# Zero cycles on a Cayley plane

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**Abstract**

Let $k$ be a field of characteristic 0. Let $G$ be an exceptional simple algebraic group over $k$ of type $\mathrm{F}_4$, ${}^1\mathrm{E}_6$ or $\mathrm{E}_7$ with trivial Tits algebras. Let $X$ be a projective $G$-homogeneous variety. If $G$ is of type $\mathrm{E}_7$ we assume in addition that the respective parabolic subgroup is of type $P_7$. The main result of the paper says that the degree map on the group of zero cycles of $X$ is injective.

## 1 Introduction

Let $k$ be a field and $G$ a simple algebraic group over $k$. Consider a projective $G$-homogeneous variety $X$ over $k$. Any such variety over the separable closure $k_s$ of $k$ becomes isomorphic to the quotient $G_s/P$, where $P$ is a parabolic subgroup of the split group $G_s = G \times_k k_s$. It is known that the conjugacy classes of parabolic subgroups of $G_s$ are in one-to-one correspondence with the subsets of the vertices $\Pi$ of the Dynkin diagram of $G_s$: we say a parabolic subgroup is of type $\theta \subset \Pi$ and denote it by $P_\theta$ if it is conjugate to a standard parabolic subgroup generated by the Borel subgroup and all unipotent subgroups corresponding to roots in the span of $\Pi$ with no $\theta$ terms (see [TW02, 42.3.1]).

In the present paper we assume $k$ has characteristic 0, $G$ is an exceptional simple algebraic group over $k$ of type $\mathrm{F}_4$, ${}^1\mathrm{E}_6$ or $\mathrm{E}_7$ with trivial Tits algebras and $X$ is a projective $G$-homogeneous variety over $k$. The goal of the paper is to compute the group of zero-cycles $\mathrm{CH}_0(X)$ which is an important geometric invariant of a variety. Namely, we prove

**Theorem.** *Let $k$ be a field of characteristic 0. Let $G$ be an exceptional simple algebraic group over $k$ of type $\mathrm{F}_4$, ${}^1\mathrm{E}_6$ or $\mathrm{E}_7$ with trivial Tits algebras and $X$ a projective $G$-homogeneous variety over $k$. If $G$ is of type $\mathrm{E}_7$ we assume in* *addition that $X$ corresponds to the parabolic subgroup of type $P_7$. Then the degree map* $\mathrm{CH}_0(X) \to \mathbb{Z}$ *is injective.*

The history of the question starts with the work of I. Panin [Pa84], where he proved the injectivity of the degree map for Severi-Brauer varieties. For quadrics this was proved by R. Swan in [Sw89]. The case of involution varieties was considered by A. Merkurjev in [Me95]. For varieties of type $\mathrm{F}_4$ it was announced by M. Rost.

Our work was mostly motivated by the paper of D. Krashen [Kr05], where he reformulated the question above in terms of $R$-triviality of certain symmetric powers and proved the injectivity for a wide class of generalized Severi-Brauer varieties and some involutive varieties, hence, generalizing the previously known results of Panin and Merkurjev. Another motivating point was the result of V. Popov [Po05], giving a full classification of generically $n$-transitive actions of a split linear algebraic group $G$ on a projective homogeneous variety $G/P$. For instance, the case of a Cayley plane $X = G/P_1$, where $G$ is split of type $\mathrm{E}_6$ (see [IM05]), provides an example of such an action for $n = 3$. As a consequence, one can identify the open orbit $(S^3X)^0$ of the induced action on the third symmetric power with a homogeneous variety $G/(T \cdot \mathrm{Spin}_8)$, where $T$ is the torus which is complementary to $\mathrm{Spin}_8$. Then the results of Krashen reduce the question of injectivity to the question of $R$-triviality of the twisted form of $(S^3X)^0/S_3$.

Apart from the main result concerning exceptional varieties we give shortened proofs for injectivity of the degree map for quadrics and Severi-Brauer varieties as well.

Recently V. Chernousov and A. Merkurjev obtained independent proof of the same results in any characteristic using Rost Invariant and Chain Lemma (paper in preparation). Our proof doesn't use these tools but only the geometry and some basic facts about projective homogeneous varieties.

The paper is organized as follows. In the first section we provide several facts about zero-cycles and symmetric powers. Then we prove the theorem for twisted forms of a Cayley plane (here prime 3 plays the crucial role). In the next section we prove injectivity for the twisted form of a homogeneous variety of type $\mathrm{E}_7$ (this deals with prime 2). In the last section, we combine these two results together with some facts about rational correspondences and finish the proof of the main theorem.

## 2 Zero cycles and symmetric powers

In this section we collect some facts from [Kr05] on the interrelation between zero cycles on a projective variety $X$ and classes of $R$-equivalence on symmetric powers of $X$.

**2.1.** We systematically use Galois descent language, i.e., identify a (quasi-projective) variety $X$ over $k$ with the variety $X_s = X \times_k k_s$ over the separable closure $k_s$ equipped with an action (by semiautomorphisms of $X_s$) of the absolute Galois group $\Gamma = \mathrm{Gal}(k_s/k)$. More explicitly, $\Gamma$ acts on $X_s$ through the second factor, and therefore every element $\sigma \in \Gamma$ defines some automorphism $\varphi_\sigma$ of $X_s$ over $k$. Moreover, the diagram

$$\begin{tikzcd} X_s \arrow[r, "\varphi_\sigma"] \arrow[d] & X_s \arrow[d] \\ \operatorname{Spec} k_s \arrow[r, "\sigma^\sharp"] & \operatorname{Spec} k_s \end{tikzcd}$$

is commutative, and $\varphi_{\sigma\tau} = \varphi_\sigma \varphi_\tau$. Conversely, a quasi-projective variety $X'$ over $k_s$ equipped with automorphisms $\varphi_\sigma$ of $X'$ over $k$ satisfying these conditions defines a variety $X$ over $k$ such that $X' \simeq X_s$ and all the $\varphi_\sigma$ come from the action of $\Gamma$ through the second factor.

The set of $k$-rational points of $X$ is precisely the set of $k_s$-rational points of $X_s$ stable under the action of $\Gamma$.

**2.2.** Let $X$ be a variety over $k$. Two rational points $p, q \in X(k)$ are called *elementary linked* if there exists a rational morphism $\varphi\colon \mathbb{P}^1_k \dashrightarrow X$ such that $p, q \in \mathrm{Im}(\varphi(k))$. The $R$-equivalence is the equivalence relation generated by this relation. A variety $X$ is called *$R$-trivial* if there exists exactly one class of $R$-equivalence on $X$, and *algebraically $R$-trivial* if $X_K = X \times_k K$ is $R$-trivial for any finite field extension $K/k$.

The $n$-th symmetric power of $X$ is by the definition the variety $S^n X = X^n/S_n$, where $S_n$ is the symmetric group acting on $X^n$ via permutations.

Let $p$ be a prime number. A field $k$ is called *prime-to-$p$ closed* if there is no proper, finite field extension $K/k$ of degree prime to $p$. For any field $k$ we denote by $k_p$ its *prime-to-$p$ closure* that is the algebraic extension of $k$ which is prime-to-$p$ closed.

Let $X$ be a projective variety over $k$. By $\widetilde{\mathrm{CH}}_0(X)$ we denote the kernel of the degree map:

$$\widetilde{\mathrm{CH}}_0(X) = \mathrm{Ker}(\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}).$$

The following results will play a crucial role in the sequel:

**2.3 Lemma.** *([Kr05, Lemma 1.3]) If* $\widetilde{\mathrm{CH}}_0(X_{k_p}) = 0$ *for each prime* $p$ *then* $\widetilde{\mathrm{CH}}_0(X) = 0$.

**2.4 Proposition.** *([Kr05, Theorem 1.4]) Suppose that* $k$ *is prime-to-*$p$ *closed and the following conditions are satisfied:*

1. $S^{p^n}X$ *is algebraically* $R$*-trivial for some* $n \geq 0$,
2. *For any field* $K/k$ *such that* $X(K) \neq \emptyset$ *the variety* $X_K$ *is* $R$*-trivial.*

*Then* $\widetilde{\mathrm{CH}}_0(X) = 0$.

**2.5.** As an easy application of these results we sketch a proof of the well-known result (first appearing in [Pa84, Theorem 2.3.7]) that $\widetilde{\mathrm{CH}}_0(\mathrm{SB}(A)) = 0$, where $A$ is a central simple algebra over $k$ (a more general case of flag varieties is settled in [Kr05]). For simplicity we assume $\deg A = p$ is prime.

By Lemma 2.3 we may assume the base field $k$ is prime-to-$p$ closed (for a prime $q$ different from $p$ the algebra $A$ splits over $k_q$). According to Proposition 2.4 it suffices to show that $S^p\,\mathrm{SB}(A_K)$ is $R$-trivial for every finite extension $K/k$ (the second hypothesis of 2.4 holds for any twisted flag variety). Changing the base we may assume $K = k$. If $A$ is split, the assertion is trivial; so we may assume $A$ is not split.

According to our conventions (see 2.1) the variety $\mathrm{SB}(A)$ is the variety of all parabolic subgroups $P$ of type $P_1$ in the group $\mathrm{PGL}_1(A \otimes_k k_s)$ with the action of $\Gamma$ coming from its action on $k_s$. Therefore, $S^pX$ is the variety of all unordered $p$-tuples $[P^{(1)}, \ldots, P^{(p)}]$ of parabolic subgroups of type $P_1$ of $\mathrm{PGL}_1(A \otimes_k k_s)$. Let $U$ be an open subset of $S^pX$ defined by the condition that the intersection $P^{(1)} \cap \ldots \cap P^{(p)}$ is a maximal torus in $\mathrm{PGL}_1(A \otimes_k k_s)$. Every maximal torus $T$ in $\mathrm{PGL}_1(A \otimes_k k_s)$ is contained in precisely $p$ parabolic subgroups of type $P_1$, whose intersection is $T$. Therefore, $U$ is isomorphic to the variety of all maximal tori in $\mathrm{PGL}_1(A)$. This variety is known to be rational (and therefore $R$-trivial since it is homogeneous). Moreover, one can check that if $A$ is not split then the embedding $U \to S^pX$ is surjective on $k$-points. So $S^pX$ is $R$-trivial, and we are done.

**2.6.** The same method can be applied to prove that $\widetilde{\mathrm{CH}}_0(Q) = 0$ for a nonsingular projective quadric $Q$ over a field of characteristic 0 (the result of Swan [Sw89]).

As above, we may assume that $p=2$ and $Q$ is anisotropic. It suffices to prove that $S^2Q$ is $R$-trivial. Let $q$ be the corresponding quadratic form on a vector space $V$. The quadric $Q$ can be viewed as the variety of lines $\langle v\rangle$, where $v \in V\otimes_k k_s$ satisfies $q(v)=0$, with the obvious action of $\Gamma$. So $S^2Q$ can be identified with the variety of pairs $[\langle v_1\rangle,\langle v_2\rangle]$ of lines of this kind, with induced action of $\Gamma$. Consider the open subset $U$ defined by the condition $b_q(v_1,v_2)\neq 0$ ($b_q$ stands for the polarization of $q$). Clearly, the embedding $U\to S^2Q$ is surjective on $k$-points (otherwise the subspace $\langle v_1,v_2\rangle$ defines a totally isotropic subspace over $k$). So it is enough to check that $U$ is $R$-trivial.

Consider the open subvariety $W$ of $\mathrm{Gr}(2,V)$ consisting of planes $H\subset V\otimes_k k_s$ such that $q|_H$ is nonsingular. For every such a plane there exists up to scalar factors exactly one hyperbolic basis $\{v_1,v_2\}$ over $k_s$. Therefore, the map from $U$ to $W$ sending $[\langle v_1\rangle,\langle v_2\rangle]$ to $\langle v_1,v_2\rangle$ is an isomorphism. But any open subvariety of $\mathrm{Gr}(2,V)$ is $R$-trivial, and we are done.

We shall use in the sequel the following observation.

**2.7 Lemma.** *Let $H\subset K\subset G$ be algebraic groups over $k$. Suppose that the map $\mathrm{H}^1(k,H)\to\mathrm{H}^1(k,K)$ is surjective. Then the morphism $G/H\to G/K$ is surjective on $k$-points.*

*Proof.* An element $x$ of $G/K(k)$ is presented by an element $g\in G(k_s)$ satisfying the condition $\gamma(\sigma)=g^{-1}\cdot{}^{\sigma}g$ lies in $K(k_s)$ for all $\sigma\in\Gamma$. But $\gamma$ is clearly a 1-cocycle with coefficients in $K$. Therefore by the assumption, there exists some $h\in K$ such that $h^{-1}\gamma(\sigma)\cdot{}^{\sigma}h=(gh)^{-1}\cdot{}^{\sigma}(gh)$ is a 1-cocycle with coefficients in $H$. But then $gh$ presents an element of $G/H(k)$ which goes to $x$ under the morphism $G/H\to G/K$, and the lemma is proved. $\square$

# 3 Twisted forms of a Cayley plane

In the present section we prove the injectivity of the degree map in the case when $X$ is a twisted form of a Cayley plane.

**3.1.** Let $J$ denote a simple exceptional 27-dimensional Jordan algebra over $k$, and $N_J$ its norm (which is a cubic form on $J$). An invertible linear map $f\colon J\to J$ is called a *similitude* if there exists some $\alpha\in k^*$ (called the *multiplier* of $f$) such that $N_J(f(v))=\alpha N_J(v)$ for all $v\in J$. The group $G=\mathrm{Sim}(J)$ of all similitudes is a reductive group whose semisimple part has type ${}^1\mathrm{E}_6$, and every group of type ${}^1\mathrm{E}_6$ with trivial Tits algebras can be obtained in this way up to isogeny (see [Ga01, Theorem 1.4]).

**3.2.** The *(twisted) Cayley plane* $\mathbb{OP}^2(J)$ is the variety of all parabolic subgroups of type $P_1$ in $\mathrm{Sim}(J)$. Since the definition is stable under a base extension, $\mathbb{OP}^2(J)$ can be obtained by Galois descent from $\mathbb{OP}^2(J_s)$ where $J_s = J \otimes_k k_s$. The last variety can be identified with the variety of all lines $\langle e \rangle$ spanned by elements $e \in J_s = J \otimes_k k_s$ satisfying the condition $e \times e = 0$ (see [Ga01, Theorem 7.2]).

The goal of the present section is to prove

**3.3 Theorem.** $\widetilde{\mathrm{CH}}_0(\mathbb{OP}^2(J)) = 0$.

We start the proof with the following easy reduction.

**3.4.** By Proposition 2.4 it is enough to prove that $(S^p\mathbb{OP}^2(J)) \times_k K$ is $R$-trivial for any prime $p$ and any finite field extension $K/k_p$. Making the base change, it suffices to prove it for $K = k_p$. Moreover, we may assume $J$ is not reduced (otherwise $\mathbb{OP}^2(J)$ is a rational homogeneous variety and, hence, is $R$-trivial).

Assume $p \neq 3$, then $\mathbb{OP}^2(J)(k_p) \neq \emptyset$ (and, hence, is $R$-trivial). Indeed, choose any cubic étale subalgebra $L$ of $J$ (see [Inv, Proposition 39.20]). It splits over $k_p$ and, therefore, $L \otimes_k k_p$ contains a primitive idempotent $e$. As an element of $J \otimes_k k_p$ it satisfies the condition $e \times e = 0$ (see [SV, Lemma 5.2.1(i)]). So we may assume $p = 3$.

**3.5.** From now on $p = 3$ and the field $k$ is prime-to-$p$ closed. By definition $S^3(\mathbb{OP}^2(J))$ is the variety of all unordered triples $[\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle]$, where $e_i$ are the elements of $J_s = J \otimes_k k_s$ satisfying the conditions $e_i \times e_i = 0$, with the natural action of $\Gamma$. Denote by $U$ the open subvariety of $\mathbb{OP}^2(J)$ defined by the condition

$$N_{J_s}(e_1, e_2, e_3) \neq 0,$$

where $N$ is the polarization of the norm.

The embedding $U \to S^3(\mathbb{OP}^2(J))$ is surjective on $k$-points. Indeed, if $[\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle]$ is stable under the action of $\Gamma$ and $N_{J_s}(e_1, e_2, e_3) = 0$, then $\langle e_1, e_2, e_3 \rangle$ gives by descent a $k$-defined subspace $V$ of $J$ such that $N|_V = 0$. But then $J$ is reduced by [SV, Theorem 5.5.2], which leads to a contradiction. So it is enough to show that $U$ is $R$-trivial.

**3.6.** Choose a cubic étale subalgebra $L$ in $J$. Over the separable closure we have

$$L \otimes_k k_s = k_s e_1 \oplus k_s e_2 \oplus k_s e_3$$

where $e_1, e_2, e_3$ are primitive idempotents. Then we have $e_i \times e_i = 0$ in $J_s$, $i = 1, 2, 3$, the norm $N_{J_s}(e_1, e_2, e_3) = N_{L\otimes_k k_s}(e_1, e_2, e_3)$ is non-trivial and the triple $[e_1, e_2, e_3]$ is stable under the action of $\Gamma$ (since so is $L$). Hence, the triple $[\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle]$ is a $k$-rational point of $U$.

By [SV68, Proposition 3.12] the group $G$ acts transitively on $U$. Therefore, we have

$$U \simeq G/\operatorname{Stab}_G([\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle]).$$

The stabilizer is defined over $k$, since it is stable under $\Gamma$. Moreover, it coincides with $\operatorname{Stab}_G(L)$. Indeed, one inclusion is obvious, and the other one follows from the fact that $e_1, e_2, e_3$ are the only elements $e$ of $L\otimes_k k_s$ satisfying the condition $e \times e = 0$, up to scalar factors (see [SV, Theorem 5.5.1]).

**3.7.** Consider the Springer decomposition $J = L \oplus V$ of $J$ with respect to $L$. The pair $(L, V)$ has a natural structure of a twisted composition, and there is a monomorphism $\operatorname{Aut}(L, V) \to \operatorname{Aut}(J)$ sending a pair $(\varphi, t)$ (where $\varphi\colon L \to L$, $t\colon V \to V$) to $\varphi \oplus t\colon J \to J$ (see [Inv, § 38.A]). Note that $\operatorname{Aut}(L, V)$ coincides with the stabilizer of $L$ in $\operatorname{Aut}(J)$.

**3.8 Lemma.** *The following sequence of algebraic groups is exact*

$$1 \longrightarrow \operatorname{Aut}(L, V) \longrightarrow \operatorname{Stab}_G(L) \longrightarrow R_{L/k}(\mathbb{G}_m) \longrightarrow 1,$$
$$f \mapsto f(1)$$

*where $R_{L/k}$ stands for the Weil restriction.*

*Proof.* Exactness at the middle term follows from 3.7 and the fact that the stabilizer of 1 in $G$ coincides with $\operatorname{Aut}(J)$ (see [SV, Proposition 5.9.4]). To prove the exactness at the last term observe that a $k_s$-point of $R_{L/k}(\mathbb{G}_m)$ is a triple of scalars $(\alpha_0, \alpha_1, \alpha_2) \in k_s^* \times k_s^* \times k_s^*$. We have to find $f$ from $\operatorname{Stab}_G(L)(k_s)$ sending 1 to $\operatorname{diag}(\alpha_0, \alpha_1, \alpha_2)$.

Assume first that $\alpha_0\alpha_1\alpha_2 = 1$. Choose a *related triple* $(t_0, t_1, t_2)$ of elements of $\mathrm{GO}^+(\mathbb{O}_d, N_{\mathbb{O}_d})$ ($\mathbb{O}_d$ is the split Cayley algebra) such that $\mu(t_i) = \alpha_i$, $i = 0, 1, 2$ (see [Inv, Corollary 35.5]). Now the transformation $f$ of $J$ defined by

$$\begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} \mapsto \begin{pmatrix} \alpha_0\varepsilon_0 & t_2(c_2) & \cdot \\ \cdot & \alpha_1\varepsilon_1 & t_0(c_0) \\ t_1(c_1) & \cdot & \alpha_2\varepsilon_2 \end{pmatrix}$$

lies in $\operatorname{Sim}(J)$ by [Ga01, (7.3)], stabilizes $L \otimes_k k_s = \operatorname{diag}(k_s, k_s, k_s)$ and sends $1 \in J_s$ to $\operatorname{diag}(\alpha_0, \alpha_1, \alpha_2)$.

In general case set $\alpha_i' = \alpha_i(\alpha_0\alpha_1\alpha_2)^{-\frac{1}{3}}$ $(i = 1, 2, 3)$, find $f'$ such that $f'(1) = \operatorname{diag}(\alpha_1', \alpha_2', \alpha_3')$, and define $f$ to be the product of $f'$ and the scalar transformation of $J_s$ with the coefficient $(\alpha_0\alpha_1\alpha_2)^{\frac{1}{3}}$ (which is an element of $\operatorname{Stab}_G(L)(k_s)$). $\square$

**3.9.** Since $\mathrm{H}^1(k, L^*) = 1$ (by Hilbert '90), the map $\mathrm{H}^1(k, \operatorname{Aut}(L, V)) \to \mathrm{H}^1(k, \operatorname{Stab}_G(L))$ is surjective. By Lemma 2.7 the morphism

$$G/\operatorname{Aut}(L, V) \to G/\operatorname{Stab}_G(L) \simeq U$$

is surjective on $k$-points. Therefore, it suffices to show that $G/\operatorname{Aut}(L, V)$ is $R$-trivial.

**3.10.** Consider the morphism

$$\psi\colon G/\operatorname{Aut}(L, V) \to G/\operatorname{Aut}(J).$$

By [Kr05, Corollary 3.14] it suffices to show that

1. $\psi$ is surjective on $k$-points;
2. $G/\operatorname{Aut}(J)$ is $R$-trivial;
3. The fibers of $\psi$ (which are isomorphic to $\operatorname{Aut}(J)/\operatorname{Aut}(L, V)$) are unirational and $R$-trivial.

**3.11.** In order to prove surjectivity of $\psi$ on $k$-points it is enough by Lemma 2.7 to prove surjectivity of the map $\mathrm{H}^1(k, \operatorname{Aut}(L, V)) \to \mathrm{H}^1(k, \operatorname{Aut}(J))$. The set $\mathrm{H}^1(k, \operatorname{Aut}(L, V))$ classifies all twisted compositions $(L', V')$ which become isomorphic to $(L, V)$ over $k_s$ and $\mathrm{H}^1(k, \operatorname{Aut}(J))$ classifies all (exceptional 27-dimensional) Jordan algebras $J'$. It is easy to verify that the morphism sends $(L', V')$ to the Jordan algebra $L' \oplus V'$ and, hence, the surjectivity follows from the fact that any Jordan algebra admits a Springer decomposition (cf. [Inv, Proposition 38.7]).

**3.12.** Let $W$ be the open subvariety of $J$ consisting of elements $v$ with $N_J(v) \neq 0$. Then $G$ acts transitively on $W$ (see [SV, Proposition 5.9.3]) and the stabilizer of the point 1 coincides with $\operatorname{Aut}(J)$. So $G/\operatorname{Aut}(J) \simeq W$ is clearly $R$-trivial.

**3.13.** Consider the variety $Y$ of all étale cubic subalgebras of $J$. By [Inv, Proposition 39.20(1)] there is a map from an open subvariety $J_0$ of regular elements in $J$ to $Y$ (sending $a$ to $k[a]$), surjective on $k$-points. Therefore $Y$ is unirational and $R$-trivial 24-dimensional irreducible variety.

The group $\mathrm{Aut}(J)$ acts on $Y$ naturally. Let $L'$ be any $k$-point of $Y$. The stabilizer of $L'$ in $\mathrm{Aut}(J)$ obviously equals to $\mathrm{Aut}(L', V')$ ($J = L' \oplus V'$ is the Springer decomposition). So the orbit of $L'$ is isomorphic to $\mathrm{Aut}(J)/\mathrm{Aut}(L', V')$ and, in particular, has dimension 24. Therefore, it is open and, since $L'$ is arbitrary, the action is transitive. So we have $\mathrm{Aut}(J)/\mathrm{Aut}(L, V) \simeq Y$ is unirational and $R$-trivial and the proof of 3.3 is completed.

# 4 Case $\mathrm{E}_7/P_7$

In the present Section we prove the injectivity of the degree map for twisted forms of a projective homogeneous variety corresponding to an exceptional group of type $\mathrm{E}_7$ and a parabolic subgroup of type $P_7$.

**4.1.** Let $\mathcal{B}$ denote a 56-dimensional Brown algebra over $k$. It defines naturally up to a scalar factor a skew-symmetric form $b$ on $\mathcal{B}$ and a trilinear map $t$ from $\mathcal{B} \times \mathcal{B} \times \mathcal{B}$ to $\mathcal{B}$ such that $(\mathcal{B}, t, b)$ is a Freudenthal triple system (see [Ga01, Definition 3.1] and [Ga01, § 4]). An invertible linear map $f : \mathcal{B} \to \mathcal{B}$ is called a *similitude* if there exists some $\alpha \in k^*$ (called the *multiplier* of $f$) such that $b(f(u), f(v)) = \alpha b(u, v)$ and $t(f(u), f(v), f(w)) = \alpha f(t(u, v, w))$ for all $u, v, w \in \mathcal{B}$. The group $G = \mathrm{Sim}(\mathcal{B})$ of all similitudes is a reductive group whose semisimple part has type $\mathrm{E}_7$ and every group of type $\mathrm{E}_7$ with trivial Tits algebras can be obtained in this way up to isogeny (cf. [Ga01, Theorem 4.16]).

An element $e$ is called *singular* (or *strictly regular* following [Fe72]) if $t(e, e, \mathcal{B}) \subseteq \langle e \rangle$. In this case $t(e, e, v) = 2b(v, e)e$ for every $v \in V$. An equivalent definition is that $t(e, e, e) = 0$ and $e \in t(e, e, \mathcal{B})$ (see [Fe72, Lemma 3.1]). $\mathcal{B}$ is called *reduced* if it contains singular elements. There do exist anisotropic groups of type $\mathrm{E}_7$ with trivial Tits algebras over certain fields (see [Ti90]).

**4.2.** Let $X(\mathcal{B})$ be the variety obtained by the Galois descent from the variety of all parabolic subgroups of type $P_7$ in $\mathrm{Sim}(\mathcal{B} \otimes_k k_s)$ (the action of $\Gamma = \mathrm{Gal}(k_s/k)$ comes from the action on $k_s$). This variety can be identified with the variety of all lines $\langle e\rangle$ spanned by singular elements $e \in \mathcal{B} \otimes_k k_s$ (see [Ga01, Theorem 7.6]).

The goal of this section is to prove

**4.3 Theorem.** $\widetilde{\mathrm{CH}}_0(X(\mathcal{B})) = 0$.

We start with the similar reduction as in the case of $\mathrm{E}_6$.

**4.4.** Assume first that $G$ has Tits index $\mathrm{E}^{66}_{7,1}$ (see [Ti66, Table II]). Its anisotropic kernel is of type $\mathrm{D}_6$ and, since $G$ has trivial Tits algebras, the anisotropic kernel corresponds to a 12-dimensional nondegenerate quadratic form $q$ with split simple factors of its Clifford algebra. A straightforward computation (see [Br05, Thm. 7.4]) shows that

$$\mathcal{M}(X(\mathcal{B})) \simeq \mathcal{M}(Q) \oplus \mathcal{M}(Y)(6) \oplus \mathcal{M}(Q)(17),$$

where $Q$ is the projective quadric corresponding to $q$, $Y$ is a twisted form of the maximal orthogonal grassmanian of a split 12-dimensional quadric, and $\mathcal{M}$ denotes Chow motive. Therefore, $\widetilde{\mathrm{CH}}_0(X(\mathcal{B})) = \widetilde{\mathrm{CH}}_0(Q) = 0$, the last equality due to Swan.

**4.5.** By Proposition 2.4 it is enough to prove that $(S^pX(\mathcal{B}))\times_k K$ is $R$-trivial for any prime $p$ and any finite field extension $K/k_p$. Making the base change, it suffices to prove it for $K = k_p$. Moreover, we may assume $\mathcal{B}$ is not reduced (otherwise $X(\mathcal{B})$ is rational and, hence, $R$-trivial).

Assume $p \neq 2$, then $\mathcal{B} \otimes k_p$ is reduced by [Fe72, Corollary 3.4] and, therefore, $X(\mathcal{B})(k_p) \neq \emptyset$. So we may assume $p = 2$.

From now on $p = 2$ and $k = k_p$. Since $\mathcal{B}$ is not reduced, the group $G$ has Tits index either $\mathrm{E}^{133}_{7,0}$ or $\mathrm{E}^{66}_{7,1}$ (see [Ti71, 6.5.5] and [Ti66, Table II]). By 4.4 we may assume $G$ is anisotropic (has index $\mathrm{E}^{133}_{7,0}$).

**4.6.** By definition $S^2(X(\mathcal{B}))$ is the variety of all unordered pairs $[\langle e_1\rangle, \langle e_2\rangle]$, where $e_i$ are singular elements of $\mathcal{B}\otimes_k k_s$, with the natural action of $\Gamma$. Denote by $U$ the open subvariety of $X(\mathcal{B})$ defined by the condition $b(e_1, e_2) \neq 0$.

**4.7 Lemma.** *The embedding $U \to S^2(X(\mathcal{B}))$ is surjective on $k$-points.*

*Proof.* Consider the diagonal action of $G$ on $X(\mathcal{B}) \times X(\mathcal{B})$ (we may assume in this proof that $G$ is simple). Over $k_s$ this action has four orbits: the minimal orbit which is the diagonal and, hence, is isomorphic to $G_s/P_7$, the open dense orbit which is isomorphic to the quotient $G_s/L(P_7)$, where $L(P_7)$ denotes the Levi part of $P_7$, and two locally closed orbits. Indeed, there is a one-to-one correspondence between the orbits of the $G_s$-action and double coset classes $P_7\backslash G_s/P_7$ given by mutually inverse maps $G_s\cdot(x,y)\mapsto P_7x^{-1}yP_7$ and $P_7wP_7\mapsto G_s\cdot(1,w)$. Observe that the minimal orbit corresponds to the class of the identity and the open dense orbit to the class of the longest element $w_0$ of the Weyl group of $G_s$.

Consider the diagonal action of $G$ on $S^2(X(\mathcal{B}))$. Over $k_s$ the subset $U$ is the open dense orbit in $S^2(X(\mathcal{B}))$ (see 4.8). Assume that there exists a $k$-rational point on $S^2(X(\mathcal{B}))\setminus U$. Then the stabilizer $H$ of this point is a subgroup of $G$ defined over $k$. Observe that over $k_s$ the connected component of the identity $H^0$ is the stabilizer of one of the non-open orbits for the action of $G$ on $X(\mathcal{B})\times X(\mathcal{B})$ considered above, i.e., can be identified with the intersection of two parabolic subgroups $H^0_s=P_7\cap wP_7w^{-1}$, where $w$ is the double coset representative corresponding to the orbit. By [DG, Exposé XXVI, Theorem 4.3.2] $H^0_s$ is reductive iff $H^0_s$ is the Levi subgroup of $P_7$, i.e., iff $P_7wP_7=P_7w_0P_7$. Therefore, $H^0_s$ is non-reductive and so is $H$. The latter implies that $G$ must have a unipotent element over $k$. But according to [Ti86, p. 265], if $G$ is anisotropic and $\operatorname{char} k\neq 2,3$, then this is impossible, a contradiction. □

According to the lemma it suffices to show that $U$ is $R$-trivial.

**4.8.** The Brown algebra $\mathcal{B}\otimes_k k_s$ is split, that is isomorphic to the Brown algebra of matrices of the form

$$\begin{pmatrix} F & J_d \\ J_d & F \end{pmatrix},$$

where $J_d$ is the split Jordan algebra. Set

$$e_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\quad e_2=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The pair $[\langle e_1\rangle,\langle e_2\rangle]$ is stable under an arbitrary semiautomorphism of $\mathcal{B}\otimes_k k_s$ (see [Ga01, Proof of Theorem 2.9]) and, in particular, under the action of $\Gamma$. Therefore, $[\langle e_1\rangle,\langle e_2\rangle]$ is a $k$-rational point of $U$. Moreover, $\langle e_1,e_2\rangle$ defines by descent the ($k$-defined) étale quadratic subalgebra $L$ of $\mathcal{B}$.

By [Fe72, Proposition 7.6] $G$ acts transitively on $U$. Therefore,

$$U\simeq G/\operatorname{Stab}_G([\langle e_1\rangle,\langle e_2\rangle]).$$

This stabilizer clearly coincides with $\mathrm{Stab}_G(L)$ (one inclusion is obvious, and the other one follows from the fact that $e_1, e_2$ are the only singular elements of $L \otimes_k k_s$ up to scalar factors).

**4.9 Lemma.** *There is an exact sequence of algebraic groups*

$$1 \longrightarrow \mathrm{Aut}(\mathcal{B}) \longrightarrow \mathrm{Stab}_G(L) \longrightarrow R_{L/k}(\mathbb{G}_m) \longrightarrow 1.$$
$$f \mapsto f(1)$$

*Proof.* This follows from the fact that the stabilizer of 1 in $G$ coincides with $\mathrm{Aut}(\mathcal{B})$. Indeed, we have an obvious injection $\mathrm{Aut}(\mathcal{B}) \to \mathrm{Stab}_G(1)$. To prove the surjectivity we can assume that $k$ is separably closed. Let $f$ be an element of $G$ preserving 1. Since a decomposition into a sum of two nonorthogonal singular elements is unique by [Fe72, Lemma 3.6] and $1 = e_1 + e_2$, the element $f$ must preserve the pair $[e_1, e_2]$. By [Fe72, Lemma 7.5] $f$ has a form $\eta_\pi^\lambda$, where $\eta$ is a similitude of $J$ with a multiplier $\rho$, $\pi$ is a permutation on $\{1, 2\}$, $\lambda \in k^*$, and $\eta_\pi^\lambda$ acts on $\mathcal{B}$ by formulae [Fe72, (15)]. Now it follows that $\lambda^{-1}\rho^{-1} = 1$ and $\lambda^2\rho = 1$ and therefore $\lambda = \rho = 1$. So $f$ is an automorphism of $\mathcal{B}$, as claimed.

The surjectivity of the last map also follows from [Fe72, Lemma 7.5]. □

**4.10.** Since $\mathrm{H}^1(k, L^*) = 1$, the map $\mathrm{H}^1(k, \mathrm{Aut}(\mathcal{B})) \to \mathrm{H}^1(k, \mathrm{Stab}_G(L))$ is surjective. By Lemma 2.7 the morphism

$$G/\mathrm{Aut}(\mathcal{B}) \to G/\mathrm{Stab}_G(L) \simeq U$$

is surjective on $k$-points. Therefore, it suffices to show that $G/\mathrm{Aut}(\mathcal{B})$ is $R$-trivial.

**4.11.** Let $W$ be the open subvariety of $\mathcal{B}$ consisting of elements $v$ such that $b(v, t(v, v, v)) \neq 0$. Then $G$ acts transitively on $W$ (it follows easily from [Fe72, Theorem 7.10] or [SK77, p. 140]) and the stabilizer of the point 1 coincides with $\mathrm{Aut}(\mathcal{B})$. So $G/\mathrm{Aut}(\mathcal{B}) \simeq W$ is clearly $R$-trivial, and we finished the proof of 4.3.

# 5 Other homogeneous varieties

In this section using the results of [Me] and [Ti66] we finish the proof of the theorem of the introduction. We start with the following

**5.1 Lemma.** *Let $X$ and $Y$ denote projective homogeneous varieties over a field $k$. Assume $X$ is isotropic over the function field of $Y$ and $Y$ is isotropic over the function field of $X$. Then the groups of zero-cycles of $X$ and $Y$ are isomorphic.*

*Proof.* The fact that $X$ is isotropic over $k(Y)$ is equivalent to the existence of a rational map $Y \dashrightarrow X$. Hence, we have two composable rational maps $f\colon Y \dashrightarrow X$ and $g\colon X \dashrightarrow Y$, and the compositions $f \circ g$ and $g \circ f$ correspond to taking a $k(X)$-point on $X$ and a $k(Y)$-point on $Y$ respectively.

Consider the category of rational correspondences $RatCor(k)$ introduced in [Me]. The objects of this category are smooth projective varieties over $k$ and morphisms $\mathrm{Mor}(X,Y) = \mathrm{CH}_0(Y_{k(X)})$. The key property of this category is that the $\mathrm{CH}_0$-functor factors through it. Namely, $\mathrm{CH}_0$ is a composition of two functors: the first is given by taking a graph of a rational map (any rational map gives rise to a morphism in $RatCor(k)$), the second is the realization functor (see [Me, Thm. 3.2]).

The maps $f$ and $g$ give rise to the morphisms $[f]$ and $[g]$ in $RatCor(k)$. By definition the compositions $[f \circ g]$ and $[g \circ f]$ give the identity maps in the category $RatCor(k)$. Hence, the realizations $[f]_*$ and $[g]_*$ give the respective mutually inverse isomorphisms between $\mathrm{CH}_0(X)$ and $\mathrm{CH}_0(Y)$. □

The next lemma finishes the proof of the theorem of introduction.

**5.2 Lemma.** *Let $X$ be an anisotropic projective $G$-homogeneous variety, where $G$ is a group of type $\mathrm{F}_4$ or ${}^1\mathrm{E}_6$ with trivial Tits algebras. Then $\widetilde{\mathrm{CH}}_0(X) = 0$.*

*Proof.* According to Lemma 2.3 it is enough to prove the lemma over fields $k_p$, where $p = 2$ or $3$.

Assume $p = 2$ and $k = k_2$. Let $G$ be a group of type ${}^1\mathrm{E}_6$. Consider a Jordan algebra $J$ corresponding to the group $G$. Since the base field $k$ is prime-to-2 closed, the algebra $J$ is reduced ([Inv, Theorem 40.8]) and, hence, comes from an octonion algebra $\mathbb{O}$. Consider the variety $Y$ of norm zero elements of $\mathbb{O}$ (which is an anisotropic Pfister quadric). Since $G$ has trivial Tits algebras, there are only two Tits diagrams allowed for $G$ and its scalar extensions, namely, ${}^1\mathrm{E}_{6,6}^{0}$ and ${}^1\mathrm{E}_{6,2}^{28}$ (see [Ti71, 6.4.5]). Since $X$ is anisotropic (by the hypothesis), extending the scalars to $k(X)$ adds additional circles to the respective Tits diagram and, hence, changes it. Therefore, $G_{k(X)}$ (equivalently $J_{k(X)}$) must be split. The fact that $G_{k(Y)}$ (equiv. $J_{k(Y)}$) is split is obvious (see [Inv, Corollary 37.18]). All this means that the varieties $X_{k(Y)}$ and $Y_{k(X)}$ are isotropic. By Lemma 5.1 we obtain $\widetilde{\mathrm{CH}}_0(X) = \widetilde{\mathrm{CH}}_0(Y) = 0$, where the last equality holds by [Sw89].

In the case $G$ is a group of type $\mathrm{F}_4$ there are three possible Tits diagrams, namely, $\mathrm{F}_{4,0}^{52}$ (anisotropic), $\mathrm{F}_{4,1}^{21}$ (isotropic) and $\mathrm{F}_{4,4}^{0}$ (split case). Consider the first case, i.e., $G$ is an anisotropic group of type $\mathrm{F}_4$ which splits by a quadratic field extension. Let $Z$ be the Pfister form corresponding to the invariant $f_5$. We claim that $X_{k(Z)}$ and $Z_{k(X)}$ are isotropic. Obviously, $k(X)$ splits $Z$. The invariants $g_3$ and $f_5$ are trivial for the respective Jordan algebra $J_{k(Z)}$. By [PR94, p. 205] this implies that the group $G_{k(Z)}$ is isotropic, i.e., corresponds to the diagram $\mathrm{F}_{4,1}^{21}$. Then, the variety $X_{k(Z)}$ is isotropic as well. Again by Lemma 5.1 and [Sw89] we conclude that $\widetilde{\mathrm{CH}}_0(X) = \widetilde{\mathrm{CH}}_0(Z) = 0$.

Hence, we may assume there are only two Tits diagrams for $G$, namely, $\mathrm{F}_{4,1}^{21}$ and $\mathrm{F}_{4,4}^{0}$. Following the same arguments as for the group of type ${}^1\mathrm{E}_6$ we prove that $\widetilde{\mathrm{CH}}_0(X) = 0$.

Assume $p = 3$ and $k = k_3$. In this case there are two Tits diagrams allowed for $G$, namely, ${}^1\mathrm{E}_{6,0}^{78}$ and ${}^1\mathrm{E}_{6,6}^{0}$ (resp. $\mathrm{F}_{4,0}^{52}$ and $\mathrm{F}_{4,4}^{0}$). Consider the pair $X$ and $Y = \mathbb{OP}^2(J)$. Again the obvious arguments with Tits diagrams show that $X_{k(Y)}$ and $Y_{k(X)}$ are isotropic. We obtain $\widetilde{\mathrm{CH}}_0(X) = \widetilde{\mathrm{CH}}_0(Y) = 0$, where the last equality holds by Theorem 3.3. $\square$

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