TITS INDICES OVER SEMILOCAL RINGS

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1. Introduction

In his famous paper [Ti66] Jacques Tits showed that any semisimple group G over a field is determined by its anisotropic kernel and a combinatorial datum called the *Tits index* of G. Some arguments were sketched or omitted there, and appeared only in [Ti71, Ti90]. The goal of the present paper is to show that the Tits classification carries over to arbitrary connected semilocal rings. We do not rely on the field case, but rather provide a shortened and simplified version of Tits' arguments.

Our proof consists of two parts: combinatorial and representation-theoretic. Combinatorial restrictions follow from the presence of the opposition involution on the extended Dynkin diagram. It is worth mentioning that already these restrictions allow to exclude most of the "wrong" indices (Proposition 3). Representation-theoretic arguments rely on the fact that, given a good enough representation of an algebraic group over a splitting field, one can twist it to a representation into some Azumaya algebra, called a *Tits algebra* (cf. [Ti71]), over the base field. We show that this holds over arbitrary schemes (Theorem 1). In Theorem 2 we give a necessary and sufficient condition in terms of Tits algebras that a semisimple group scheme H can be embedded into a larger semisimple group G as the derived subgroup of a Levi subgroup of a parabolic subgroup of G. Combining this result with the combinatorial restrictions, we obtain the list of all possible indices, and show that the existence of a group with a given index is equivalent to the existence of an anisotropic group (its anisotropic kernel) subject to certain explicitly stated restrictions (§ 6, Theorem 3).

2. Semisimple group schemes

In this section we reproduce some definitions and results of [SGA]. Throughout the paper, all references starting with Exp. YZ refer to this source.

Let S be a scheme (not necessarily separated). A group scheme G over S is called *reductive* if it is affine and smooth over S, and its geometric fibers $G_{\overline{k(s)}}$ are connected reductive groups in the usual sense for all $s \in S$ (Exp. XIX Déf. 4.7).When S is reduced, the smoothness can be replaced by the condition that G is finitely presented over S and the dimension of a fiber is locally constant (see Exp. $\mathrm{VI_B}$, Cor. 4.4). The *type* of G at $s \in S$ is the root datum of $G_{\overline{k(s)}}$. The type is locally constant (Exp. XXII Prop. 2.8). To simplify the exposition, in the sequel we consider reductive group schemes of constant type only. Thus the type of a reductive group scheme G is a root datum $\mathcal{R} = (\Phi, \Lambda, \Phi^*, \Lambda^*)$, where Φ is a root system, called the *root system* of G, Λ is a $\mathbb{Z}$-lattice containing Φ, called the *lattice of weights* of G, and Φ^* and Λ^* are the dual objects (Exp. XXI Déf. 1.1.1). A reductive group G is *semisimple*, if the rank of Φ equals that of Λ. We also usually include in the type a fixed subset of positive roots Φ^+ in Φ, which determines a system of simple roots of Φ and, therefore, a Dynkin diagram D.

Over any scheme S there exists a unique *split* group scheme G_0 of a given type $\mathcal{R}$, which actually comes from a group scheme over $\operatorname{Spec}\mathbb{Z}$ known as the Chevalley – Demazure group scheme (Exp. XXV Thm. 1.1). *Quasi-split* group schemes over S of the same type as G_0 are parametrized by $\mathrm{H}^1(S,\, \mathrm{Aut}\,(\mathcal{R},\, \Phi^+))$, where $\mathrm{Aut}\,(\mathcal{R},\, \Phi^+)$ is the group of automorphisms of $\mathcal{R}$ preserving Φ^+ (cf. Exp. XXIV Thm. 3.11). All cohomology groups we consider are with respect to the fpqc topology (but note that $\mathrm{H}^1(S,\, H) = \mathrm{H}^1_{\text{ét}}(S,\, H)$ when H is smooth).

Every semisimple group scheme G is an inner twisted form of a uniquely determined quasi-split group G_{qs}, given by a cocycle $\xi \in \mathrm{Z}^1(S,\, G_{qs}^{ad})$, where G_{qs}^{ad} is the adjoint group acting on G_{qs} by

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inner automorphisms; cocycles in the same class in $\mathrm{H}^1(S,\, G_{qs}^{ad})$ produce isomorphic group schemes (Exp. XXIV 3.12.1).

A Dynkin diagram D is nothing but a finite set of vertices together with a subset $E \subseteq D \times D$ of edges and a length function $D \to \{1,\, 2,\, 3\}$ (in other words, a colored graph). The scheme-theoretic counterpart of this notion is called a *Dynkin scheme* (Exp. XXIV § 3). So a Dynkin scheme over S is a twisted finite scheme $\mathcal{D}$ over S together with a subscheme $\mathcal{E} \subseteq \mathcal{D} \times_S \mathcal{D}$ and a map $\mathcal{D} \to \{1,\, 2,\, 3\}_S$. Isomorphisms, base extensions and constant Dynkin schemes are defined in an obvious way. We denote by D_S the constant Dynkin scheme over S corresponding to a Dynkin diagram D. By $\mathrm{Aut}\,(\mathcal{D})$ we always mean the scheme of automorphisms of $\mathcal{D}$ over S as a Dynkin scheme; it is a twisted constant group scheme over S.

To any semisimple group scheme G one associates the Dynkin scheme $\mathrm{Dyn}(G)$ in such a way that $\mathrm{Dyn}(G)$ is isomorphic to $\mathrm{Dyn}(G_{qs})$; and $\mathrm{Dyn}(G_{qs})$ is a twisted form of D_S corresponding to the image in $\mathrm{Z}^1(S,\, \mathrm{Aut}\,(D))$ of a cocycle $\xi \in \mathrm{Z}^1(S,\, \mathrm{Aut}\,(\mathcal{R},\, \Phi^+))$ defining G_{qs} under the map induced by the canonical map $\mathrm{Aut}\,(\mathcal{R},\, \Phi^+) \to \mathrm{Aut}\,(D)$ (Exp. XXIV 3.7). When G_{qs} is simply connected or adjoint, the latter map is an isomorphism.

Let T/S be a Galois covering that splits $\mathrm{Dyn}(G)$, i.e. $\mathrm{Dyn}(G)_T \simeq D_T$. For example, one can take as T the torsor corresponding to the cocycle in $\mathrm{Z}^1(S,\, \mathrm{Aut}\,(D))$. Every element $\sigma \in \mathrm{Aut}\,(T/S)$ acts on $\mathrm{Dyn}(G)_T$ and therefore defines some $\varphi_\sigma \in \mathrm{Aut}\,(D)(T)$ such that the diagram

$$\begin{tikzcd} D_T \arrow[r, "\varphi_\sigma"] \arrow[d] & D_T \arrow[d] \\ T \arrow[r, "\sigma"] & T \end{tikzcd}$$

commutes. By Galois descent this action (which is called *the $*$-action*) completely determines $\mathrm{Dyn}(G)$. If S is connected, the $*$-action can be considered as an action of $\mathrm{Aut}\,(T/S)$ on the Dynkin diagram D, and extends by $\mathbb{Q}$-linearity to the $*$-action on Λ.

A subgroup scheme P of G is called *parabolic* if it is smooth and $P_{\overline{k(s)}}$ is a parabolic subgroup of $G_{\overline{k(s)}}$ in the usual sense for every $s \in S$ (Exp. XXVI Déf. 1.1). To a parabolic subgroup P one can attach the *type* $\mathfrak{t}(P)$ of P which is a clopen subscheme of $\mathrm{Dyn}(G)$ (Exp. XXVI 3.2). Note that the clopen subschemes of $\mathrm{Dyn}(G)$ are in one-to-one correspondence with the $*$-invariant clopen subschemes of D_T, where T/S is as above.

3. Representation-theoretic lemmas

By a *representation* of a group scheme G over S we mean a homomorphism of algebraic groups $\rho\colon G \to \mathrm{GL}_1(A)$, where A is an Azumaya algebra (more formally, a sheaf of Azumaya algebras) over S.

Let G_0 be a split semisimple group scheme over a scheme S, and let $G_0 \to \mathrm{GL}(V)$ be a representation of G_0 on a projective module (more formally, a locally free sheaf of modules) V of finite rank over S. Fix a maximal split torus T_0 of G_0 and let Λ and Λ_{r} be its lattices of weights and roots respectively. Then V decomposes into a direct sum $\bigoplus_{\lambda \in \Lambda} V_\lambda$ so that for any scheme S' over S, any $t \in T_0(S')$, and any $v \in V_\lambda(S')$ one has $\rho(t)v = \lambda(t)v$ (Exp. I Prop. 4.7.3). A character λ with $V_\lambda \neq 0$ is called a *weight* of V.

The *cocenter* $\mathrm{Cocent}(G)$ of G is the group scheme $\mathrm{Hom}\,(\mathrm{Cent}(G),\, \mathbf{G}_{\mathrm{m}})$. When G is split it can be identified with the constant group scheme $(\Lambda \,/\, \Lambda_{\mathrm{r}})_S$. If G is given by a cocycle $\xi \in \mathrm{Z}^1(S,\, G_{qs}^{ad})$ then $\mathrm{Cent}(G)$ is isomorphic to $\mathrm{Cent}(G_{qs})$, and therefore $\mathrm{Cocent}(G)$ is isomorphic to $\mathrm{Cocent}(G_{qs})$. Of course, the isomorphism depends on the choice of ξ (or rather on the corresponding element in $\mathrm{Isomext}\,(G_{qs},\, G)(S)$, see Exp. XXIV Rem. 1.11).

A representation $\rho\colon G \to \mathrm{GL}_1(A)$ will be called *center preserving* if $\rho(\mathrm{Cent}(G)) \subseteq \mathrm{Cent}(\mathrm{GL}_1(A))$. In this case ρ induces a homomorphism $\rho^{ad}\colon G^{ad} \to \mathrm{PGL}_1(A)$ and determines an element $\lambda_\rho \in \mathrm{Cocent}(G)(S)$, which is the restriction of ρ to $\mathrm{Cent}(G)$ composed with the natural isomorphism $\mathrm{Cent}(\mathrm{GL}_1(A)) \simeq \mathbf{G}_{\mathrm{m}}$.

Lemma 1. (1) *$G \to \mathrm{GL}(V)$ is center preserving if and only if over a splitting covering $\coprod S_\tau \to S$ every two weights of V differ by an element of Λ_{r}.*

(2) *The dual* $G \to \mathrm{GL}(V^*)$ *of a center preserving representation* $G \to \mathrm{GL}(V)$ *is center preserving.*
(3) *The tensor product* $G \to \mathrm{GL}(V_1 \otimes V_2)$ *of center preserving representations* $G \to \mathrm{GL}(V_1)$ *and* $G \to \mathrm{GL}(V_2)$ *is center preserving.*
(4) *For any representation* $\rho\colon G \to \mathrm{GL}(V)$ *and an element* $\lambda \in \mathrm{Cocent}(G)(S)$*, the submodule* $W \subseteq V$ *defined by*

$$W(S') = \{v \in V \times_S S' \mid c \cdot v = \lambda(c)v \text{ for all fpqc } S''/S' \text{ and } c \in \mathrm{Cent}(G)(S'')\}$$

is a G*-invariant direct summand of* V*. Moreover, the representation* $\rho'\colon G \to \mathrm{GL}(W)$ *is center preserving and* $\lambda_{\rho'} = \lambda$ *if* $W \neq 0$.

Proof. For (1) observe that since the condition $\rho(\mathrm{Cent}(G)) \subseteq \mathrm{Cent}(\mathrm{GL}(V))$ is local with respect to fpqc topology, we can assume that G is split. Then V is center preserving if and only if restrictions of every two weights λ and μ of V to $\mathrm{Cent}(G)$ coincide. This means exactly that $\lambda - \mu$ belongs to Λ_{r} (Exp. XXII Rem. 4.1.8).

Parts (2) and (3) follow from (1).

To prove (4), define $W'(S')$ as the set of all $v \in V(S')$ such that there exist an fpqc covering $\coprod S'_\tau \to S'$ and, for each τ, a finite number of elements $\lambda_1, \ldots, \lambda_k \in \mathrm{Cocent}(G)(S'_\tau)$ distinct from λ and elements $v_1, \ldots, v_k \in V \times_S S'_\tau$ such that $v = v_1 + \ldots + v_k$ and $cv_i = \lambda_i(c)v_i$ for all fpqc S''_τ/S'_τ and $c \in \mathrm{Cent}(G)(S''_\tau)$. Obviously W and W' are G-invariant (sheaves of) submodules of V. Over a splitting covering of G it is easily seen that $V = W \oplus W'$; therefore it is also true over the base S. By construction the representation $\rho'\colon G_{qs} \to W$ is center preserving and $\lambda_{\rho'} = \lambda$. □

Lemma 2. *Let* G_{qs} *be a quasi-split group over* S*. Then any element of* $\mathrm{Cocent}(G_{qs})(S)$ *appears as* λ_ρ *for some center preserving representation* $\rho\colon G_{qs} \to \mathrm{GL}(V)$.

Proof. Over a splitting covering of G_{qs} choose a weight $\lambda \in \Lambda$ that represents a given element of $\mathrm{Cocent}(G_{qs})(S)$. Obviously $\lambda + \Lambda_{\mathrm{r}}$ is $*$-invariant. It is known (see [B, Ch. VI, Exerc. 5 du §2]) that any weight is equivalent modulo Λ_{r} to a minuscule weight. On the other hand, by [Ti71, 3.1] we have $(\Lambda / \Lambda_{\mathrm{r}})^* = \Lambda^* / {\Lambda_{\mathrm{r}}}^*$. So we may assume that λ is a $*$-invariant minuscule weight.

Consider first the split group G_0 over $\mathbb{Z}$. Recall briefly the construction of a Weyl module $V(\lambda)$ for G_0 (see [Jan] for details). We start from a finite dimensional irreducible $(G_0)_{\mathbb{C}}$-module with the highest weight λ; we fix a vector v_+ of the weight λ (which is unique up to a scalar). Denote by $\mathfrak{U}$ the universal enveloping algebra of the Lie algebra of $(G_0)_{\mathbb{C}}$, by $\mathfrak{U}^+$ and $\mathfrak{U}^-$ its subalgebras generated by the positive (respectively, negative) root subspaces, and by $\mathfrak{U}_{\mathbb{Z}}$, $\mathfrak{U}^+_{\mathbb{Z}}$, $\mathfrak{U}^-_{\mathbb{Z}}$ their $\mathbb{Z}$-forms used in the Chevalley's construction of split reductive groups. Then $V(\lambda)$ is defined as $\mathfrak{U}^-_{\mathbb{Z}}\, v_+$. Note that $V(\lambda)$ is center preserving by Lemma 1, (1).

Let Γ be a subgroup of $\mathrm{Aut}\,(\mathcal{R}, \Phi^+)$ preserving λ. Then any element $\gamma \in \Gamma$ induces an automorphism of $\mathfrak{U}_{\mathbb{Z}}$ which preserves $\mathfrak{U}^+_{\mathbb{Z}}$ and $\mathfrak{U}^-_{\mathbb{Z}}$. Since γ preserves λ, the representations $\rho\colon (G_0)_{\mathbb{C}} \to \mathrm{GL}(V(\lambda)_{\mathbb{C}})$ and $\rho \circ \gamma\colon (G_0)_{\mathbb{C}} \to \mathrm{GL}(V(\lambda)_{\mathbb{C}})$ are equivalent, and their differentials are equivalent as well. Therefore, there exists $\varphi \in \mathrm{GL}(V(\lambda)_{\mathbb{C}})$ such that $\gamma(g)\varphi(v) = \varphi(gv)$ for every $v \in V(\lambda)_{\mathbb{C}}$ and $g \in \mathfrak{U}$; moreover, φ is unique up to a scalar. It is easy to see that φ preserves the line spanned by v_+, and we can normalize φ so that $\varphi(v_+) = v_+$. Now,

$$\varphi(\mathfrak{U}^-_{\mathbb{Z}}\, v_+) \leq \gamma(\mathfrak{U}^-_{\mathbb{Z}})\varphi(v_+) = \mathfrak{U}^-_{\mathbb{Z}}\, v_+,$$

so φ induces an automorphism $\varphi_{\mathbb{Z}}$ of $V(\lambda)$ compatible with γ and preserving v_+. Since $\mathbb{Z}[G_0]$ is a Hopf subalgebra of $\mathbb{Q}[G_0]$ and $V(\lambda)$ is a subcomodule of $V(\lambda)_{\mathbb{Q}}$, and $\mathbb{C}/\mathbb{Q}$ is faithfully flat, $\varphi_{\mathbb{Z}}$ is an equivalence of the representations $\rho\colon G_0 \to \mathrm{GL}(V(\lambda))$ and $\rho \circ \gamma\colon G_0 \to \mathrm{GL}(V(\lambda))$. Moreover, since $\varphi_{\mathbb{Z}}$ is uniquely determined by γ, we obtain a homomorphism $\psi\colon \Gamma \to \mathrm{GL}(V(\lambda))$.

Now let ξ be a cocycle in $\mathrm{Z}^1(S, \Gamma)$ producing G_{qs}. The cocycle $\psi_*(\xi)$ then defines a projective module V together with a representation $G_{qs} \to \mathrm{GL}(V)$ we need. □

4. Tits algebras

Theorem 1. *Let* G *be a semisimple group scheme of constant type over* S *given by a cocycle* $\xi \in \mathrm{Z}^1(S, G^{ad}_{qs})$.

(1) *There exist two natural mutually quasi-inverse equivalences* F_ξ, F'_ξ *between the categories of group schemes over* S *with* G^{ad}_{qs}*-action (by group automorphisms) and group schemes over* S *with* G^{ad}*-action. They depend only on the class of* ξ *in* $\mathrm{Isomext}\,(G_{qs},\,G)(S)$ *(cf. Exp. XXIV Rem. 1.11). In particular, each center preserving representation* $\rho\colon G_{qs}\to \mathrm{GL}(V)$ *gives rise to a center preserving representation* $F_\xi(\rho)\colon G\to \mathrm{GL}_1(A_{\xi,\,\rho})$ *for some Azumaya algebra* $A_{\xi,\,\rho}$.

(2) *The class* $[A_{\xi,\,\rho}]$ *in the Brauer group* $\mathrm{Br}(S)$ *depends only on* $\lambda_{F_\xi(\rho)}$*, and not on the particular choices of* ρ *and* ξ*. Its image in* $\mathrm{H}^2(S,\,\mathbf{G}_{\mathrm{m}})$ *coincides with* $(\lambda_\rho)_*\delta([\xi])$*, where*

$$(\lambda_\rho)_*\colon\ \mathrm{H}^2(S,\,\mathrm{Cent}(G_{qs}))\to \mathrm{H}^2(S,\,\mathbf{G}_{\mathrm{m}}),$$

and δ *is the connecting homomorphism in the long exact sequence arising from the sequence*

$$1\longrightarrow \mathrm{Cent}(G_{qs})\longrightarrow G_{qs}\longrightarrow G^{ad}_{qs}\longrightarrow 1.$$

Proof. **1.** Let u be the class of ξ in $\mathrm{Isomext}\,(G_{qs},\,G)(S)$. Consider the left G^{ad}- and right G^{ad}_{qs}-torsor $I=\mathrm{Isomint}_{\,u}(G_{qs},\,G)$ (see Exp. XXIV Rem. 1.11). Let H be a group scheme with a G^{ad}_{qs}-action. Then $F_\xi(H)=I\times^{G^{ad}_{qs}}H$ is a group scheme over $I/G^{ad}_{qs}\simeq S$ with a left G^{ad}-action. Similarly, F'_ξ is defined by $F'_\xi(H')=I'\times^{G^{ad}}H'$, where $I'=\mathrm{Isomint}_{\,u^{-1}}(G,\,G_{qs})$. Further, we have isomorphisms $I'\times^{G^{ad}}I\simeq G^{ad}_{qs}$ and $I\times^{G^{ad}_{qs}}I'\simeq G^{ad}$, hence F_ξ and F'_ξ are mutually quasi-inverse.

2. The cohomological class in $\mathrm{H}^1(S,\,\mathrm{PGL}(V))$ corresponding to $A_{\xi,\,\rho}$ is nothing but $\rho^{ad}_*([\xi])$, where $\rho^{ad}:G^{ad}_{qs}\to\mathrm{PGL}(V)$ is the representation induced by ρ. Now the last assertion of the Theorem follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{H}^1(S,\,G^{ad}_{qs}) \arrow[r, "\delta"] \arrow[d, "\rho^{ad}_*"'] & \mathrm{H}^2(S,\,\mathrm{Cent}(G_{qs})) \arrow[d, "(\lambda_\rho)_*"] \\
\mathrm{H}^1(S,\,\mathrm{PGL}(V)) \arrow[r] & \mathrm{H}^2(S,\,\mathbf{G}_{\mathrm{m}}),
\end{tikzcd}$$

which comes from the diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Cent}(G_{qs}) \arrow[r] \arrow[d, "\lambda_\rho"'] & G_{qs} \arrow[r] \arrow[d, "\rho"'] & G^{ad}_{qs} \arrow[r] \arrow[d, "\rho^{ad}"'] & 1 \\
1 \arrow[r] & \mathbf{G}_{\mathrm{m}} \arrow[r] & \mathrm{GL}(V) \arrow[r] & \mathrm{PGL}(V) \arrow[r] & 1.
\end{tikzcd}$$

Thus, once ξ is fixed, the class of $A_{\xi,\,\rho}$ depends only on λ_ρ. Note that, in view of the isomorphism $\mathrm{Cocent}(G)\simeq\mathrm{Cocent}(G_{qs})$, λ_ρ can be identified with $\lambda_{F_\xi(\rho)}$. We need to show that $[A_{\xi,\,\rho}]$ does not depend on ξ. Let η be another cocycle producing G. Then $\rho'=F'_\eta(F_\xi(\rho))$ is ρ composed with the corresponding outer automorphism of G_{qs}; in particular, its target is still $\mathrm{GL}(V)$. Obviously $F_\eta(\rho')\simeq F_\xi(\rho)$, and we are done. $\square$

The Azumaya algebra $A_{\xi,\rho}$ will be called the *Tits algebra* of G corresponding to a center preserving representation $\rho\colon G_{qs}\to\mathrm{GL}(V)$. We denote by β_G the homomorphism

$$\begin{aligned}\beta_G\colon\ &\mathrm{Cocent}(G)(S)\to\mathrm{Br}(S)\\ &\qquad\lambda\mapsto[A_{\xi,\,\rho}]\text{ with }\lambda_{F_\xi(\rho)}=\lambda.\end{aligned}$$

It is well-defined in view of Lemma 2 and Theorem 1. To see that β_G is indeed a homomorphism one can use either the tensor product of representations or the fact that $\mathrm{Br}(S)$ is a subgroup in $\mathrm{H}^2(S,\,\mathbf{G}_{\mathrm{m}})$.

If the element in $\mathrm{Isomext}\,(G_{qs},\,G)(S)$ is fixed, we will consider β_G as a homomorphism from $\mathrm{Cocent}(G_{qs})$ to $\mathrm{Br}(S)$. Further, for an element λ of Λ^* we will write $\beta_G(\lambda)$ instead of $\beta_G(\lambda|_{\mathrm{Cent}(G_{qs})})$.

The Dynkin scheme $\mathrm{Dyn}(G)$ is the disjoint union of its minimal clopen subschemes which will be called *orbits* for brevity; they indeed correspond to orbits of the $*$-action on a set of simple roots.

Assume that G is simply connected. Let T_{qs} be a fixed maximal torus of G_{qs}, T_{qs}^{ad} be the respective torus in G_{qs}^{ad}. Over a splitting covering we have two canonical homomorphisms

$$\omega\colon \operatorname{Dyn}(G) \to \operatorname{Hom}(T_{qs}, \mathbf{G}_{\mathrm{m}}),$$
$$\alpha\colon \operatorname{Dyn}(G) \to \operatorname{Hom}(T_{qs}^{ad}, \mathbf{G}_{\mathrm{m}}),$$

that associate to each vertex i of the Dynkin diagram the fundamental weight ω_i or, respectively, the simple root α_i. By faitfully flat descent these homomorphisms are defined over the base scheme S.

Let O be an orbit in $\operatorname{Dyn}(G)$. Composing ω (resp., α) with the inclusion $O \to \operatorname{Dyn}(G)$, we obtain a weight $\omega_O\colon (T_{qs})_O \to \mathbf{G}_{\mathrm{m}}$ (resp., a root $\alpha_O\colon (T_{qs}^{ad})_O \to \mathbf{G}_{\mathrm{m}}$), which will be called the *canonical weight* (resp., the *canonical root*) corresponding to O (cf. Exp. XXIV 3.8). It is easy to see that α_O and ω_O are $*$-invariant weights of G_O.

Note that we also have homomorphisms (where $R_{S'/S}$ stands for the Weil restriction, $\prod_{S'/S}$ in the notation of [SGA])

$$\bar{\omega}_O\colon T_{qs} \to R_{O/S}(\mathbf{G}_{\mathrm{m}}),$$
$$\bar{\alpha}_O\colon T_{qs}^{ad} \to R_{O/S}(\mathbf{G}_{\mathrm{m}}),$$

which are the compositions of $R_{O/S}(\omega_O)$ and $R_{O/S}(\alpha_O)$ with the canonical homomorphisms $T_{qs} \to R_{O/S}((T_{qs})_O)$ and $T_{qs}^{ad} \to R_{O/S}((T_{qs}^{ad})_O)$.

If O splits over an extension S'/S into a disjoint union $\coprod_i O_i$, then $(\bar{\omega}_O)_{S'}$ (resp. $(\bar{\alpha}_O)_{S'}$) is equal to $\prod_i \omega_{O_i}$ (resp., $\prod_i \alpha_{O_i}$) composed with the natural isomorphism $\prod_i R_{O_i/S'}(\mathbf{G}_{\mathrm{m}}) \simeq R_{\coprod_i O_i/S'}(\mathbf{G}_{\mathrm{m}})$. In particular, over a splitting covering $\bar{\omega}_O$ (resp. $\bar{\alpha}_O$) can be identified with an appropriate product of ω_i (resp., α_i).

Proposition 1. (1) *In the above setting we have the isomorphism*

$$\prod_O \bar{\omega}_O\colon T_{qs} \simeq \prod_O R_{O/S}(\mathbf{G}_{\mathrm{m}})$$

(cf. Exp. XXIV Prop. 3.13).

(2) *If L'_{qs} is the standard Levi subgroup of a standard parabolic subgroup P in G_{qs}^{ad}, then we have the isomorphism*

$$\prod_{O\colon O\not\subset \mathfrak{t}(P)} \bar{\alpha}_O\colon \operatorname{Cent}(L'_{qs}) \simeq \prod_{O\colon O\not\subset \mathfrak{t}(P)} R_{O/S}(\mathbf{G}_{\mathrm{m}}).$$

(3) *We have*

$$L'_{qs} = \operatorname{Cent}_{G_{qs}}(Q) = \operatorname{Cent}_{G_{qs}}(Q_{diag}),$$

where Q is the natural split subtorus $\prod_{O\colon O\not\subset \mathfrak{t}(P)} \mathbf{G}_{\mathrm{m}}$ of $\prod_{O\colon O\not\subset \mathfrak{t}(P)} R_{O/S}(\mathbf{G}_{\mathrm{m}})$, and Q_{diag} is the split torus of rank 1 embedded diagonally into Q.

Proof. Let's prove (2). Note that $\operatorname{Cent}(L'_{qs})$ is contained in T_{qs}^{ad}, so the map is well-defined. Over each element S_τ of a splitting covering of S the Dynkin scheme can be identified with a set D and $\mathfrak{t}(P)$ with a subset $D \setminus J$. The map $\prod_{O\colon O\not\subset \mathfrak{t}(P)} \bar{\alpha}_O$ becomes $\prod_{i\in J} \alpha_i$, and $\operatorname{Cent}(L'_{qs})_{S_\tau}$ equals $\bigcap_{i\in D\setminus J} \operatorname{Ker} \alpha_i$. But

$$\prod_{i\in D} \alpha_i\colon (T_{qs}^{ad})_{S_\tau} \to \prod_{i\in D} \mathbf{G}_{\mathrm{m}}$$

is an isomorphism, and (2) follows. Part (1) can be proved similarly and even easier.

We have obvious inclusions

$$L'_{qs} \le \operatorname{Cent}_{G_{qs}}(Q) \le \operatorname{Cent}_{G_{qs}}(Q_{diag}),$$

so to prove (3) it suffices to show that $H = \operatorname{Cent}_{G_{qs}}(Q_{diag})$ is contained in L'_{qs}. We can pass to a splitting covering. By Exp. XXVI Prop. 6.1 H_{S_τ} is smooth with connected fibers; clearly it contains $(T_{qs}^{ad})_{S_\tau}$. By Exp. XXII 5.4.1 such subgroup is uniquely determined by the set of roots α such that the generator e_α of $\operatorname{Lie}((G_{qs})_{S_\tau})$ is contained in its Lie algebra. Note that the restriction of a simple root α_i to Q_{diag} is identity when $i \in J$ and is trivial otherwise. So e_α belongs to $\operatorname{Lie}(H_{S_\tau})$ if and only if the sum of its coefficients at α_i with $i \in J$ is zero. But $(L'_{qs})_{S_\tau}$ is also smooth with connected fibers and corresponds to the same set of roots, hence $L'_{qs} = H$. □

Proposition 2. *In the setting of Theorem 1, assume moreover that G is simply connected and* $\mathrm{Pic}(\mathrm{Dyn}(G)) = 0$*. Then $[\xi]$ comes from an element in $\mathrm{H}^1(S, G_{qs})$ if and only if $\beta_{G_O}(\omega_O) = 0$ for each orbit O.*

Proof. If $[\xi]$ belongs to the image of $\mathrm{H}^1(S, G_{qs}) \to \mathrm{H}^1(S, G_{qs}^{ad})$ then $\delta([\xi]_O) = 0$ and therefore $\beta_{G_O} = 0$ for each O. Conversely, assume that $\beta_{G_O}(\omega_O) = 0$ for each O. Proposition 1 applied to the Borel subgroup implies that $T_{qs} \simeq \prod_O R_{O/S}(\mathbf{G}_\mathrm{m})$ and $T_{qs}^{ad} \simeq \prod_O R_{O/S}(\mathbf{G}_\mathrm{m})$. Now the Shapiro lemma (cf. Exp. XXIV Prop. 8.2) implies that the image of $\delta([\xi])$ in $\mathrm{H}^2(S, T_{qs})$ is trivial, while $\mathrm{H}^1(S, T_{qs}^{ad}) = \mathrm{Pic}(\mathrm{Dyn}(G)) = 0$. Now the claim follows from the exact sequence

$$\mathrm{H}^1(S, T_{qs}^{ad}) \longrightarrow \mathrm{H}^2(S, \mathrm{Cent}(G_{qs})) \longrightarrow \mathrm{H}^2(S, T_{qs}),$$

which comes from the sequence

$$1 \longrightarrow \mathrm{Cent}(G_{qs}) \longrightarrow T_{qs} \longrightarrow T_{qs}^{ad} \longrightarrow 1.$$

□

Theorem 2. (1) *Let G be a semisimple group scheme of constant type over S, P be its parabolic subgroup admitting a Levi subgroup L, H be the derived subgroup of L. Denote by G_{qs} and H_{qs} the corresponding quasi-split groups and by Λ the lattice of weights of G_{qs}. For every $\lambda \in \Lambda^*$ denote by λ' the restriction of λ to $\mathrm{Cent}(H_{qs})$. Then $\beta_G(\lambda) = \beta_H(\lambda')$. In particular, for any $\alpha \in {\Lambda_\mathrm{r}}^*$ one has $\beta_H(\alpha') = 0$.*

(2) *Let G_{qs} be a quasi-split simply connected group, P_{qs} be a standard parabolic subgroup of G_{qs}, L_{qs} be its standard Levi subgroup, H_{qs} be the derived subgroup of L_{qs}. Assume that H is an inner form of H_{qs}, satisfying $\beta_{H_O}(\alpha'_O) = 0$ for all $O \not\subset \mathfrak{t}(P_{qs})$. Then there exist an inner form G of G_{qs} and its parabolic subgroup P admitting a Levi subgroup L, such that over a quasi-splitting covering the pair $L \le G$ becomes isomorphic to $L_{qs} \le G_{qs}$, and the derived subgroup of L is isomorphic to H.*

(3) *In the setting of* (2)*, assume that* $\mathrm{Pic}(\mathrm{Dyn}(S)) = 0$*. Then G is unique up to an isomorphism.*

Proof. **1.** Let ξ be a cocycle in $\mathrm{Z}^1(S, G_{qs}^{ad})$ corresponding to G, given by elements $g_{\sigma\tau} \in G_{qs}^{ad}(S_\sigma \times_S S_\tau)$ for some covering $\coprod S_\tau \to S$ that quasi-splits G. Over each S_τ one can (possibly, passing to a finer covering) conjugate P_{S_τ} and L_{S_τ} by some element of G_{qs}^{ad} to P_{qs} and L_{qs}, where P_{qs} is a standard parabolic subgroup of G_{qs} and L_{qs} is its standard Levi subgroup. Adjusting ξ by the coboundary given by these elements, we can assume that all $g_{\sigma\tau}$'s belong to L'_{qs}, where L'_{qs} is the image of L_{qs} in G_{qs}^{ad}, by Exp. XXVI Prop. 1.15 and Cor. 1.8 (cf. Exp. XXVI 3.21)

Let $\rho\colon G_{qs} \to \mathrm{GL}(V)$ be a center preserving representation with a weight λ. Consider its restriction to H_{qs} and denote by U the center preserving direct summand corresponding to λ' and by U' its complement invariant under H_{qs} (see Lemma 1, (4)). Denote by T_{qs} the standard maximal torus of L_{qs} and by T'_{qs} its intersection with H_{qs}. Note that U and U', being sums of weight subspaces of T'_{qs}, are stable under T_{qs} and, therefore, are invariant under the action of L_{qs}. Hence the map $\mathrm{H}^1(S, L'_{qs}) \to \mathrm{H}^1(S, \mathrm{PGL}(V))$ factors through $\mathrm{H}^1(S, (\mathrm{GL}(U) \times \mathrm{GL}(U'))/\,\mathbf{G}_\mathrm{m})$, where $\mathbf{G}_\mathrm{m}$ is embedded into $\mathrm{GL}(U) \times \mathrm{GL}(U')$ diagonally.

Now the claim is obtained by comparing the diagrams

$$\begin{array}{ccc} \mathrm{H}^1(S, (\mathrm{GL}(U) \times \mathrm{GL}(U'))/\,\mathbf{G}_\mathrm{m}) & \longrightarrow & \mathrm{H}^2(S, \mathbf{G}_\mathrm{m}) \\ \downarrow & & \| \\ \mathrm{H}^1(S, \mathrm{PGL}(V)) & \longrightarrow & \mathrm{H}^2(S, \mathbf{G}_\mathrm{m}) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{H}^1(S, (\mathrm{GL}(U) \times \mathrm{GL}(U'))/\,\mathbf{G}_\mathrm{m}) & \longrightarrow & \mathrm{H}^2(S, \mathbf{G}_\mathrm{m}) \\ \downarrow & & \| \\ \mathrm{H}^1(S, \mathrm{PGL}(U)) & \longrightarrow & \mathrm{H}^2(S, \mathbf{G}_\mathrm{m}), \end{array}$$

which come from the sequences

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{G}_{\mathrm{m}} \arrow[r] \arrow[d, equal] & \mathrm{GL}(U)\times\mathrm{GL}(U') \arrow[r] \arrow[d] & (\mathrm{GL}(U)\times\mathrm{GL}(U'))/\,\mathbf{G}_{\mathrm{m}} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbf{G}_{\mathrm{m}} \arrow[r] & \mathrm{GL}(V) \arrow[r] & \mathrm{PGL}(V) \arrow[r] & 1. \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{G}_{\mathrm{m}} \arrow[r] \arrow[d, equal] & \mathrm{GL}(U)\times\mathrm{GL}(U') \arrow[r] \arrow[d] & (\mathrm{GL}(U)\times\mathrm{GL}(U'))/\,\mathbf{G}_{\mathrm{m}} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbf{G}_{\mathrm{m}} \arrow[r] & \mathrm{GL}(U) \arrow[r] & \mathrm{PGL}(U) \arrow[r] & 1. \end{tikzcd}$$

2. Let ζ be a cocycle in $\mathrm{Z}^1(S,\,H_{qs}^{ad})=\mathrm{Z}^1(S,\,L_{qs}^{ad})$ corresponding to H. Denote by L'_{qs} and H'_{qs} the images of L_{qs} and H_{qs} in G_{qs}^{ad}. Let us compute the image $\delta([\zeta])\in\mathrm{H}^2(S,\,\mathrm{Cent}(L'_{qs}))$. Using the assumption, Theorem 1 (3), and the commutative diagram

$$\begin{tikzcd} \mathrm{H}^1(S,\,H_{qs}^{ad}) \arrow[r, "\delta"] \arrow[d, equal] & \mathrm{H}^2(S,\,\mathrm{Cent}(H'_{qs})) \arrow[d] \\ \mathrm{H}^1(S,\,L_{qs}^{ad}) \arrow[r, "\delta"] & \mathrm{H}^2(S,\,\mathrm{Cent}(L'_{qs})), \end{tikzcd}$$

we see that $(\alpha_O)_*\delta([\zeta_O]))=0$ for any $O\not\subset\mathfrak{t}(P_{qs})$. Now Proposition 1 (2) and the Shapiro lemma show that $\delta([\zeta])=0$. It means that $[\zeta]$ comes from some $[\xi]\in\mathrm{H}^1(S,\,L'_{qs})$, and the image of $[\xi]$ in $\mathrm{H}^1(S,\,G_{qs}^{ad})$ defines the desired G.

3. Let G be such a group; denote by ξ a corresponding cocycle in $\mathrm{Z}^1(S,\,G_{qs}^{ad})$. As we have seen earlier, $[\xi]$ comes from an element of $\mathrm{H}^1(S,\,L'_{qs})$, say $[\zeta]$. We have to show that $[\zeta]$ (and a fortioti $[\xi]$) is completely determined by its image in $\mathrm{H}^1(S,\,L_{qs}^{ad})$, or, in other words, that the canonical map $\pi_*\colon\ \mathrm{H}^1(S,\,L'_{qs})\to\mathrm{H}^1(S,\,L_{qs}^{ad})$ is injective. Since $\mathrm{Cent}(L'_{qs})$ is central in L'_{qs}, $\mathrm{H}^1(S,\,\mathrm{Cent}(L'_{qs}))$ acts on $\mathrm{H}^1(S,\,L'_{qs})$, and the orbits of the action coincide with the fibers of π_*. But $\mathrm{H}^1(S,\,\mathrm{Cent}(L'_{qs}))$ by Proposition 1 (2) and the Shapiro lemma injects into $\mathrm{Pic}(\mathrm{Dyn}(G))$, which is trivial by the assumption. □

5. Combinatorial restrictions

From now on we assume that $S=\operatorname{Spec}R$, where R is a connected semilocal ring. Recall that in this case all minimal parabolic subgroups $P_{\min}$ of G are conjugate under $G(S)$ and hence have the same type $\mathfrak{t}_{\min}=\mathfrak{t}(P_{\min})$, which is a clopen subscheme of $\mathrm{Dyn}(G)$ (Exp. XXVI Cor. 5.7). If T/S is a Galois covering that splits $\mathrm{Dyn}(G)$, $(\mathfrak{t}_{\min})_T$ is a clopen $*$-invariant subscheme of D_T. By Exp. XXVI Lemme 3.8 $P\mapsto\mathfrak{t}(P)$ is a bijection between parabolic subgroups P of G containing $P_{\min}$ and clopen subschemes $\mathfrak{t}$ of $\mathrm{Dyn}(G)$ containing $\mathfrak{t}_{\min}$.

Since S is affine, for any parabolic subgroup P of G there exists a Levi subgroup L (Exp. XXVI Cor. 2.3) of P, and a unique parabolic subgroup P^- which is opposite to P with respect to L, i.e. satisfies $P^-\cap P=L$ (Exp. XXVI Th. 4.3.2). The type $\mathfrak{t}(P^-)$ is the image $s_G(\mathfrak{t}(P))$ of $\mathfrak{t}(P)$ under an automorphism s_G of $\mathrm{Dyn}(G)$ called the *opposition involution* (Exp. XXIV Prop. 3.16.6 and Exp. XXVI 4.3.1; cf. [Ti66] 1.5.1). The corresponding automorphism $s_G\in\mathrm{Aut}\,(D)$ is induced by the automorphism $\alpha\mapsto -w_0(\alpha)$ of the root system Φ of G_0, where w_0 is the unique element of maximal length in the Weyl group of Φ. In fact s_G acts nontrivially only on irreducible components of Φ of type A_n, $n\geq 2$, D_{2n+1}, $n\geq 1$, or E_6, where it coincides with the unique nontrivial automorphism of the component.

The assumption that $S=\operatorname{Spec}R$ is connected allows us to identify D_T with D, and a clopen $*$-invariant subscheme of D_T with a $*$-invariant subset of D. Let $J\subseteq D$ be the complement of the subset corresponding to $\mathfrak{t}_{\min}$. Then the *Tits index* of G is the pair $(D,\,J)$ together with a $*$-action on D, represented by a subgroup Γ of $\mathrm{Aut}\,(D)$. Usually we indicate Γ by writing its order as the upper left index attached to D (for example, ${}^2E_6$, ${}^6D_4$ and so on). The group G is of *inner type*, if $\Gamma = \{1\}$. The group G is quasi-split, if $J = D$, and split, if it is quasi-split and the $*$-action is trivial. When $J = \emptyset$ we say that G is *anisotropic*.

Exp. XXVI Prop. 1.20 implies that a parabolic subgroup is minimal if and only if its Levi subgroup is anisotropic. The derived subgroup of the Levi subgroup is then called the *anisotropic kernel* of G and denoted by G_{an} (it is determined uniquely up to an isomorphism).

Clearly, we have $\mathfrak{t}_{\min} = s_G(\mathfrak{t}_{\min})$, since if $P = P_{\min}$ is a minimal parabolic subgroup, then P^- is also minimal. Thus, if (D, J, Γ) is a Tits index, we have $J = s_G(J)$.

From now on, we fix a minimal parabolic subgroup $P = P_{\min}$ of G, a Levi subgroup L of P, and a maximal split subtorus Q of G such that $L = \mathrm{Cent}_G(Q)$, which exists by Exp. XXVI Cor. 6.11 (or by Proposition 1 (3) and descent). Let M be the lattice of characters of Q. The Lie algebra $\mathrm{Lie}(G)$ of G decomposes under the action of Q into a direct sum of weight subspaces:

$$\mathrm{Lie}(G) = \mathrm{Lie}(L) \oplus \bigoplus_{\alpha \in M \setminus \{0\}} \mathrm{Lie}(G)^{\alpha}.$$

We denote by Ψ the set of elements $\alpha \in M \setminus \{0\}$ such that $\mathrm{Lie}(G)^{\alpha} \neq 0$. By Exp. XXVI Th. 7.4 Ψ is a root system, which is called the *relative root system* of G with respect to Q. One readily sees that the simple roots of Ψ correspond bijectively to the $*$-orbits contained in J.

Denote by $\hat{D}$ the extended Dynkin diagram (one adds a vertex corresponding to minus the maximal root to each irreducible component of D), and by $\hat{J}$ the union $J \cup (\hat{D} \setminus D)$.

Lemma 3. *Let G be a semisimple algebraic group over S, and let (D, J) be the Tits index of G. Then any $*$-orbit $O \subseteq \hat{J}$ is invariant under the opposition involution of the Dynkin diagram $(\hat{D} \setminus \hat{J}) \cup O$.*

Proof. Let $A \in \Psi$ be the relative root corresponding to O (it is simple if $O \subseteq J$ and the opposite to the maximal otherwise). By Exp. XXVI Prop. 6.1 the subsets $\mathbb{Z} A \cap \Psi$ and $\mathbb{Z} A \cap \Psi^+$ correspond to certain subgroups G' and P' of G; moreover, G' is reductive and P' is a parabolic subgroup of G' having L as a Levi subgroup. Since L is anisotropic, P' is a minimal parabolic subgroup of G'. Passing to a splitting covering one sees that the Dynkin diagram of G' is $(\hat{D} \setminus \hat{J}) \cup O$, and the type of P' is given by O. The Lemma follows. □

In the next Proposition we list all possible cases when the conclusion of Lemma 3 holds for an irreducible root system Φ. Our numbering of the vertices of Dynkin diagrams follows [B].

Proposition 3. *Let Φ be a reduced irreducible root system, D the corresponding Dynkin diagram, $J \neq \emptyset$ a subset of D and Γ a group of automorphisms of D. A triple (Φ, J, Γ) satisfies that any Γ-orbit $O \subseteq \hat{J}$ is invariant under the opposition involution of the Dynkin diagram $(\hat{D} \setminus \hat{J}) \cup O$, if and only if it is, up to an automorphism of D, one in the following list:*

(1) $\Phi = A_n$, $n \geq 1$; $|\Gamma| = 1$; $J = \{d, 2d, \ldots, rd\}$ *for some* $d, r \geq 1$ *such that* $d \cdot (r+1) = n+1$.
(2) $\Phi = A_n$, $n \geq 2$; $|\Gamma| = 2$; $J = \{d, 2d, \ldots, rd, n+1-d, n+1-2d, \ldots, n+1-rd\}$ *for some* $d, r \geq 1$ *such that* $d \mid n+1$, $2rd \leq n+1$.
(3) $\Phi = B_n$, $n \geq 2$; $|\Gamma| = 1$; $J = \{d, 2d, \ldots, rd\}$ *for some* $d, r \geq 1$ *such that* d *is even or* $d = 1$, $rd \leq n$.
(4) $\Phi = C_n$, $n \geq 2$; $|\Gamma| = 2$; $J = \{d, 2d, \ldots, rd\}$ *for some* $d, r \geq 1$ *such that* $rd \leq n$.
(5) $\Phi = D_n$, $n \geq 4$; $|\Gamma| = 1$; $J = \{d, 2d, \ldots, rd\}$ *for some* $d, r \geq 1$ *such that* d *is even or* $d = 1$, $rd \leq n$, $rd \neq n-1$.
(6) $\Phi = D_n$, $n \geq 4$; $|\Gamma| = 2$; $J = \{d, 2d, \ldots, rd\}$ *(or* $J = \{d, 2d, \ldots, (r-2)d, n-1, n\}$ *in the case* $rd = n-1$*) for some* $d, r \geq 1$ *such that* d *is even or* $d = 1$, $rd \leq n-1$.
(7) $\Phi = D_4$; $|\Gamma| = 3$ *or* $|\Gamma| = 6$; $J = \{2\}$, D.
(8) $\Phi = E_6$; $|\Gamma| = 1$; $J = \{2\}$, $\{1, 6\}$, $\{2, 4\}$, D.
(9) $\Phi = E_6$; $|\Gamma| = 2$; $J = \{2\}$, $\{1, 6\}$, $\{2, 4\}$, $\{1, 6, 2\}$, D.
(10) $\Phi = E_7$; $|\Gamma| = 1$; $J = \{1\}$, $\{6\}$, $\{7\}$, $\{1, 3\}$, $\{1, 6\}$, $\{1, 6, 7\}$, $\{1, 3, 4, 6\}$, D.
(11) $\Phi = E_8$; $|\Gamma| = 1$; $J = \{1\}$, $\{8\}$, $\{1, 8\}$, $\{7, 8\}$, $\{1, 6, 7, 8\}$, D.
(12) $\Phi = F_4$; $|\Gamma| = 1$; $J = \{1\}$, $\{4\}$, $\{1, 4\}$, D.
(13) $\Phi = G_2$; $|\Gamma| = 1$; $J = \{2\}$, D.

Proof. If Φ is an exceptional root system or D_4, the result is verified by an easy try-out. Consider the case $\Phi = A_n$, $|\Gamma| = 1$. The opposition involution of A_n is the non-trivial automorphism of D, hence if $|J| = 1$ then $n = 2k + 1$ and $J = \{k + 1\}$, the middle vertex. Proceeding by induction on $|J|$, we see that $J = \{d, 2d, \ldots, rd\}$ for some $d \geq 1$ such that $d|n + 1$, $d \cdot (r + 1) = n + 1$, and any such J is valid. If $|\Gamma| = 2$ then since J is Γ-invariant, J contains a vertex k if and only if it contains $n + 1 - k$; the opposition involution condition implies that $J = \{d, 2d, \ldots, rd\} \cup \{n + 1 - d, n + 1 - 2d, \ldots, n + 1 - rd\}$, and any such J is valid.

Now consider the case $\Phi = B_n, C_n, D_n$ and $|\Gamma| = 1$. Let $J = \{i_1, i_2, \ldots, i_r\}$, $i_1 < i_2 < \ldots < i_r$. If $\Phi = D_n$ and $i_r > n - 2$, we may assume $i_r = n$ applying an automorphism of D. Then $J \setminus \{i_r\}$ lies in the connected component of $D \setminus \{i_r\}$ of type $A_{i_r - 1}$. Since $J \setminus \{i_r\}$ satisfies the opposition involution condition, by the A_n case $J \setminus \{i_r\}$ is of the form $\{d, 2d, \ldots, (r - 1)d\}$ for some $d \geq 1$ such that $i_r = rd$. Therefore, $J = \{d, 2d, \ldots, rd\}$, as required. If $\Phi = C_n$, this finishes the proof, since any such J satisfies the opposition involution condition. If $\Phi = D_n$ or B_n, such J does not satisfy the opposition involution condition for $O = \hat{J} \setminus J$ if d is odd > 1, so this case is excluded. The case $\Phi = D_n$, $|\Gamma| = 2$ is verified analogously. □

6. Tits indices

We now start the classification of semisimple algebraic groups over $S = \operatorname{Spec} R$, where R is a connected semilocal ring. The problem allows two immediate reductions. First, every semisimple group G is completely determined by its root datum and the corresponding simply connected group G^{sc}, so we can assume that G is simply connected.

Second, if the Dynkin diagram D of G is not connected (that is, the root system is not irreducible), we can present D as the disjoint union of its *isotypic* components D_t (it means that we collect isomorphic components together), and then we have a canonical decomposition $G \simeq \prod G_t$, where G_t is a group over S with the Dynkin diagram D_t (Exp. XXIV Prop. 5.5). Further, if D_t is the disjoint union of n_t copies of a connected graph $D_{0,t}$, there exists a canonical étale extension S_t/S of degree n_t and a group $G_{0,t}$ over S_t such that $G_t \simeq R_{S_t/S}(G_{0,t})$ (Exp. XXIV Prop. 5.9). So we can assume that D is connected, that is, G is a *simple* algebraic group.

Our reasoning will be based on Theorem 2, which implies that a semisimple algebraic group G is determined, up to an isomorphism, by the quasi-split group G_{qs}, the type $\mathfrak{t}_{min} = D \setminus J$ of a minimal parabolic subgroup, and the anisotropic kernel G_{an} subject to certain conditions on the Tits algebras. In its turn, G_{qs} is determined by the Dynkin diagram D and the $*$-action on it. Thus the classification consists in listing all possible Tits indices of simple algebraic groups, and, for any given index, the conditions on the corresponding anisotropic kernels. The necessary combinatorial restriction on a Tits index stated in Lemma 3 reduces possibilities to those listed in Proposition 3. For some of them conditions on the Tits algebras lead to a contradiction; for the rest they give criteria that anisotropic kernels must satisfy. Whenever it does not require any extra technique, we describe isomorphism classes of admissible anisotropic kernels in terms of more intuitive algebraic structures, like Azumaya algebras over R or étale extensions R'/R of a given degree.

We represent Tits indices graphically by Dynkin diagrams D with the vertices in J being circled; nontrivial $*$-action is indicated by arrows $\longleftrightarrow$. We also use the Tits notation ${}^m X_{n,r}^k$ for the groups of specific indices (see [Ti66]). Unless explicitly stated otherwise, E denotes an Azumaya algebra over R.

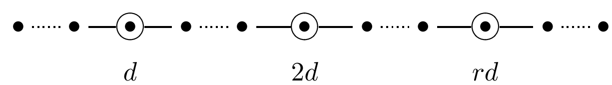
We begin with simple groups of type A_n. The split simple simply connected group of type A_n over R is $\mathrm{SL}_{n+1}(R)$; the corresponding adjoint group is $\mathrm{PGL}_{n+1}(R) = \mathrm{Aut}\,(\mathrm{M}_{n+1}(R))$. So the simple simply connected groups of inner type A_n are of the form $\mathrm{SL}_1(A)$, where A is an Azumaya algebra over R of degree $n + 1$, uniquely determined up to an isomorphism. Obviously A is the Tits algebra of $\mathrm{SL}_1(A)$ corresponding to the natural representation of $\mathrm{SL}_{n+1}(R)$ in R^{n+1}; so $[A] = \beta_{\mathrm{SL}_1(A)}(\omega_1)$.

Lemma 4. *Assume that* $\mathrm{SL}_1(E)$ *and* $\mathrm{SL}_1(E')$ *are anisotropic, and* $[E] = [E']$ *in* $\mathrm{Br}(R)$*. Then* $E \simeq E'$*.*

Proof. Since projective modules over R are free, $[E] = [E']$ means that $\mathrm{M}_n(E) \simeq \mathrm{M}_m(E')$ for some n and m. Consider the simple group $G = \mathrm{SL}_n(E) \simeq \mathrm{SL}_m(E')$. Then $\mathrm{SL}_1(E)^n$ and $\mathrm{SL}_1(E')^m$ are both anisotropic kernels of G, so they are isomorphic. In particular, they have the same type, that is $m = n$ and the degrees of E and E' are equal. This implies $E \simeq E'$ by [K, Ch. III Prop. 5.2.3 2)]. □

Theorem 3 ($^1\mathbf{A_n}$). *Every simple simply connected group G of inner type A_n over R is isomorphic to* $\mathrm{SL}_{r+1}(E)$ *for a uniquely determined $r \geq 0$ and an Azumaya algebra E over R such that* $\mathrm{SL}_1(E)$ *is anisotropic. The Tits index of G is $({}^1A_n,\, J)$, where $J = \{d,\, 2d,\, \ldots,\, rd\}$, d is the degree of E and $n+1 = (r+1)d$:*

$({}^1A_{n,r}^{(d)})$



Proof. Let $({}^1A_n,\, J)$ be the Tits index of G; we have $J = \{d,\, 2d,\, \ldots,\, rd\}$ for some d with $n+1 = (r+1)d$ by Lemma 3 and Proposition 3. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(E_1) \times \ldots \mathrm{SL}_1(E_{r+1})$ for some Azumaya algebras $E_1,\, \ldots,\, E_r$. The Cartan matrix of A_n shows that $\alpha_{i \cdot d} = 2\omega_{i\cdot d} - \omega_{i \cdot d-1} - \omega_{i\cdot d+1}$ for $i = 1,\, \ldots,\, r$. By Theorem 2, we have

$$0 = \beta_{G_{an}}(\alpha'_{i\cdot d}) = \beta_{\mathrm{SL}_1(E_i)}(\omega_1) - \beta_{\mathrm{SL}_1(E_{i+1})}(\omega_1) = [E_i] - [E_{i+1}].$$

Now Lemma 4 implies that all E_i are isomorphic. Set $E = E_1$; then $\mathrm{SL}_{r+1}(E)$ has the same Tits index and the same anisotropic kernel as G, so by Theorem 2 we have $G \simeq \mathrm{SL}_{r+1}(E)$, as claimed. □

The above result implies that for any Azumaya algebra A over R, the group $G = \mathrm{SL}_1(A)$ is isomorphic to $\mathrm{SL}_{r+1}(E)$, where E is an Azumaya algebra such that $\mathrm{SL}_1(E)$ is anisotropic. In this case the degree of E is called the *index* of A and is denoted by $\operatorname{ind} A$; obviously $\operatorname{ind} A$ divides $\deg A$. The *exponent* $\exp A$ of A is the order of $[A]$ in $\mathrm{Br}(R)$. We will need the following result:

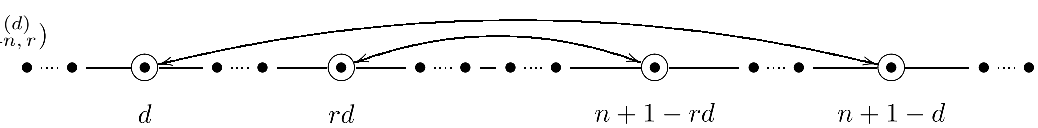
Proposition 4. *Let A be an Azumaya algebra. Then* $\exp A$ *divides* $\operatorname{ind} A$*, and they have the same prime factors.*

Proof. The first part follows from the fact that $[A] = [E] = \beta_{\mathrm{SL}_1(E)}(\omega_1)$, and $(\deg E)\omega_1$ belongs to the root lattice of $\mathrm{SL}_1(A)$. The second part follows from [Gab, Ch. II, Thm. 1]. □

Let R'/R be an étale extension of degree n. We can interpret the corestriction homomorphism $\mathrm{cores}_{R'/R}\colon \mathrm{Br}(R') \to \mathrm{Br}(R)$ as follows. If A is an Azumaya algebra over R' of degree d, $R_{R'/R}(\mathrm{SL}_1(A))$ is a group of type nA_{d-1} over R, with the $*$-action permuting the copies of A_{d-1}. Now $\mathrm{cores}_{R'/R}([A]) = \beta_{R_{R'/R}(\mathrm{SL}_1(A))}(\omega)$, where ω is the sum of the fundamental weights ω_1 of all copies of A_{d-1} (cf. [Ti71, § 5.3]).

Theorem 4 ($^2\mathbf{A_n}$). *Every simple simply connected group G of type ${}^2A_n$ over R has the Tits index $({}^2A_n,\, J)$, where $J = \{d,\, 2d,\, \ldots,\, rd,\, n+1-rd,\, \ldots,\, n+1-2d,\, n+1-d\}$ for some $r \geq 0$, $d > 0$ such that $d \mid n+1$, $2rd \leq n+1$:*

$({}^2A_{n,r}^{(d)})$



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type ${}^2A_{n-2rd}$ over R with $\beta_{H_O}(\omega_1) = [E]$,* $\operatorname{ind} E = d$*, where O is the orbit corresponding to $\{1,\, n-2rd\}$, when $n - 2rd \geq 2$;*
- *pairs consisting of an Azumaya algebras A over R and a connected quadratic étale extension R'/R such that* $\operatorname{ind} A = \deg A = 2$ *and* $\operatorname{ind} A_{R'} = d$*, when $n - 2rd = 1$;*
- *Azumaya algebras E over a connected quadratic étale extension R'/R with* $\operatorname{ind} E = \deg E = d$ *and* $\mathrm{cores}_{R'/R}([E]) = 0$*, when $n - 2rd \leq 0$.*

Proof. Let $({}^2A_n,\, J)$ be the Tits index of G; we have $J = \{d,\, 2d,\, \ldots,\, rd,\, n+1-rd,\, \ldots,\, n+1-2d,\, n+1-d\}$ for some $r \geq 0$, $d > 0$ with $d \mid n+1$, $2rd \leq n+1$ by Lemma 3 and Proposition 3. The anisotropic kernel G_{an} is isomorphic to $H_1 \times \ldots \times H_r \times H$, where H_i are groups of outer type $A_{d-1} + A_{d-1}$ with the $*$-action permuting two summands, and H is a group of outer type

${}^2A_{n-2rd}$ when $n-2rd \geq 2$, is isomorphic to $\mathrm{SL}_1(A)$ for some Azumaya algebra A over R with $\operatorname{ind} A = \deg A = 2$ when $n - 2rd = 1$, and is trivial otherwise. Over a quadratic étale extension R'/R every H_i becomes inner, hence we have $H_i \simeq R_{R'/R}(\mathrm{SL}_r(E_i))$ for some Azumaya algebra E_i over R', $\operatorname{ind} E_i = \deg E_i = d$.

Denote the orbit corresponding to $\{i \cdot d,\, n+1-i\cdot d\}$ by O_i, $i = 1, \ldots, r$. The Cartan matrix of A_n shows that $\alpha_{i\cdot d} = 2\omega_{i\cdot d} - \omega_{i\cdot d-1} - \omega_{i\cdot d+1}$. When $i < r$, by Theorem 2 we have

$$0 = \beta_{(G_{an})_{O_i}}(\alpha'_{O_i}) = \beta_{\mathrm{SL}_1(E_i)}(\omega_1) - \beta_{\mathrm{SL}_1(E_{i+1})}(\omega_1) = [E_i] - [E_{i+1}].$$

Lemma 4 implies now that all E_i are isomorphic; we set $E = E_1$.

In the case $n - 2rd \geq 2$ by Theorem 2 we have

$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_{H_{O_r}}(\omega_1) = [E] - \beta_{H_{O_r}}(\omega_1).$$

In the case $n - 2rd = 1$ we have

$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_{\mathrm{SL}_1(A)_{O_r}}(\omega_1) = [E] - [A_{R'}],$$

for $O_r \simeq \operatorname{Spec} R'$ as a scheme.

In the case $n - 2rd = 0$ we have

$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) = [E],$$

hence $E \simeq R'$. G is quasi-split in this case.

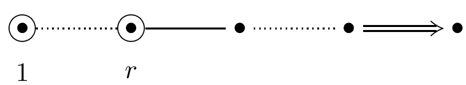
Finally, in the case $n - 2rd = -1$ we have $O_r \simeq \operatorname{Spec} R$, and hence

$$0 = \beta_{G_{an}}(\alpha'_{O_r}) = \mathrm{cores}_{R'/R}(\beta_{\mathrm{SL}_1(E)}(\omega_1)) = \mathrm{cores}_{R'/R}([E]).$$

□

Theorem 3 ($\mathbf{B_n}$). *Every simple simply connected group of type B_n over R, $n \geq 2$, has the Tits index $(B_n,\, J)$, where $J = \{1, 2, \ldots, r\}$ for some $r \geq 0$:*

$(B_{n,r})$



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups of type B_{n-r} over R, when $n - r \geq 2$;*
- *Azumaya algebras A over R with $\operatorname{ind} A = \deg A = 2$, when $n - r = 1$.*

If $n = r$ then G is split.

Proof. Let $(B_n,\, J)$ be the Tits index of G ; we have $J = \{d, 2d, \ldots, rd\}$ for some $r \geq 0$, $d > 0$ with $rd \leq n$ by Lemma 3 and Proposition 3. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(E_1) \times \ldots \mathrm{SL}_1(E_r) \times H$, where H is a group of type B_{n-rd} when $n - rd \geq 2$, is isomorphic to $\mathrm{SL}_1(A)$ for some Azumaya algebra A over R with $\operatorname{ind} A = \deg A = 2$ when $n - rd = 1$, or is trivial when $n = rd$.

In the case $n - rd \geq 2$ the Cartan matrix of B_n shows that $\alpha_{rd} = 2\omega_{rd} - \omega_{rd-1} - \omega_{rd+1}$. By Theorem 2, we have

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{\mathrm{SL}_1(E_r)}(\omega_1) - \beta_H(\omega_1) = [E_r].$$

So $E_r = R$, hence $d = 1$.

In the case $n - rd = 1$ we have $\alpha_{rd} = 2\omega_{n-1} - \omega_{n-2} - 2\omega_n$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{\mathrm{SL}_1(E_r)}(\omega_1) - 2\beta_H(\omega_1) = [E_r],$$

and again $d = 1$.

Finally, in the case $n = rd$ we have $\alpha_{rd} = 2\omega_n - \omega_{n-1}$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{\mathrm{SL}_1(E_r)}(\omega_1) = [E_r],$$

$d = 1$, and G is split in this case. □

The split simple simply connected group scheme of type C_n over R is $\mathrm{Sp}_{2n}(R)$.

Proposition 5. *Assume that G is a simple simply connected group of type C_n over R, $\beta_G(\omega_1) = [E]$, $\operatorname{ind} E = d$. Then $d = 2^k$ for some $k \geq 0$ and $d \mid 2n$. If $d = 1$ then G is split.*

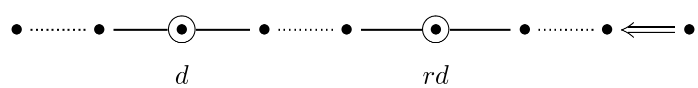
Proof. We have $2[E]=0$, since $2\omega_1$ belongs to Λ_{r}. Now Proposition 4 implies that $d=2^k$.

The vector representation $\rho\colon \mathrm{Sp}_{2n}(R)\to \mathrm{GL}(R^{2n})$ is center preserving and has a weight ω_1; so $[A_\rho]=[E]$. But A_ρ has degree $2n$, so $d\mid 2n$.

If $d=1$ then by Proposition 2 G corresponds to an element of $\mathrm{H}^1(R,\mathrm{Sp}_{2n})$, and the latter is trivial by [K, Ch. I, Cor. 4.1.2]. □

Theorem 3 ($\mathbf{C_n}$). *Every simple simply connected group G of type C_n over R, $n\geq 2$, has the Tits index $(C_n,\,J)$, where $J=\{d,\,2d,\,\dots,\,rd\}$ for some $r\geq 0$, $d>0$ such that $d=2^k\mid 2n$, $rd\leq n$, and $r=n$ when $d=1$:*

$(C^{(d)}_{n,\,r})$



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type C_{n-rd} over R with $\beta_H(\omega_1)=[E]$, $\operatorname{ind}E=d$, when $n-rd\geq 2$;*
- *Azumaya algebras E over R with $\operatorname{ind}E=\deg E=d$ and $\exp E\leq 2$, when $n-rd\leq 1$.*

Proof. Let $(C_n,\,J)$ be the Tits index of G; we have $J=\{d,\,2d,\,\dots,\,rd\}$ for some $r\geq 0$, $d>0$ with $rd\leq n$ by Lemma 3 and Proposition 3. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(E_1)\times\dots\mathrm{SL}_1(E_r)\times H$, where H is a group of type C_{n-rd} when $n-rd\geq 2$, is isomorphic to $\mathrm{SL}_1(A)$ for some Azumaya algebra A over R with $\operatorname{ind}A=\deg A=2$ when $n-rd=1$, or is trivial when $n=rd$.

The Cartan matrix of C_n shows that $\alpha_{i\cdot d}=2\omega_{i\cdot d}-\omega_{i\cdot d-1}-\omega_{i\cdot d+1}$ for $i=1,\,\dots,\,r-1$. By Theorem 2, we have

$$0=\beta_{G_{an}}(\alpha'_{i\cdot d})=\beta_{\mathrm{SL}_1(E_i)}(\omega_1)-\beta_{\mathrm{SL}_1(E_{i+1})}(\omega_1)=[E_i]-[E_{i+1}].$$

Lemma 4 implies now that all E_i are isomorphic; set $E=E_1$. Note that $[E]=\beta_G(\omega_1)$, hence by Proposition 5 $d=2^k\mid 2n$, and G is split when $d=1$.

In the case $n-rd\geq 2$ the Cartan matrix of C_n shows that $\alpha_{rd}=2\omega_{rd}-\omega_{rd-1}-\omega_{rd+1}$, so

$$0=\beta_{G_{an}}(\alpha'_{rd})=\beta_{\mathrm{SL}_1(E)}(\omega_1)-\beta_H(\omega_1)=[E]-\beta_H(\omega_1).$$

In the case $n-rd=1$ we have $\alpha_m=2\omega_{n-1}-\omega_{n-2}-\omega_n$, so

$$0=\beta_{G_{an}}(\alpha'_{rd})=\beta_{\mathrm{SL}_1(E)}(\omega_1)-\beta_{\mathrm{SL}_1(A)}(\omega_1)=[E]-[A].$$

Hence $[E]=[A]$ and $d=2$.

Finally, in the case $n=rd$ we have $\alpha_m=2\omega_n-2\omega_{n-1}$, so

$$0=\beta_{G_{an}}(\alpha'_{rd})=2\beta_{\mathrm{SL}_1(E)}(\omega_1)=2[E],$$

that is $\exp E\leq 2$. □

The split simple simply connected group scheme of type D_n over R is $\mathrm{Spin}_{2n}(R)$.

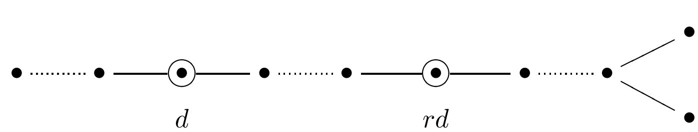
Proposition 6. *Assume that G is a simple simply connected group of type ${}^1D_n$ or ${}^2D_n$ over R, $n\geq 4$, $\beta_G(\omega_1)=[E]$, $\operatorname{ind}E=d$. Then $d=2^k$ for some $k\geq 0$ and $d\mid 2n$.*

Proof. We have $2[E]=0$, since $2\omega_1$ belongs to Λ_{r}. Now Proposition 4 implies that $d=2^k$.

The vector representation $\rho\colon \mathrm{Spin}_{2n}(R)\to \mathrm{GL}(R^{2n})$ is center preserving and has a weight ω_1; so $[A_\rho]=[E]$. But A_ρ has degree $2n$, so $d\mid 2n$. □

Theorem 3 (${}^{\mathbf{1}}\mathbf{D_n}$). *Every simple simply connected group G of inner type D_n over R, $n\geq 4$, has the Tits index $({}^1D_n,\,J)$, where $J=\{d,\,2d,\,\dots,\,rd\}$ (possibly, after interchanging $n-1$ and n) for some $r\geq 0$, $d>0$ such that $d=2^k\mid 2n$, $rd\leq n$, $n\neq rd+1$:*

$({}^1D^{(d)}_{n,\,r})$



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of inner type D_{n-rd} over R with $\beta_H(\omega_1) = [E]$, $\operatorname{ind} E = d$, when $n - rd \geq 4$;*
- *Azumaya algebras A over R with $\operatorname{ind} A = \deg A = 4$ and $2[A] = [E]$, $\operatorname{ind} E = d$, when $n - rd = 3$;*
- *pairs of Azumaya algebras A_1 and A_2 over R with $\operatorname{ind} A_1 = \deg A_1 = \operatorname{ind} A_2 = \deg A_2 = 2$ and $[A_1] + [A_2] = [E]$, $\operatorname{ind} E = d$, when $n - rd = 2$;*
- *Azumaya algebras E over R with $\operatorname{ind} E = \deg E = d$ and $\exp E \leq 2$, when $n = rd$.*

Proof. Let $({}^1D_n,\, J)$ be the Tits index of G; we have $J = \{d,\, 2d,\, \ldots,\, rd\}$ for some $r \geq 0$, $d > 0$ with $rd \leq n$, $rd \neq n - 1$ by Lemma 3 and Proposition 3. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(E_1) \times \ldots \mathrm{SL}_1(E_r) \times H$, where H is a group of inner type D_{n-rd} when $n - rd \geq 4$, is isomorphic to $\mathrm{SL}_1(A)$ for some Azumaya algebra A over R with $\operatorname{ind} A = \deg A = 4$ when $n - rd = 3$ is isomorphic to $\mathrm{SL}_1(A_1) \times \mathrm{SL}_1(A_2)$ for some Azumaya algebras A_1, A_2 over R with $\operatorname{ind} A_1 = \deg A_1 = \operatorname{ind} A_2 = \deg A_2 = 2$, or is trivial when $n = rd$.

The Cartan matrix of D_n shows that $\alpha_{i\cdot d} = 2\omega_{i\cdot d} - \omega_{i\cdot d-1} - \omega_{i\cdot d+1}$ for $i = 1,\, \ldots,\, r-1$. By Theorem 2, we have

$$0 = \beta_{G_{an}}(\alpha'_{i\cdot d}) = \beta_{\mathrm{SL}_1(E_i)}(\omega_1) - \beta_{\mathrm{SL}_1(E_{i+1})}(\omega_1) = [E_i] - [E_{i+1}].$$

Lemma 4 implies now that all E_i are isomorphic; set $E = E_1$. Note that $[E] = \beta_G(\omega_1)$, hence by Proposition 6 $d = 2^k \mid 2n$.

In the case $n - rd \geq 4$ the Cartan matrix of D_n shows that $\alpha_{rd} = 2\omega_{rd} - \omega_{rd-1} - \omega_{rd+1}$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_H(\omega_1) = [E] - \beta_H(\omega_1).$$

In the case $n - rd = 3$ we still have $\alpha_{rd} = 2\omega_{rd} - \omega_{rd-1} - \omega_{rd+1}$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_{\mathrm{SL}_1(A)}(\omega_2) = [E] - 2[A].$$

In the case $n - rd = 2$ we have $\alpha_{rd} = 2\omega_{n-2} - \omega_{n-3} - \omega_{n-1} - \omega_n$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_{\mathrm{SL}_1(A_1)}(\omega_1) - \beta_{\mathrm{SL}_1(A_1)}(\omega_2) = [E] - [A_1] - [A_2].$$

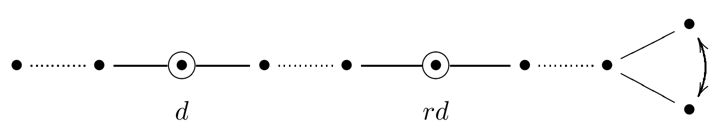
Finally, in the case $n = rd$ we have $\alpha_{rd} = 2\omega_n - \omega_{n-2}$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{\mathrm{SL}_1(E)}(\omega_2) = 2[E],$$

hence $\exp E \leq 2$. □

Theorem 3 (${}^2\mathbf{D_n}$). *Every simple simply connected group G of type ${}^2D_n$, $n \geq 4$, has the Tits index $({}^2D_n,\, J)$, where $J = \{d,\, 2d,\, \ldots,\, rd\}$ for some $r \geq 0$, $d > 0$ such that $d = 2^k \mid 2n$, $rd < n - 1$, or $J = \{d,\, 2d,\, \ldots,\, (r-1)d,\, n-1,\, n\}$ for some $r \geq 0$, $d \in \{1, 2\}$ such that $rd = n - 1$.*

$({}^2D^{(d)}_{n,r})$



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type ${}^2D_{n-rd}$ over R with $\beta_H(\omega_1) = [E]$, $\operatorname{ind} E = d$, when $n - rd \geq 4$;*
- *simple simply connected anisotropic groups H of type ${}^2A_3$ over R with $\beta_H(\omega_2) = [E]$, $\operatorname{ind} E = d$, when $n - rd = 3$;*
- *Azumaya algebras A over a connected quadratic étale extension R'/R with $\operatorname{ind} A = \deg A = 2$ and $\operatorname{cores}_{R'/R}([A]) = [E]$, $\operatorname{ind} E = d$, when $n - rd = 2$;*
- *pairs consisting of an Azumaya algebra E over R with $\operatorname{ind} E = \deg E = d$ and a connected quadratic étale extension R'/R such that $[E_{R'}] = 0$, when $n - rd = 1$.*

Proof. Let $({}^2D_n,\, J)$ be the Tits index of G; by Lemma 3 and Proposition 3 we have $J = \{d,\, 2d,\, \ldots,\, rd\}$ (or $J = \{d, 2d, \ldots, (r-2)d, n-1, n\}$ in the case $rd = n - 1$) for some $r \geq 0$, $d > 0$ with $rd \neq n - 1$. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(E_1) \times \ldots \mathrm{SL}_1(E_r) \times H$, where H is a group of type ${}^2D_{n-rd}$ when $n - rd \geq 4$, of type ${}^2A_3$ when $n - rd = 3$, is isomorphic to $R_{R'/R}(\mathrm{SL}_1(A))$ for some Azumaya algebras A over a connected quadratic étale extension R'/R with $\operatorname{ind} A = \deg A = 2$, or is trivial when $n - rd = 1$.

Denote the orbit corresponding to $\{i \cdot d,\, n+1-i\cdot d\}$ by O_i, $i = 1, \ldots, r$. The Cartan matrix of D_n shows that $\alpha_{i\cdot d} = 2\omega_{i\cdot d} - \omega_{i\cdot d-1} - \omega_{i\cdot d+1}$ for $i = 1, \ldots, r-1$. By Theorem 2, we have

$$0 = \beta_{(G_{an})_{O_i}}(\alpha'_{O_i}) = \beta_{\mathrm{SL}_1(E_i)}(\omega_1) - \beta_{\mathrm{SL}_1(E_{i+1})}(\omega_1) = [E_i] - [E_{i+1}].$$

Lemma 4 implies now that all E_i are isomorphic; set $E = E_1$. Note that $[E] = \beta_G(\omega_1)$, hence by Proposition 6 $d = 2^k \mid 2n$.

In the case $n - rd \geq 4$ the Cartan matrix of D_n shows that $\alpha_{rd} = 2\omega_{rd} - \omega_{rd-1} - \omega_{rd+1}$, so

$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_H(\omega_1) = [E] - \beta_H(\omega_1).$$

In the case $n - rd = 3$ we still have $\alpha_{rd} = 2\omega_{rd} - \omega_{rd-1} - \omega_{rd+1}$, so

$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_H(\omega_2).$$

In the case $n - rd = 2$ $O_r \simeq \operatorname{Spec} R$, and we have $\alpha_{rd} = 2\omega_{n-2} - \omega_{n-3} - \omega_{n-1} - \omega_n$, so

$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \mathrm{cores}_{R'/R}(\beta_{\mathrm{SL}_1(A)}(\omega_1)) = [E] - \mathrm{cores}_{R'/R}([A]).$$

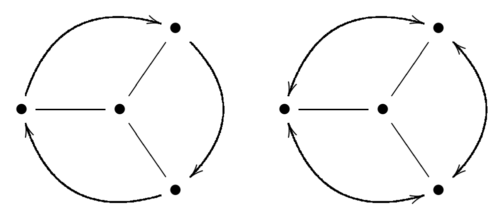
Finally, in the case $n - rd = 1$ the condition $d|2n$ implies $d \in \{1, 2\}$; also, $O_r \simeq \operatorname{Spec} R'$, and we have $\alpha_{rd} = 2\omega_n - \omega_{n-2}$, so

$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\mathrm{SL}_1(E)_{O_r}}(\omega_1) = [E_{R'}].$$

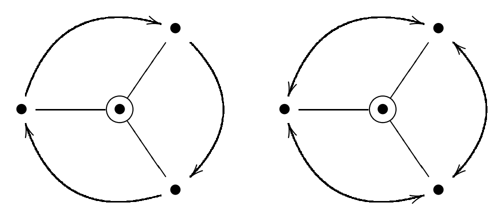
□

Theorem 3 ($\mathbf{^3D_4}$ and $\mathbf{^6D_4}$). *Every simple simply connected group G of type ${}^3D_4$ or ${}^6D_4$ over R has one of the following Tits indices:*

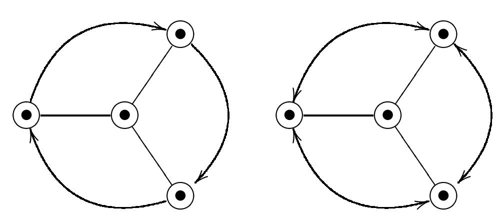
$({}^3D^{28}_{4,0},\ {}^6D^{28}_{4,0})$



$({}^3D^{9}_{4,1},\ {}^6D^{9}_{4,1})$



$({}^3D^{2}_{4,2},\ {}^6D^{2}_{4,2})$



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *Azumaya algebras A over a connected cubic cyclic (resp., noncyclic) étale extension R'/R with $\operatorname{ind} A = \deg A = 2$ and $\mathrm{cores}_{R'/R}([A]) = 0$, in the case of ${}^3D^9_{4,1}$ (resp., ${}^6D^9_{4,1}$);*
- *connected cubic cyclic (resp., noncyclic) étale extensions R'/R, in the case of ${}^3D^2_{4,2}$ (resp., ${}^6D^2_{4,2}$).*

Proof. By Lemma 3 and Proposition 3 G is anisotropic or quasi-split, or has the Tits index ${}^3D^9_{4,1}$ or ${}^6D^9_{4,1}$. Quasi-split groups are obviously in one-to-one correspondence with connected cubic étale extensions R'/R.

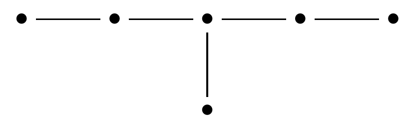
Let the Tits index be ${}^3D_{4,1}^9$ or ${}^6D_{4,1}^9$. The anisotropic kernel G_{an} is isomorphic to $R_{R'/R}(\mathrm{SL}_1(A))$ for some Azumaya algebra over a connected cubic étale extension R'/R with $\operatorname{ind} A = \deg A = 2$. The Cartan matrix of D_4 shows that $\alpha_2 = 2\omega_2 - \omega_1 - \omega_3 - \omega_4$, so by Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha_2') = -\operatorname{cores}_{R'/R}(\beta_{\mathrm{SL}_1(A)}(\omega_1)) = -\operatorname{cores}_{R'/R}([A]).$$

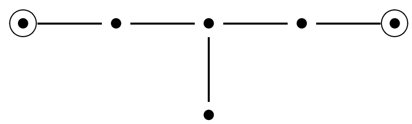
□

Theorem 3 ($^1\mathbf{E_6}$). *Every simple simply connected group G of inner type E_6 over R has one of the following Tits indices:*

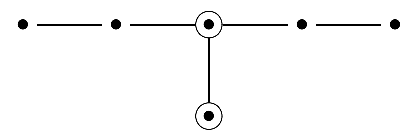
$({}^1E_{6,0}^{78})$



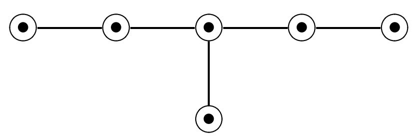
$({}^1E_{6,2}^{28})$



$({}^1E_{6,2}^{16})$



$({}^1E_{6,6}^{0})$

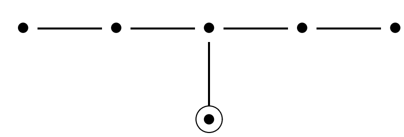


Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type D_4 over R with $\beta_H = 0$, in the case of ${}^1E_{6,2}^{28}$;*
- *Azumaya algebras A over R with* $\operatorname{ind} A = \deg A = 3$, *in the case of ${}^1E_{6,2}^{16}$.*

The only group of index ${}^1E_{6,6}^{0}$ is split.

Proof. By Lemma 3 and Proposition 3 the Tits index of G is either one of the listed above or the following:



Let us first exclude the latter case. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(A)$ for some Azumaya algebra A over R with $\operatorname{ind} A = \deg A = 6$. The Cartan matrix of E_6 shows that $\alpha_2 = 2\omega_2 - \omega_4$. By Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha_2') = -\beta_{\mathrm{SL}_1(A)}(\omega_3) = -3[A].$$

Hence $\exp A = 3$, but this contradicts Proposition 4.

In the case of ${}^1E_{6,2}^{28}$ the anisotropic kernel G_{an} is of type ${}^1D_4$. We have $\alpha_1 = 2\omega_1 - \omega_3$, $\alpha_6 = 2\omega_6 - \omega_5$, so

$$0 = \beta_{G_{an}}(\alpha_1') = -\beta_{G_{an}}(\omega_1);$$
$$0 = \beta_{G_{an}}(\alpha_6') = -\beta_{G_{an}}(\omega_4).$$

It follows that $\beta_{G_{an}} = 0$.

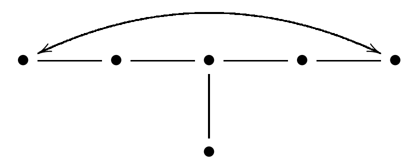
In the case of ${}^1E_{6,2}^{16}$ the anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(A_1) \times \mathrm{SL}_1(A_2)$ for some Azumaya algebras A_1, A_2 over R with $\operatorname{ind} A_1 = \deg A_1 = \operatorname{ind} A_2 = \deg A_2 = 3$. We have $\alpha_4 = 2\omega_4 - \omega_2 - \omega_3 - \omega_5$, so

$$0 = \beta_{G_{an}}(\alpha_4') = \beta_{\mathrm{SL}_1(A_1)}(\omega_1) - \beta_{\mathrm{SL}_1(A_2)}(\omega_1) = [A_1] - [A_2].$$

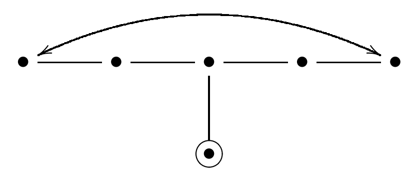
By Lemma 4 $A_1 \simeq A_2$. □

Theorem 3 ($\mathbf{^2E_6}$). *Every simple simply connected group G of type ${}^2E_6$ over R has one of the following Tits indices:*

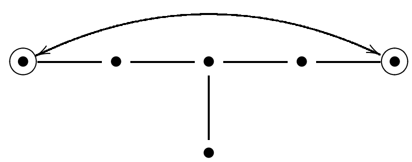
$({}^2E_{6,0}^{78})$



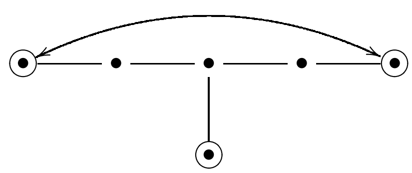
$({}^2E_{6,1}^{35})$



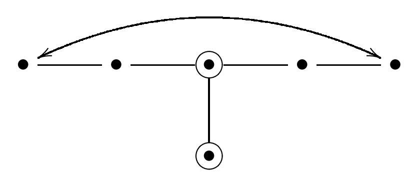
$({}^2E_{6,1}^{29})$



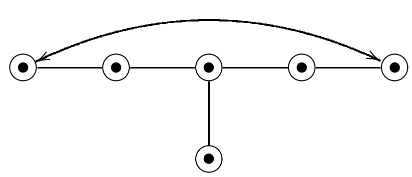
$({}^2E_{6,2}^{16'})$



$({}^2E_{6,2}^{16''})$



$({}^2E_{6,4}^{2})$



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type ${}^2A_5$ over R with $\beta_H(\omega_3) = 0$, in the case of ${}^2E_{6,1}^{35}$;*
- *simple simply connected anisotropic groups H of type ${}^2D_4$ over R with $\beta_{H_O}(\omega_3) = 0$, where O is the orbit corresponding to $\{3, 4\}$, in the case of ${}^2E_{6,1}^{29}$;*
- *simple simply connected anisotropic groups H of type ${}^2A_3$ over R with $\beta_H(\omega_2) = 0$ and $\beta_{H_O}(\omega_1) = 0$, where O is the orbit corresponding to $\{1, 3\}$, in the case of ${}^2E_{6,2}^{16'}$;*
- *Azumaya algebras A over a connected quadratic étale extension R'/R with* $\operatorname{ind} A = \deg A =$ 3 *and* $\mathrm{cores}_{R'/R}([A]) = 0$, *in the case ${}^2E_{6,2}^{16''}$;*
- *connected quadratic étale extensions R'/R, in the case of ${}^2E_{6,4}^{2}$.*

Proof. By Lemma 3 and Proposition 3 the Tits index of G is one of the listed above.

In the case of ${}^2E^{35}_{6,1}$ the anisotropic kernel G_{an} is a group of type ${}^2A_5$. The Cartan matrix of E_6 shows that $\alpha_2 = 2\omega_2 - \omega_4$. By Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha_2') = -\beta_{G_{an}}(\omega_3).$$

In the case of ${}^2E^{29}_{6,1}$ the anisotropic kernel G_{an} is a group of type ${}^2D_4$. Denote by O the orbit corresponding to $\{1, 6\}$. We have $\alpha_1 = 2\omega_1 - \omega_2$, so

$$0 = \beta_{G_{an\,O}}(\alpha_O') = -\beta_{G_{an\,O}}(\omega_3).$$

In the case of ${}^2E^{16'}_{6,1}$ the anisotropic kernel G_{an} is a group of type ${}^2A_3$. We have $\alpha_1 = 2\omega_1 - \omega_2$, $\alpha_2 = 2\omega_2 - \omega_4$, so

$$0 = \beta_{G_{an\,O}}(\alpha_O') = -\beta_{G_{an\,O}}(\omega_1);$$
$$0 = \beta_{G_{an}}(\alpha_4') = -\beta_{G_{an}}(\omega_2).$$

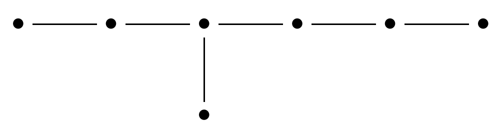
In the case of ${}^2E^{16''}_{6,1}$ the anisotropic kernel G_{an} is isomorphic to $R_{R'/R}(\mathrm{SL}_1(A))$, where A is an Azumaya algebra over R' with $\operatorname{ind} A = \deg A = 3$, $O \simeq \operatorname{Spec} R'$. We have $\alpha_4 = 2\omega_4 - \omega_2 - \omega_3 - \omega_5$, so

$$0 = \beta_{G_{an}}(\alpha_4') = \mathrm{cores}_{R'/R}(\beta_{\mathrm{SL}_1(A)}(\omega_1)) = \mathrm{cores}_{R'/R}([A]).$$

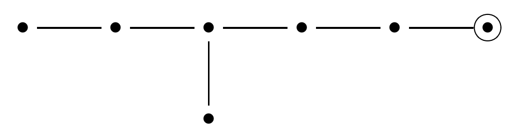
□

Theorem 3 ($\mathbf{E_7}$)**.** *Every simple simply connected group G of type E_7 over R has one of the following Tits indices:*

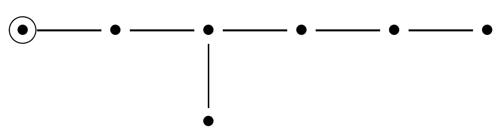
$(E^{133}_{7,0})$



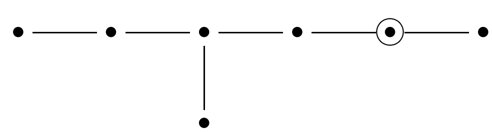
$(E^{78}_{7,1})$



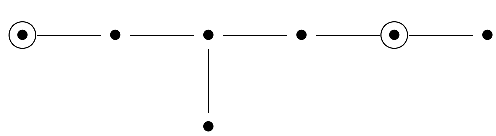
$(E^{66}_{7,1})$



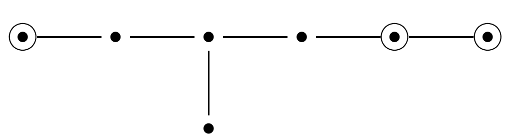
$(E^{48}_{7,1})$



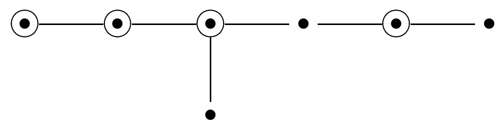
$(E^{31}_{7,2})$



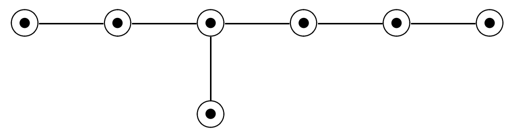
$(E^{28}_{7,3})$



$(E^{9}_{7,4})$



$(E^{0}_{7,7})$

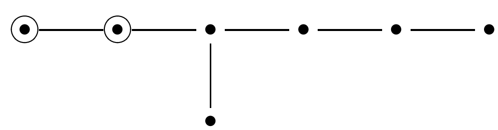


Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type ${}^1E_6$ over R with $\beta_H = 0$, in the case of $E^{78}_{7,1}$;*
- *simple simply connected anisotropic groups H of type ${}^1D_6$ over R with $\beta_H(\omega_5) = 0$, in the case of $E^{66}_{7,1}$;*
- *simple simply connected anisotropic groups H of type ${}^1D_5$ over R with $\beta_H(\omega_4) = [E]$, $\operatorname{ind} E = 2$, in the case of $E^{48}_{7,1}$;*
- *simple simply connected anisotropic groups H of type ${}^1D_4$ over R with $\beta_H(\omega_1) = 0$ and $\beta_H(\omega_3) = [E]$, $\operatorname{ind} E = 2$, in the case of $E^{31}_{7,2}$;*
- *simple simply connected anisotropic groups H of type ${}^1D_4$ over R with $\beta_H = 0$, in the case of $E^{28}_{7,3}$;*
- *Azumaya algebras A over R with $\operatorname{ind} A = \deg A = 2$, in the case of $E^{9}_{7,4}$.*

The only group of index $E^{0}_{7,7}$ is split.

Proof. By Lemma 3 and Proposition 3 the Tits index of G is either one of the listed above or the following:



Let us first exclude the latter case. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(A)$ for some Azumaya algebra A over R with $\operatorname{ind} A = \deg A = 6$. The Cartan matrix of E_7 shows that $\alpha_3 = 2\omega_3 - \omega_1 - \omega_4$. By Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha_3') = -\beta_{\mathrm{SL}_1(A)}(\omega_2) = 2[A].$$

Hence $\exp A = 2$, but this contradicts Proposition 4.

In the case of $E^{78}_{7,1}$ the anisotropic kernel G_{an} is of type ${}^1E_6$. We have $\alpha_7 = 2\omega_7 - \omega_6$, so

$$0 = \beta_{G_{an}}(\alpha_7') = -\beta_{G_{an}}(\omega_6).$$

It follows that $\beta_{G_{an}} = 0$.

In the case of $E^{66}_{7,1}$ the anisotropic kernel G_{an} is of type ${}^1D_6$. We have $\alpha_1 = 2\omega_1 - \omega_3$, so

$$0 = \beta_{G_{an}}(\alpha_1') = -\beta_{G_{an}}(\omega_5).$$

In the case of $E^{48}_{7,1}$ the anisotropic kernel G_{an} is isomorphic to $H \times \mathrm{SL}_1(E)$, where H is a group of type ${}^1D_6$, E is an Azumaya algebra over R with $\operatorname{ind} E = \deg E = 2$. We have $\alpha_6 = 2\omega_6 - \omega_5 - \omega_7$, so

$$0 = \beta_{G_{an}}(\alpha_6') = -\beta_H(\omega_4) - \beta_{\mathrm{SL}_1(E)}(\omega_1) = -\beta_H(\omega_4) + [E].$$

In the case of $E^{31}_{7,2}$ the anisotropic kernel G_{an} is isomorphic to $H \times \mathrm{SL}_1(E)$, where H is a group of type ${}^1D_4$, E is an Azumaya algebra over R with $\operatorname{ind} E = \deg E = 2$. We have $\alpha_1 = 2\omega_1 - \omega_3$, $\alpha_6 = 2\omega_6 - \omega_5 - \omega_7$, so

$$0 = \beta_{G_{an}}(\alpha_1') = -\beta_H(\omega_1);$$
$$0 = \beta_{G_{an}}(\alpha_6') = -\beta_H(\omega_3) - \beta_{\mathrm{SL}_1(E)}(\omega_1) = -\beta_H(\omega_3) + [E].$$

In the case of $E^{28}_{7,3}$ the anisotropic kernel G_{an} is of type ${}^1D_4$. We have $\alpha_1 = 2\omega_1 - \omega_3$, $\alpha_6 = 2\omega_6 - \omega_5 - \omega_7$, so

$$0 = \beta_{G_{an}}(\alpha'_1) = -\beta_{G_{an}}(\omega_1);$$
$$0 = \beta_{G_{an}}(\alpha'_6) = -\beta_{G_{an}}(\omega_3).$$

It follows that $\beta_{G_{an}} = 0$.

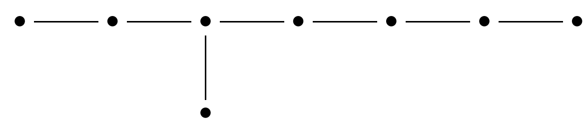
In the case of $E^{28}_{7,3}$ the anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(A_1)\times\mathrm{SL}_1(A_2)\times\mathrm{SL}_1(A_3)$ for some Azumaya algebras A_1, A_2, A_3 over R with $\operatorname{ind} A_1 = \deg A_1 = \operatorname{ind} A_2 = \deg A_2 = \operatorname{ind} A_3 = \deg A_3 = 2$. We have $\alpha_4 = 2\omega_4 - \omega_2 - \omega_3 - \omega_5$, $\alpha_6 = 2\omega_6 - \omega_5 - \omega_7$, so

$$0 = \beta_{G_{an}}(\alpha'_4) = -\beta_{\mathrm{SL}_1(A_1)}(\omega_1) - \beta_{\mathrm{SL}_1(A_2)}(\omega_1) = [A_1] - [A_2];$$
$$0 = \beta_{G_{an}}(\alpha'_6) = -\beta_{\mathrm{SL}_1(A_2)}(\omega_1) - \beta_{\mathrm{SL}_1(A_3)}(\omega_1) = [A_2] - [A_3].$$

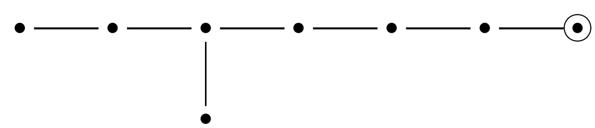
By Lemma 4 $A_1 \simeq A_2 \simeq A_3$. $\square$

Theorem 3 ($\mathbf{E_8}$). *Every simple simply connected group G of type E_8 over R has one of the following Tits indices:*

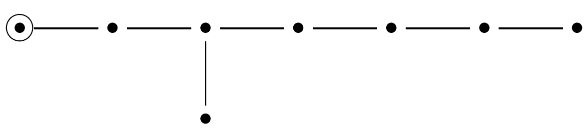
($E^{248}_{8,0}$)



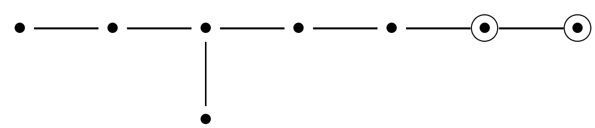
($E^{133}_{8,1}$)



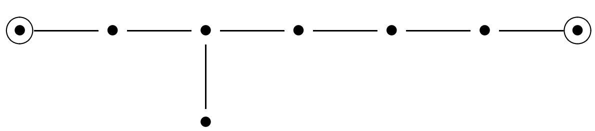
($E^{91}_{8,1}$)



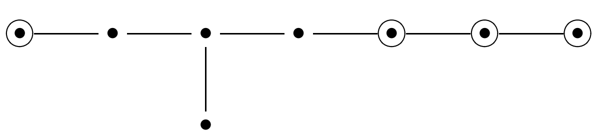
($E^{78}_{8,2}$)



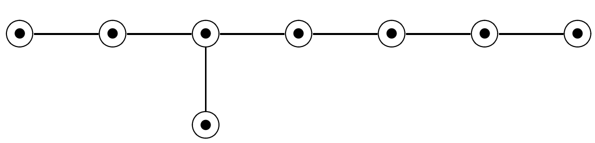
($E^{66}_{8,2}$)



($E^{28}_{8,4}$)



($E^{0}_{8,8}$)



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type E_7 over R with $\beta_H = 0$, in the case of $E^{133}_{8,1}$;*
- *simple simply connected anisotropic groups H of type ${}^1D_7$ over R with $\beta_H = 0$, in the case of $E^{91}_{8,1}$;*

- *simple simply connected anisotropic groups H of type ${}^1E_6$ over R with $\beta_H = 0$, in the case of $E^{78}_{8,\,2}$;*
- *simple simply connected anisotropic groups H of type ${}^1D_6$ over R with $\beta_H = 0$, in the case of $E^{66}_{8,\,2}$;*
- *simple simply connected anisotropic groups H of type ${}^1D_4$ over R with $\beta_H = 0$, in the case of $E^{28}_{8,\,4}$.*

The only group of index $E^{0}_{8,\,8}$ is split.

Proof. By Lemma 3 and Proposition 3 the Tits index of G is one of the listed above.

In the case of $E^{133}_{8,\,1}$ the anisotropic kernel G_{an} is of type E_7. The Cartan matrix of E_8 shows that $\alpha_8 = 2\omega_8 - \omega_7$. By Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha'_8) = -\beta_{G_{an}}(\omega_7).$$

It follows that $\beta_{G_{an}} = 0$.

In the case of $E^{91}_{8,\,1}$ the anisotropic kernel G_{an} is of type ${}^1D_7$. We have $\alpha_1 = 2\omega_1 - \omega_3$, so

$$0 = \beta_{G_{an}}(\alpha'_1) = -\beta_{G_{an}}(\omega_6).$$

It follows that $\beta_{G_{an}} = 0$.

In the case of $E^{78}_{8,\,2}$ the anisotropic kernel G_{an} is of type ${}^1E_6$. We have $\alpha_7 = 2\omega_7 - \omega_6 - \omega_8$, so

$$0 = \beta_{G_{an}}(\alpha'_7) = -\beta_{G_{an}}(\omega_6).$$

It follows that $\beta_{G_{an}} = 0$.

In the case of $E^{66}_{8,\,2}$ the anisotropic kernel G_{an} is of type ${}^1D_6$. We have $\alpha_1 = 2\omega_1 - \omega_3$, $\alpha_8 = 2\omega_8 - \omega_7$, so

$$0 = \beta_{G_{an}}(\alpha'_1) = -\beta_{G_{an}}(\omega_5);$$
$$0 = \beta_{G_{an}}(\alpha'_8) = -\beta_{G_{an}}(\omega_1).$$

It follows that $\beta_{G_{an}} = 0$.

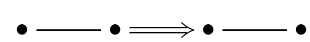
In the case of $E^{28}_{8,\,4}$ the anisotropic kernel G_{an} is of type ${}^1D_4$. We have $\alpha_1 = 2\omega_1 - \omega_3$, $\alpha_6 = 2\omega_6 - \omega_5 - \omega_7$, so

$$0 = \beta_{G_{an}}(\alpha'_1) = -\beta_{G_{an}}(\omega_1);$$
$$0 = \beta_{G_{an}}(\alpha'_6) = -\beta_{G_{an}}(\omega_3).$$

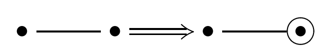
It follows that $\beta_{G_{an}} = 0$. □

Theorem 3 ($\mathbf{F_4}$). *Every simple simply connected group G of type F_4 over R has one of the following Tits indices:*

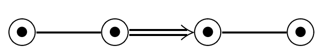
$(F^{52}_{4,\,0})$



$(F^{21}_{4,\,1})$

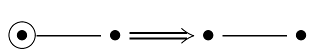


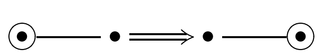
$(F^{0}_{4,\,4})$



Isomorphism classes of groups of index $F^{21}_{4,\,1}$ bijectively correspond to isomorphism classes of simple simply connected anisotropic groups H of type B_3 over R with $\beta_H = 0$. The only group of index $F^{0}_{4,\,4}$ is split.

Proof. By Lemma 3 and Proposition 3 the Tits index of G is either one of the listed above or one of the following:





Let us exclude the two latter cases. In the first of them the anisotropic kernel G_{an} is of type C_3. The Cartan matrix of F_4 shows that $\alpha_1 = 2\omega_1 - \omega_2$. By Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha_1') = -\beta_{G_{an}}(\omega_3).$$

It follows that $\beta_{G_{an}} = 0$, in contradiction with Proposition 5.

In the second case the anisotropic kernel G_{an} is of type C_2. We have $\alpha_4 = 2\omega_4 - \omega_3$, so

$$0 = \beta_{G_{an}}(\alpha_4') = -\beta_{G_{an}}(\omega_1).$$

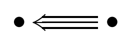
It follows that $\beta_{G_{an}} = 0$, in contradiction with Proposition 5.

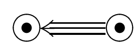
In the case of $F_{4,1}^{21}$ G_{an} is of type B_3. We have $\alpha_4 = 2\omega_4 - \omega_3$, so

$$0 = \beta_{G_{an}}(\alpha_4') = -\beta_{G_{an}}(\omega_3).$$

It follows that $\beta_{G_{an}} = 0$. ☐

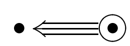
Theorem 3 ($\mathbf{G_2}$). *Every simple simply connected group G of type G_2 over R has one of the following Tits indices:*

($G_{2,0}^{14}$) 

($G_{2,2}^{0}$) 

The only group of index $G_{2,2}^{0}$ is split.

Proof. By Lemma 3 and Proposition 3 the Tits index of G is either one of the listed above or the following:



We need to exclude the latter case. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(A)$ for some Azumaya algebra A over R with $\operatorname{ind} A = \deg A = 2$. The Cartan matrix of G_2 shows that $\alpha_2 = 2\omega_2 - 3\omega_1$. By Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha_2') = -3\beta_{\mathrm{SL}_1(A)}(\omega_1) = -3[A].$$

But by Proposition 4 $2[A] = 0$, hence $[A] = 0$, a contradiction. ☐

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