[Fri58] A. Friedman, *Remarks on the maximum principle for parabolic equations and its applications*, Pacific J. Math. **8** (1958), 201–211. MR 102655

[KN09] V. Kozlov and A. Nazarov, *The Dirichlet problem for non-divergence parabolic equations with discontinuous in time coefficients*, Math. Nachr. **282** (2009), no. 9, 1220–1241.

[Kry76] N.V. Krylov, *Sequences of convex functions, and estimates of the maximum of the solution of a parabolic equation*, Sibirsk. Mat. Ž. **17** (1976), no. 2, 290–303, [Russian]; English transl. in *Sib. Math. J.*, 17:226–236, 1976.

[Naz05] A.I. Nazarov, *The maximum principle of A.D. Aleksandrov*, Sovrem. Mat. Prilozh. **29** (2005), 127–143, [Russian]; English transl. in: *J. Math. Sci. (N. Y.)*, 142(3):2154–2171, 2007.

[Výb57] R. Výborný, *The properties of the solutions of certain boundary problems for parabolic equations*, Dokl. Akad. Nauk SSSR **117** (1957), 563–565, [Russian].

[Zha96] Q. Zhang, *A Harnack inequality for the equation* $\nabla(a\nabla u) + b\nabla u = 0$*, when* $|b| \in K_{n+1}$, Manuscripta Math. **89** (1996), no. 1, 61–77.

Darya E. Apushkinskaya
Peoples' Friendship University of Russia (RUDN University)
6 Miklukho-Maklaya St, Moscow, 117198, Russia
and
St. Petersburg Department of Steklov Institute
Fontanka 27, St. Petersburg 191023, Russia
apushkinskaya@gmail.com

Alexander I. Nazarov
St. Petersburg Department of Steklov Institute
Fontanka 27, St. Petersburg 191023, Russia
and
St. Petersburg State University
Universitetskii pr. 28, St. Petersburg 198504, Russia
al.il.nazarov@gmail.com