GROTHENDIECK—SERRE CONJECTURE FOR ADJOINT GROUPS OF TYPES E_6 AND E_7

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Abstract. Assume that R is a semi-local regular ring containing an infinite perfect field, or that R is a semi-local ring of several points on a smooth scheme over an infinite field. Let K be the field of fractions of R. Let H be a strongly inner adjoint simple algebraic group of type E_6 or E_7 over R. We prove that the kernel of the map

$$\mathrm{H}^1_{\acute{e}t}(R,H) \to \mathrm{H}^1_{\acute{e}t}(K,H)$$

induced by the inclusion of R into K is trivial. This continues the recent series of papers [PaSV], [Pa] on the Grothendieck—Serre conjecture [Gr, Rem. 1.11].

In what follows we use the notation and terminology of [PS]. Our numbering of vertices of Dynkin diagrams follows [B].

Lemma 1. *Let R be a semi-local domain, and let H be a strongly inner adjoint simple group scheme of type E_6 (resp., E_7) over R. There exists an inner* adjoint *simple group scheme G of type E_7 (resp., E_8) over R, together with a maximal parabolic subgroup P of type $\{1,2,3,4,5,6\}$ (resp., $\{1,2,3,4,5,6,7\}$), such that H is isomorphic to the quotient $L/\operatorname{Cent}(L)$ for a Levi subgroup L of P. In this setting $\operatorname{Cent}(L)$ is isomorphic to $\mathbf{G}_{\mathrm{m}}$.*

Proof. Let H^{sc} be a simply-connected algebraic group over R of the same type as H, and let H^{sc}_0 be the corresponding split group. Let G^{sc}_0 be a split simply-connected algebraic group over R of type E_7 (respectively, E_8) if H is of type E_6 (respectively, E_7). Let P_0 be a standard maximal parabolic subgroup of G^{sc}_0 corresponding to the 7th (respectively, the 8th) vertex of the Dynkin diagram of G^{sc}_0. Then H^{sc}_0 is isomorphic to the derived subgroup of a standard Levi subgroup L_0 of P_0. By [PS, Th. 2 (2)] for the strongly inner form H^{sc} of H^{sc}_0 there exist an inner form $\tilde{G}$ of G^{sc}_0, a parabolic subgroup $\tilde{P}$ of $\tilde{G}$ of the same type as P_0 in G^{sc}_0, and a Levi subgroup $\tilde{L}$ of P, such that H^{sc} is isomorphic to the derived subgroup of $\tilde{L}$. Now set $G = \tilde{G}^{ad} = \tilde{G}/\operatorname{Cent}(\tilde{G})$, and let P, L be the images of $\tilde{P}$, $\tilde{L}$ in G. Note that the image of H^{sc} in G is isomorphic to H^{sc}, for H^{sc}_0 meets the center of G^{sc}_0 trivially. The fact that L is an inner form of the corresponding split group and [PS, Prop. 1 (2)] imply that the center of L is isomorphic to $\mathbf{G}_{\mathrm{m}}$, and that $L/\operatorname{Cent}(L)$ is isomorphic to $H = H^{ad}$. □

Theorem 1. *Let R be a semi-local domain. Assume moreover that R is regular and contains a infinite perfect field k, or that R is a semi-local ring of several points on a k-smooth scheme over an infinite field k. Let K be the field of fractions of R. Let H be an adjoint strongly inner simple group scheme of type E_6 or E_7 over R. Then the map*

$$\mathrm{H}^1_{\acute{e}t}(R,H) \to \mathrm{H}^1_{\acute{e}t}(K,H)$$

induced by the inclusion of R into K has trivial kernel.

Proof. Take G, P and L as in Lemma 1. By [SGA, Exp. XXVI Cor. 5.10 (i)] the map

$$\mathrm{H}^1_{\acute{e}t}(R,L) \to \mathrm{H}^1_{\acute{e}t}(K,G)$$

is injective. According to [Pa, Th. 1.0.1, Th. 1.0.2] the map

$$\mathrm{H}^1_{\acute{e}t}(R,G) \to \mathrm{H}^1_{\acute{e}t}(K,G)$$

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has trivial kernel, hence the map

$$\mathrm{H}^1_{\acute{e}t}(R, L) \to \mathrm{H}^1_{\acute{e}t}(K, L)$$

has trivial kernel as well. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1_{\acute{e}t}(R, \mathbf{G}_\mathrm{m}) \arrow[r] \arrow[d] & \mathrm{H}^1_{\acute{e}t}(R, L) \arrow[r] \arrow[d] & \mathrm{H}^1_{\acute{e}t}(R, H) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^2_{\acute{e}t}(R, \mathbf{G}_\mathrm{m}) \arrow[d] \\
\mathrm{H}^1_{\acute{e}t}(K, \mathbf{G}_\mathrm{m}) \arrow[r] & \mathrm{H}^1_{\acute{e}t}(K, L) \arrow[r] & \mathrm{H}^1_{\acute{e}t}(K, H) \arrow[r, "\delta"] & \mathrm{H}^2_{\acute{e}t}(K, \mathbf{G}_\mathrm{m}),
\end{tikzcd} \tag{1}$$

induced by the short exact sequence

$$1 \to \mathbf{G}_\mathrm{m} \to L \to H \to 1.$$

Note that the connecting map δ passes through $\mathrm{Br}(R)$ and $\mathrm{Br}(K)$ respectively. Indeed, let $V = U/[U, U]$, where U is the unipotent radical of the parabolic subgroup P. We may consider V as a finitely generated projective R-module (cf. [SGA, Exp.XXVI Cor. 2.5]). The natural representation $\rho\colon L \to \mathrm{GL}(V)$ maps $\mathrm{Cent}(L) \simeq \mathbf{G}_\mathrm{m}$ isomorphically onto $\mathrm{Cent}(\mathrm{GL}(V))$, since this is so for the corresponding split groups. Then the diagram

$$\begin{tikzcd}
\mathrm{H}^1_{\acute{e}t}(R, L) \arrow[r] \arrow[d] & \mathrm{H}^1_{\acute{e}t}(R, H) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^2_{\acute{e}t}(R, \mathbf{G}_\mathrm{m}) \arrow[d, equal] \\
\mathrm{H}^1_{\acute{e}t}(R, \mathrm{GL}(V)) \arrow[r] & \mathrm{H}^1_{\acute{e}t}(R, \mathrm{PGL}(V)) \arrow[r] & \mathrm{H}^2_{\acute{e}t}(R, \mathbf{G}_\mathrm{m})
\end{tikzcd}$$

commutes, and hence δ passes through $\mathrm{Br}(R)$. By [Gr, Corollaire 1.10] the map $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is injective. Now the assertion of the Theorem follows from the diagram (1) by an easy diagram chasing. □

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