Algebra Colloquium **16** : 4 (2009) 631–648

Algebra Colloquium

Normal Structure of Maximal Parabolic Subgroups in Chevalley Groups over Rings

Anastasia Stavrova
Department of Mathematics and Mechanics, St. Petersburg State University
198504 Peterhof, Russia
E-mail: a_stavrova@mail.ru

Received 12 December 2006
Revised 12 March 2007

Communicated by Fu-an Li

Abstract. We study the normal structure of maximal parabolic subgroups of a Chevalley group over a commutative ring. More precisely, we describe the subgroups of a maximal parabolic subgroup P normalized by the elementary part of its Levi subgroup. As a corollary, we obtain a description of the subgroups in P normalized by its elementary subgroup EP.

2000 Mathematics Subject Classification: 20G35, 20G15

Keywords: Chevalley group, parabolic subgroup, unipotent radical, Levi subgroup

1 Introduction

In the present paper, we describe the normal subgroups of maximal parabolic subgroups of a Chevalley group over an (almost) arbitrary commutative ring. On one hand, this is an extension of results on the internal structure of parabolic subgroups of Chevalley groups over fields, such as of Azad, Barry and Seitz [5] and Röhrle [12–14] (see also [10]). On the other hand, it can be considered as a first step towards the generalization of the main structure theorem for Chevalley groups to non-reductive algebraic groups.

To be more specific, we introduce the following notation (see also Section 2). Let Φ be a reduced irreducible root system of rank greater than one, and R be a commutative ring with 1. We denote by $G = G(\Phi, R)$ and $E = E(\Phi, R)$ a Chevalley group of type Φ over R and its elementary subgroup, respectively. The main structure theorem for Chevalley groups [1, 2, 4, 7, 8, 17] (see also [19] and [20] for further references and discussion) asserts that, roughly speaking, the $E(\Phi, R)$-normalized subgroups of $G(\Phi, R)$ are parametrized by ideals of the ring R. Namely, in most cases (for example, under the condition that 2 is invertible in R when $\Phi = B_l, C_l, F_4$ and $2, 3$ are invertible in R when $\Phi = G_2$, which we assume in this paper), for any $E(\Phi, R)$-normalized subgroup H, there exists a unique ideal $I \trianglelefteq R$ such that $E(\Phi, R, I) \le H \le C(\Phi, R, I)$, where $E(\Phi, R, I)$ and $C(\Phi, R, I)$ are respectively the relative elementary subgroup and the full congruence subgroup of level

I in G. Here we prove an analogous result for the normal subgroups in maximal parabolic subgroups of a Chevalley group. These subgroups can be described in terms of ideals in R too, with the only difference that we assign to each subgroup a system of ideals instead of a single ideal.

Let P_r be a maximal (standard) parabolic subgroup of a Chevalley group $G(\Phi, R)$ corresponding to the r-th simple root of Φ. We denote by EP_r the subgroup of P_r generated by those elementary root unipotents that are contained in P_r by its definition. The main goal of this paper is to describe the EP_r-normalized subgroups in P_r. Recall that the subgroup P_r can be represented as a semidirect product $P_r = L_r \ltimes U_r$ of its Levi subgroup L_r and its unipotent radical U_r. To obtain our main result, we actually describe the EL_r-normalized subgroups of P_r, where EL_r is the elementary subgroup of L_r.

The paper is organized as follows. In §2, we introduce the basic notation and recall some known results. In particular, we recall one of the forms of the main structure theorem for Chevalley groups, and prove its extension to reductive groups (Theorem 2.3), that we use later to describe the structure of L_r. In §3, we state our main results. In Theorem 3.1, we explicitly describe the EL_r-normalized subgroups in U_r, and Theorem 3.3 completes the description of the EL_r-normalized subgroups in P_r. The condition for an EL_r-normalized subgroup to be EP_r-normalized appears in Corollary 3.4. In §4, we prove some technical lemmas, mostly concerning the structure of the unipotent radical. In §5, we prove Theorem 3.1 in all cases except $\Phi = G_2$ and $r = 1$, and in §6, we investigate this special case. Finally, in §7, we establish Theorem 3.3.

2 Preliminaries

Let G be a group. For two elements $x, y \in G$, we denote by $[x, y] = xyx^{-1}y^{-1}$ their commutator, by $x^y = y^{-1}xy$ and ${}^y x = yxy^{-1}$ the conjugates of x by y and y^{-1}, respectively. We will use relations $[xy, z] = {}^x[y, z] \cdot [x, z]$ and $[x, yz] = [x, y] \cdot {}^y[x, z]$ without any further reference. We write $H \le G$ to denote that H is a subgroup of G and $H \trianglelefteq G$ to denote that H is a normal subgroup of G. For a subset $X \subseteq G$, we denote by $\langle X \rangle$ the subgroup of G generated by X, and by $\langle X \rangle^H$ the smallest overgroup of X normalized by $H \subseteq G$. For two subgroups $F, H \le G$, we denote by $[F, H]$ the corresponding relative commutator subgroup, i.e., the subgroup of G generated by all commutators $[f, h]$ for $f \in F$ and $h \in H$.

For a commutative ring R with 1, we denote by R^* the group of its invertible elements. We also write $I \trianglelefteq R$ to signify that I is an ideal of R.

Let Φ be a reduced irreducible root system of rank $l \ge 2$ in an l-dimensional Euclidean space with the inner product $(\ ,\ )$. Fix a system $\Pi = \{\alpha_1, \dots, \alpha_l\}$ of simple roots in Φ (the labelling follows Bourbaki [6]). Then any root $\alpha \in \Phi$ can be uniquely expressed as $\alpha = \sum_{r=1}^{l} m_r(\alpha)\alpha_r$. The integer $m_r(\alpha)$ is called the r-level of α and the sum $\sum_{r=1}^{l} m_r(\alpha)$ is called the height of α. We denote by $\widetilde{\alpha}$ the unique root of maximal height in Φ. We also write Φ^+ and Φ^- for the sets of positive and negative roots, respectively.

We denote by $G = G(\Phi, R)$ a Chevalley group of type Φ over a commutative ring R with identity, i.e., the group of R-points of the corresponding Chevalley–

Demazure group scheme. We fix a split maximal torus $T = T(\Phi, R)$ of G. We also write $E = E(\Phi, R)$ for the elementary subgroup of G, and $G_{\mathrm{ad}} = G_{\mathrm{ad}}(\Phi, R)$ and $E_{\mathrm{ad}} = E_{\mathrm{ad}}(\Phi, R)$ for the adjoint groups of the corresponding type.

The elementary subgroup $E = E(\Phi, R)$ is generated by all unipotent root elements $x_\alpha(\xi)$ for $\alpha \in \Phi$ and $\xi \in R$. With any additive subgroup A in R, we associate the following subgroups of E: $X_\alpha(A) = \{x_\alpha(\xi) \mid \xi \in A\}$ for all $\alpha \in \Phi$ (we also write X_α for $X_\alpha(R)$) and $E(\Phi, A) = \langle x_\alpha(\xi) \mid \alpha \in \Phi,\ \xi \in A\rangle$. When $A = I$ is an ideal in R, we denote by $E(\Phi, R, I)$ the relative elementary subgroup $E(\Phi, I)^{E(\Phi,R)}$ of level I.

Unless explicitly stated otherwise, we suppose that the ring R satisfies

NVB $$\begin{cases} 2 \in R^* & \text{when } \Phi = B_l,\, C_l,\, F_4, \\ 2, 3 \in R^* & \text{when } \Phi = G_2, \end{cases}$$

that is, all constants in the Chevalley commutator relations are invertible.

A subset S of Φ is called a (standard) parabolic subset if it is closed under addition and contains Φ^+. The maximal parabolic subsets are precisely the sets $S_r = \{\alpha \in \Phi \mid m_r(\alpha) \geq 0\}$ for $1 \leq r \leq l$. Any S_r can be written as a disjoint union $S_r = \Delta_r \cup \Sigma_r$, where $\Sigma_r = \{\alpha \in \Phi^+ \mid m_r(\alpha) > 0\}$ and $\Delta_r = \{\alpha \in \Phi \mid m_r(\alpha) = 0\}$.

For a fixed r, we set $\Delta_r^\pm = \Delta_r \cap \Phi^\pm$ and $\Sigma_r(m) = \{\alpha \in \Phi^+ \mid m_r(\alpha) = m\}$ for any $0 \leq m \leq m_r(\widetilde{\alpha})$. We introduce on $\Sigma_r(m)$ the following partial order:

$$\alpha \prec \beta \ \text{ if there exist } \ \beta_1, \beta_2, \dots, \beta_n \in \Delta_r^+ \ \text{ such that } \ \alpha + \beta_1 + \cdots + \beta_n = \beta.$$

We call a linear order $\leq$ on Φ^+ *level-adapted* (with respect to the fixed r) if $\alpha < \beta$ when $m_r(\alpha) < m_r(\beta)$ or $m_r(\alpha) = m_r(\beta)$ and $\alpha \prec \beta$.

It is well known (see, for example, [5]) that in $\Sigma_r(m)$, there exists a unique root of minimal height. By the same token, there exists a unique root of maximal height, which we denote $\widetilde{\alpha}_m$. We summarize some other properties of roots proved in [10] in the following lemma.

Lemma 2.1. [10, Lemmas 1–4] *Let* $\alpha_r \in \Pi$.

(i) *For any* $1 \leq m \leq m_r(\widetilde{\alpha})$, *there exists a unique root* $\widetilde{\alpha}_m \in \Sigma_r(m)$ *maximal with respect to* $\prec$.

(ii) *If one of the following holds*:
- $\Phi \neq A_l$, α_r *is the unique simple root joined with* $-\widetilde{\alpha}$ *in the extended Dynkin diagram of* Φ, *and* $m = m_r(\widetilde{\alpha}) = 2$,
- $\Phi = G_2$, $\alpha_r = \alpha_1$, *and* $m = 2$,

then $|\Sigma_r(m)| = 1$. *Otherwise*, $|\Sigma_r(m)| \geq 2$.

(iii) *For any* $1 \leq m \neq n \leq m_r(\widetilde{\alpha})$, *one has* $\widetilde{\alpha}_m - \widetilde{\alpha}_n \in \Phi$.

(iv) *For any* $1 \leq m \neq n \leq m_r(\widetilde{\alpha})$ *except* $m = 1$ *and* $n = 3$ *in the case when* $\Phi = G_2$ *and* $\alpha_r = \alpha_1$, *there exists* $\beta \in \Delta_r^-$ *such that exactly one of* $\widetilde{\alpha}_m + \beta$, $\widetilde{\alpha}_n + \beta$ *is a root.*

With any closed set $S \subseteq \Phi$, one can associate the subgroup $E(S) = E(S, R) = \langle x_\alpha(\xi) \mid \xi \in R,\ \alpha \in S\rangle$ of $E = E(\Phi, R)$. We denote $E(S_r)$ by EP_r. It follows directly from the Chevalley commutator relations that EP_r can be represented as the semidirect product $EP_r = EL_r \ltimes U_r$ of its unipotent radical $U_r = E(\Sigma_r)$ and its Levi subgroup $EL_r = E(\Delta_r)$. We will use the fact that any element $u \in U_r$ can be written in the form $u = \prod x_\alpha(u_\alpha)$, where the product is taken over all roots in Σ_r in a level-adapted order. For any $m \geq 1$, we set

$$U_r^m = E(\{\alpha \in \Phi \mid m_r(\alpha) \geq m\}) = \langle x_\alpha(\xi) \mid \xi \in R,\, \alpha \in \Sigma_r(k),\, k \geq m \rangle.$$

For any closed set of roots S, there exists also a natural algebraic subgroup $G(S) = G(S, R)$ of G corresponding to S (see [9] and [21]). When $S = S_r$ is a maximal parabolic set of roots, we denote this subgroup by P_r and call it a *maximal parabolic subgroup* of $G(\Phi, R)$. If $R = K$ is a field, it is just the usual r-th standard maximal parabolic subgroup of $G(\Phi, K)$, i.e., a maximal subgroup containing the standard Borel subgroup. In the case of a general commutative ring, however, P_r does not have to be maximal as an abstract subgroup (see [16] and [18]).

By definition, the subgroup P_r contains EP_r. It admits a Levi decomposition $P_r = L_r \ltimes U_r$ with the unipotent radical $U_r = E(\Sigma_r, R)$, where $L_r = G(\Delta_r, R)$ is a reductive algebraic group of type Δ_r. The elementary subgroup of L_r coincides with $EL_r = E(\Delta_r, R)$.

A parabolic subgroup P_r is called *extraspecial* if the corresponding simple root α_r is the unique root joined with $-\widetilde{\alpha}$ in the extended Dynkin diagram of Φ.

Since our focus is on the structure of individual parabolic subgroups and not on that of the ambient Chevalley group, we usually omit the index and write simply P, EP, L, U etc. instead of P_r, EP_r, L_r, U_r etc., keeping in mind that all these objects correspond to the same r between 1 and $l = \operatorname{rank} \Phi$.

To describe the normal structure of a parabolic subgroup, we have to consider the normal structure of its Levi subgroup, which is not a Chevalley group but a reductive algebraic group. Below we give an extension to reductive groups of the main structure theorem for Chevalley groups, more precisely, of the following statement of [2]: *A subgroup H of a Chevalley group $G(\Phi, R)$ is normalized by $E(\Phi, R)$ if and only if there exists an ideal $I \trianglelefteq R$ such that*

$$E(\Phi, R, I) \leq H \leq E^*(\Phi, R, I).$$

Here $E^*(\Phi, R, I) = \{x \in G \mid [x, E(\Phi, R)] \subseteq E(\Phi, R, I)\}$ is actually the full congruence subgroup of level I in $G(\Phi, R)$.

From now and only until the end of this section, let $G = G(\Psi, -)$ denote a *reductive* algebraic group of type Ψ, where the root system $\Psi = \Psi_1 + \cdots + \Psi_n$ has several irreducible components Ψ_i ($1 \leq i \leq n$). We use the fact (see [9, Exp. XXII 4.3] for details) that there exists a morphism $\pi^{\mathrm{ad}} : G \to G_{\mathrm{ad}}(\Psi, -)$ of schemes from G to the adjoint group of same type, whose (scheme-theoretic) kernel equals $\mathrm{Cent}(G)$, the group scheme centre of G. Denote the induced morphism $G(R) \to G_{\mathrm{ad}}(\Psi, R)$ by π_R^{ad}. Then $\mathrm{Cent}(G)(R) = (\ker \pi^{\mathrm{ad}})(R) = \ker \pi_R^{\mathrm{ad}}$ is a central subgroup of $G(R)$. It is also known from [9, Exp. XXII 4.3.1] that the restriction $\pi_R^{\mathrm{ad}}|_{E(\Psi,R)} : E(\Psi, R) \to E_{\mathrm{ad}}(\Psi, R)$ given by $x_\alpha(\xi) \mapsto x_\alpha^{\mathrm{ad}}(\xi)$ is surjective. In particular, this implies:

Lemma 2.2. *Let $G = G(\Psi, -)$ be a reductive group scheme of type*

$$\Psi = \Psi_1 + \cdots + \Psi_n,$$

where each Ψ_i is an irreducible root system of rank greater than one. Let R be a commutative ring with 1. *Let* $\mathrm{Cent}(G)$, $C(G(R))$, *and* $C_{G(R)}(E(R))$ *denote the group scheme centre of G, the abstract centre of the group $G(R)$, and the centralizer of the elementary group $E(R)$ in $G(R)$, respectively. Then*

$$C(G(R)) = C_{G(R)}(E(R)) = \mathrm{Cent}(G)(R) = \ker(G(R) \xrightarrow{\pi_R^{\mathrm{ad}}} G_{\mathrm{ad}}(R)).$$

Proof. The definition of $\mathrm{Cent}(G)$ implies that $\mathrm{Cent}(G)(R) \leq C(G(R))$. On the other hand, $C(G(R)) \leq C_{G(R)}(E(R))$. Since π_R^{ad} is surjective on elementary subgroups, we get

$$\pi_R^{\mathrm{ad}}(C_{G(R)}(E(R))) \leq C_{G_{\mathrm{ad}}(R)}(E_{\mathrm{ad}}(R)).$$

Since $G_{\mathrm{ad}}(R) \cong \prod_{i=1}^n G_{\mathrm{ad}}(\Psi_i, R)$ and $E_{\mathrm{ad}}(R) \cong \prod_{i=1}^n E_{\mathrm{ad}}(\Psi_i, R)$, one has

$$C_{G_{\mathrm{ad}}(R)}(E_{\mathrm{ad}}(R)) = \prod_{i=1}^n C_{G_{\mathrm{ad}}(\Psi_i,R)}(E_{\mathrm{ad}}(\Psi_i, R)).$$

Each $G_{\mathrm{ad}}(\Psi_i, R)$ is a Chevalley group, therefore, we can use the main theorem of [3], which says in particular that the centralizer of the elementary subgroup in a Chevalley group coincides with its center and is trivial if the group is of adjoint type. Hence, $\prod_{i=1}^n C_{G_{\mathrm{ad}}(\Psi_i,R)}(E_{\mathrm{ad}}(\Psi_i, R)) = 1$, and $C_{G(R)}(E(R)) \leq \ker \pi_R^{\mathrm{ad}}$, which completes the proof. □

Theorem 2.3. *Let $G = G(\Psi, R)$ be a reductive Chevalley–Demazure group scheme with a root system $\Psi = \Psi_1 + \cdots + \Psi_n$, where each Ψ_i is an irreducible root system of rank greater than one. Let R be a commutative ring with identity satisfying* **NVB**. *Then a subgroup H of G is normalized by the elementary subgroup $E = E(\Psi, R)$ of G if and only if there exist ideals $I_i \trianglelefteq R$ $(1 \leq i \leq n)$ such that*

$$E(I_1, \dots, I_n) \leq H \leq E^*(I_1, \dots, I_n),$$

where $E(I_1, \dots, I_n) = \prod_{i=1}^n E(\Psi_i, R, I_i)$ *and* $E^*(I_1, \dots, I_n) = \{x \in G \,|\, [x, E] \subseteq E(I_1, \dots, I_n)\}$.

Proof. When G is the adjoint reductive group $G_{\mathrm{ad}}(\Psi, R)$ of type Ψ, we have $G \cong \prod_{i=1}^n G_{\mathrm{ad}}(\Psi_i, R)$ and $E \cong \prod_{i=1}^n E_{\mathrm{ad}}(\Psi_i, R)$, and the statement follows easily from the one for Chevalley groups. Now let $G = G(\Psi, R)$ be an arbitrary reductive group. For any subgroup $H \leq G$ normalized by E, its image $\pi_R^{\mathrm{ad}}(H)$ under the morphism $\pi_R^{\mathrm{ad}} : G \to G_{\mathrm{ad}}(\Psi, R)$ is normalized by $E_{\mathrm{ad}}(\Psi, R)$, and therefore satisfies

$$\prod_{i=1}^n E_{\mathrm{ad}}(\Psi_i, R, I_i) \leq \pi_R^{\mathrm{ad}}(H) \leq \prod_{i=1}^n E^*_{\mathrm{ad}}(\Psi_i, R, I_i)$$

for some ideals $I_i \trianglelefteq R$ $(1 \leq i \leq n)$. Denote by C the centre of $G(\Psi, R)$. Then $\prod_{i=1}^n E(\Psi_i, R, I_i) \leq CH$ and hence

$$\big[E, \prod_{i=1}^n E(\Psi_i, R, I_i)\big] = \prod_{i=1}^n E(\Psi_i, R, I_i) = E(I_1, \dots, I_n) \leq H.$$

Moreover, $[H,E] \le C \cdot \prod_{i=1}^n E(\Psi_i,R,I_i)$ and thus $[[H,E],E] \le \prod_{i=1}^n E(\Psi_i,R,I_i)$. Therefore, it is enough to prove $[[H,E],E] = [H,E]$. We first show $[H,E] \le E$. Observe that $[[H,E],E] \le E$ implies that $E \cap [H,E]$ is normal in $[H,E]$. Further, $[[H,E],[H,E]] \le E \cap [H,E]$ implies that the quotient group $[H,E]/(E \cap [H,E])$ is abelian. For any $h \in H$, consider the map $\psi_h : E \to [H,E]/(E \cap [H,E])$ given by $x \mapsto [h,x] \cdot (E \cap [H,E])$. Since H is normalized by E, for any $x,y \in E$, we have $[h,xy]\,([h,x][h,y])^{-1} = [[h,y^{-1}],x] \in [[H,E],E] \le E \cap [H,E]$, hence ψ_h is a homomorphism. But since $E = [E,E]$, its image under ψ_h must be trivial, that is, $[h,x] \in E \cap [H,E]$ for all $x \in E$. Therefore, $[H,E] = E \cap [H,E]$. Now we see that $[[H,E],[H,E]] \le [[H,E],E]$, and to prove $[H,E] = [[H,E],E]$, it suffices to repeat the same trick for the quotient group $[H,E]/[[H,E],E]$. □

Corollary 2.4. *Under the hypothesis of Theorem* 2.3:

(i) $E(\Psi,R) \trianglelefteq G(\Psi,R)$.

(ii) *For any subgroup H of $G(\Psi,R)$ normalized by $E(\Psi,R)$, one has $[H,E(\Psi,R)] = E(I_1,\dots,I_n)$, where $I_1,\dots,I_n \trianglelefteq R$ are the ideals in the statement of Theorem* 2.3.

3 Statement of the Main Results

Recall that we have fixed some r between 1 and $l = \operatorname{rank}\Phi$ and consider the corresponding maximal parabolic subgroup $P = P_r$ with the unipotent radical $U = U_r$ and the Levi subgroup $L = L_r$.

We need two more definitions to state our results. Set $n = m_r(\widetilde{\alpha})$. We call a collection $\sigma = (\sigma_1,\dots,\sigma_n)$ of additive subgroups in R an *α_r-ladder* in R if

(1) $\sigma_i\sigma_j \subseteq \sigma_{i+j}$ whenever $1 \le i,\ j,\ i+j \le n$,

(2) $\sigma_i \trianglelefteq R$ whenever $|\Sigma_r(i)| \ge 2$.

We recall that by Lemma 2.1, one has $|\Sigma_r(i)| = 1$ only when P is extraspecial and $i = 2 = n$, or when $\Phi = G_2$, $P = P_1$ and $i = 2$. Hence, for the fixed r, there exists at most one σ_i which is not an ideal. We assign to a ladder σ in R the *ladder subgroup* $U(\sigma) = \langle x_\alpha(\xi)\,|\,\alpha \in \Sigma_r,\ \xi \in \sigma_{m_r(\alpha)}\rangle \le U$. In most cases, the EL-normalized subgroups in U are exhausted by the subgroups $U(\sigma)$. But in the exceptional case of $\Phi = G_2$ and $P = P_1$, we need a slightly larger class of subgroups which can be described as follows.

Let $\Phi = G_2$ and $P = P_1$. We call a pair (M,A) consisting of an R-submodule M of the free R-module $R \times R$ and an additive subgroup A of R a *coherent pair* if $\pi_1(M)^2 \le A$ and $\{0\} \times \pi_1(M)A \le M$, where π_1 is the projection of $R \times R$ onto the first factor. For any coherent pair (M,A) we set

$$\begin{aligned} U(M,A) = \big\langle & x_{\alpha_1}(\xi)\,x_{3\alpha_1+\alpha_2}(\eta),\ x_{\alpha_1+\alpha_2}(-\xi')\,x_{3\alpha_1+2\alpha_2}(\eta'),\ x_{2\alpha_1+\alpha_2}(\theta)\,| \\ & (\xi,\eta),(\xi',\eta') \in M,\ \theta \in A\big\rangle. \end{aligned}$$

Recall that α_1 and α_2 denote respectively the short and long simple roots of G_2.

Theorem 3.1. *Let H be a subgroup of U normalized by EL.*

(i) *If $\Phi \ne G_2$ or $\Phi = G_2$ and $P = P_2$, then $H = U(\sigma)$ for some α_r-ladder σ in R.*

(ii) *If $\Phi = G_2$ and $P = P_1$ corresponds to the short simple root, then $H = U(M, A)$ for some coherent pair (M, A).*

Theorem 3.1 is an essential ingredient of the description of all EL-normalized subgroups of P. Another major ingredient is the description of the normal structure of the Levi subgroup L. Since L is a reductive algebraic group, we can apply Theorem 2.3, which follows from the main structure theorem for Chevalley groups. Since the main structure theorem is known only for Chevalley groups of rank greater than one, we have to impose that all irreducible components of Δ_r are of rank greater than one. Thus, for the rest of the section, we assume that the pair (Φ, r) satisfies

$$\textbf{NRO} \qquad \begin{cases} r \neq 2,\, l-1 & \text{when } \Phi = A_l,\, B_l,\, C_l, \\ r \neq 2,\, l-2 & \text{when } \Phi = D_l, \\ r \neq 3,\, 4,\, 5 & \text{when } \Phi = E_6, \\ r \neq 3,\, 4,\, 6 & \text{when } \Phi = E_7, \\ r \neq 3,\, 4,\, 7 & \text{when } \Phi = E_8, \\ r \neq 2,\, 3 & \text{when } \Phi = F_4, \\ \Phi \neq G_2. \end{cases}$$

It is easy to see that in each of these cases, Δ_r consists of at most two irreducible components. Henceforth, we write $\Delta_r = \Delta_r^1 + \Delta_r^2$, allowing one of Δ_r^1, Δ_r^2 to be empty. We also suppose Δ_r^1 to be spanned by the part of $\Pi \setminus \{\alpha_r\}$ that contains simple roots with smaller indices. In particular, $\Delta_r^1 = \emptyset$, $\Delta_r^2 = \Delta_r$ whenever $r = 1$. We set

$$\begin{aligned} EL^i &= E(\Delta_r^i, R) = \langle x_\alpha(\xi) \mid \alpha \in \Delta_r^i,\, \xi \in R \rangle, \\ EL^i(R, I) &= E(\Delta_r^i, R, I) = \langle x_\alpha(\xi) \mid \alpha \in \Delta_r^i,\ \xi \in I \rangle^{EL^i} \end{aligned}$$

for any $I \trianglelefteq R$ and $i = 1, 2$.

Lemma 3.2. *Suppose that all irreducible components of Δ_r are of rank greater than one* (**NRO**). *Then a subgroup H of L is normalized by EL if and only if there exists a (unique) pair of ideals $I_1, I_2 \trianglelefteq R$ such that*

$$EL^1(R, I_1) \times EL^2(R, I_2) \leq H \leq EL^*(I_1, I_2),$$

where $EL^*(I_1, I_2) = \{x \in L \mid [x, EL] \subseteq EL^1(I_1) \times EL^2(I_2)\}$. Moreover, one has $[H, EL] = EL^1(R, I_1) \times EL^2(R, I_2)$.

Proof. It follows directly from Theorem 2.3. □

Now let H be an arbitrary subgroup of P normalized by EL, and let H_L denote the image of H in $P/U \cong L$. Then Lemma 3.2 and Theorem 3.1 imply that there exist a unique pair of ideals $I_1, I_2 \trianglelefteq R$ such that $EL^1(R, I_1) \times EL^2(R, I_2) \leq H_L \leq EL^*(I_1, I_2)$ and a unique α_r-ladder σ in R such that $H \cap U = U(\sigma)$. With the above notation, our main result looks as follows.

Theorem 3.3. *Let G be a Chevalley group of type Φ over a commutative ring R with 1 satisfying* **NVB**. *Suppose that all irreducible components of Δ_r are of rank* *greater than one, that is,* **NRO** *holds. Let H be a subgroup of P normalized by EL.*

(i) *If P is not extraspecial, then $H = H_L \ltimes (H \cap U)$, and therefore,*

$$\big(EL^1(R, I_1) \times EL^2(R, I_2)\big) \ltimes U(\sigma) \leq H \leq EL^*(I_1, I_2) \ltimes U(\sigma).$$

(ii) *If P is extraspecial, then*

$$\big(EL^1(R, I_1) \times EL^2(R, I_2)\big) \ltimes U(\sigma) \leq H \leq EL^*(I_1, I_2) \ltimes U(\sigma) X_{\widetilde{\alpha}}.$$

Conversely, any subgroup H of P satisfying the inclusions in (i) *or* (ii) *for some I_1, I_2, σ is normalized by EL.*

Remark 1. In (ii), $EL^*(I_1, I_2)$ does not necessarily normalize $U(\sigma)$, and H does not necessarily contain H_L.

Remark 2. In view of [15], Theorem 3.1 immediately implies that the description of EL-normalized subgroups in P is "standard" in the sense that there exists a family of subgroups H_α such that $[H_\alpha, EL] = H_\alpha$ and, for any EL-normalized subgroup H, one has $H_\alpha \leq H \leq \{x \in EP \,|\, [x, EL] \subseteq H_\alpha\}$ for a unique index α. However, the methods of [15] do not allow to find H_α explicitly.

Theorem 3.3 implies the following description of the EP-normalized subgroups in P.

Corollary 3.4. *Under the hypothesis of Theorem* 3.3*, an EL-normalized subgroup H of P is normalized by EP if and only if $I_1 + I_2 \subseteq \sigma_1 \subseteq \cdots \subseteq \sigma_{m_r(\widetilde{\alpha})}$ and $H_L \leq E^*(\Phi, R, \sigma_1)$.*

4 Preliminary Lemmas

Lemma 4.1. *For any $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$, and any $\xi \in R$, $I \trianglelefteq R$, one has $\big\langle [x_\alpha(\xi), X_\beta(I)],\ [x_\alpha(\xi), [x_\alpha(\xi), X_\beta(I)]],\ [x_\alpha(\xi), [x_\alpha(\xi), [x_\alpha(\xi), X_\beta(I)]]]\big\rangle^{X_\beta(I)} \geq X_{\alpha+\beta}(\xi I)$.*

Proof. It follows directly from the Chevalley commutator formula. Calculations can be found in the proof of [10, Lemma 5] or [18, Lemma 1]. □

We often use this lemma in the form $\langle x_\alpha(\xi)\rangle^{X_\beta(I)} \geq X_{\alpha+\beta}(\xi I)$.

Lemma 4.1 together with Lemma 2.1 implies the following two statements, which give us an intuition of what an EL-normalized subgroup of U looks like when it is generated by elementary root unipotents (in fact, all such subgroups are precisely the ladder subgroups $U(\sigma)$).

Lemma 4.2. *Let $\alpha \in \Sigma_r(m)$ for some $1 \leq m \leq m_r(\widetilde{\alpha})$, $\xi \in R$. If $|\Sigma_r(m)| \geq 2$, then $\langle x_\alpha(\xi)\rangle^{EL} \geq X_\gamma(\xi R)$ for all $\gamma \in \Sigma_r(m)$.*

Proof. It follows immediately from Lemma 4.1 and the existence of the unique $\preceq$-maximal root $\widetilde{\alpha}_m \in \Sigma_r(m)$ (Lemma 2.1). □

Lemma 4.3. *Let $\alpha \in \Sigma_r(n)$ and $\beta \in \Sigma_r(m)$ for some $1 \le m, n \le m_r(\widetilde{\alpha})$ such that $m + n \le m_r(\widetilde{\alpha})$. Let $\xi, \eta \in R$. Then $\langle x_\alpha(\xi), x_\beta(\eta)\rangle^{EL} \ge X_\gamma(\xi\eta R)$ for all $\gamma \in \Sigma_r(m+n)$.*

Proof. Denote $H = \langle x_\alpha(\xi), x_\beta(\eta)\rangle^{EL}$. By Lemma 2.1, at least one of $\Sigma_r(n)$, $\Sigma_r(m)$ consists of more than one element, suppose that it is $\Sigma_r(n)$ (in the case when $m = n$ and $|\Sigma_r(m)| = 1$, the inequality $m + n \le m_r(\widetilde{\alpha})$ fails by Lemma 2.1). Then $X_{\widetilde{\alpha}_n}(\xi R) \le H$ by Lemma 4.2. By Lemma 2.1, we have $\delta = \widetilde{\alpha}_{n+m} - \widetilde{\alpha}_n \in \Sigma_r(m)$. If $\Sigma_r(m)$ consists of one element, then we have $\delta = \beta$ and $x_\delta(\eta) \in H$, otherwise $x_\delta(\eta) \in H$ by Lemma 4.2. Since $X_{\widetilde{\alpha}_n}(\xi R)$ normalizes H, it follows from Lemma 4.1 that $X_{\widetilde{\alpha}_n+\delta}(\xi\eta R) = X_{\widetilde{\alpha}_{n+m}}(\xi\eta R) \le H$. If $\Sigma_r(n+m)$ has more than one element, then we use Lemma 4.2 again to finish the proof. □

Recall that for any $1 \le m \le m_r(\widetilde{\alpha})$, one has $U^{m+1} \trianglelefteq U^m$ and $(U^m)^{EL} = U^m$. The quotient $V_m = U^m/U^{m+1}$ is isomorphic to the direct product of all $X_\alpha(R)$, $\alpha \in \Sigma_r(m)$, and possesses a natural structure of an $R[EL]$-module: the action of EL is induced from U^m, and scalars $c \in R$ act as $c \cdot x_\alpha(\xi) = x_\alpha(c\xi)$. These modules V_m are called *internal Chevalley modules*. In the case when R is a field, their structure is very well understood (see [5, 10, 11]). In particular, in this case, all V_m are irreducible; but this does not hold in general.

We observe that it follows from Lemma 4.1 and the existence of the unique $\preceq$-maximal root $\widetilde{\alpha}_m \in \Sigma_r(m)$ that EL acts on V_m non-trivially if and only if $|\Sigma_r(m)| \ge 2$.

Lemma 4.4. *Suppose $1 \le m \le m_r(\widetilde{\alpha})$ and $|\Sigma_r(m)| \ge 2$. Then the $\mathbb{Z}[EL]$-submodules of V_m are just the direct products $V_m(I) = \prod_{\alpha\in\Sigma_r(m)} X_\alpha(I)$ for $I \trianglelefteq R$.*

Proof. Let W be a $\mathbb{Z}[EL]$-submodule of V_m. We will prove that $u = \prod x_\alpha(u_\alpha) \in W$ implies $x_\alpha(u_\alpha) \in W$ for all $\alpha \in \Sigma_r(m)$. Then Lemma 4.2 applied to the inverse image of W in U^m tells that $W = V_m(I)$, where I coincides with the ideal generated by all u_α for $\alpha \in \Sigma_r(m)$ and $u \in W$. For any $u = \prod x_\alpha(u_\alpha) \in V_m$, we set $R(u) = \{\alpha \in \Sigma_r(m) \mid \exists\, \beta \preceq \alpha : u_\beta \ne 0\}$. We argue by induction on $R(u)$ (with respect to inclusion of sets). If $R(u)$ is the least possible, i.e., $R(u) = \{\widetilde{\alpha}_m\}$, the statement is trivial. Otherwise consider any $\preceq$-minimal root α in $R(u)$. By Lemma 2.1, there exists a root $\beta \in \Delta_r^+$ such that $\alpha + \beta \in \Sigma_r(m)$. Let $v = [x_\beta(1), u] = \prod x_\alpha(v_\alpha)$. It is easy to see that $R(v) \subseteq R(u)$. Moreover, since α is minimal, we have $v_\alpha = 0$, which implies $R(v) \ne R(u)$, and $v_{\alpha+\beta} = cu_\alpha$, where c is a structure constant of the Chevalley group. Applying the induction hypothesis to v, we find that $x_{\alpha+\beta}(cu_\alpha) \in W$. Since by the **NVB** condition, c is invertible, it follows from Lemma 4.2 that $x_\alpha(u_\alpha) \in W$. □

We have defined in §3 a ladder in R, which is a collection $\sigma = (\sigma_1, \dots, \sigma_n)$ of additive subgroups of R with certain properties. More precisely, since by Lemma 2.1 one has $|\Sigma_r(i)| = 1$ only when P is extraspecial and $i = 2 = n$, or when $\Phi = G_2$, $P = P_1$ and $i = 2$, all these additive subgroups, save maybe one, are ideals. If $\sigma_i \trianglelefteq R$ for all $1 \le i \le m_r(\widetilde{\alpha})$, we say that σ is an *ideal ladder*.

Recall that we assign to a ladder σ the ladder subgroup $U(\sigma)$. Condition (1) (and the fact that each σ_i is an additive subgroup) in the definition of a ladder implies that $U(\sigma)$ contains no unipotent root elements but those which were prescribed. It admits the following explicit description.

Lemma 4.5. *For any α_r-ladder σ in R, one has*

$$U(\sigma)=\big\{u\in U \,\big|\, \text{for any decomposition } u=\prod_{\alpha\in\Sigma_r}x_\alpha(u_\alpha),\ u_\alpha\in\sigma_{m_r(\alpha)} \text{ for all } \alpha\big\}.$$

Proof. One inclusion is obvious. Now suppose $u=x_{\gamma_1}(\xi_1)x_{\gamma_2}(\xi_2)\cdots x_{\gamma_k}(\xi_k)$ for some $\gamma_1,\dots,\gamma_k\in\Sigma_r$ and $\xi_i\in\sigma_{m_r(\gamma_i)}$. It is easy to see that if $u=\prod_{\alpha\in\Sigma_r}x_\alpha(u_\alpha)$, where the product is taken in some fixed order, then by Chevalley commutator relations, $u_\alpha-\sum_{\gamma_i=\alpha}\xi_i$ is an integral linear combination of products of the form $\xi_{i_1}\cdots\xi_{i_k}$, where $\gamma_{i_1}+\cdots+\gamma_{i_k}=\alpha$. Since this implies $m_r(\gamma_{i_1})+\cdots+m_r(\gamma_{i_k})=m_r(\alpha)$, we have $u_\alpha\in\sigma_{m_r(\alpha)}$ by the definition of a ladder. □

On the other hand, condition (2) in the definition of a ladder guarantees that $U(\sigma)$ is normalized by EL.

Lemma 4.6. *For any α_r-ladder σ, the group $U(\sigma)$ is normalized by EL. If σ is an ideal ladder, $U(\sigma)$ is also normalized by L.*

Proof. The first assertion follows directly from the Chevalley commutator relations. Consider the case when σ is an ideal ladder. Let us study the action of the whole Levi subgroup $L=L(R)$ on the unipotent radical $U=U(R)$. Fixing a level-adapted order on Φ^+, we get an isomorphism of the scheme U onto an affine variety, which induces the isomorphism of $U(R)$ onto R^k via the choice of basis $e_1=x_{\beta_1}(1),\dots,e_k=x_{\beta_k}(1)$, where $\Sigma=\{\beta_1,\dots,\beta_k\}$, the roots listed in the corresponding order. Since any element of L acts on U as an algebraic automorphism, for any $1\le i\le k$, we can define a map (of sets) $\varphi_i(R):L(R)\to R[x_1,\dots,x_k]$ so that for any $g\in L(R)$ and any $\lambda_i\in R$ $(1\le i\le k)$, we have

$$g\cdot\Big(\sum_{i=1}^k\lambda_ie_i\Big)=\sum_{i=1}^k\varphi_i(g)(\lambda_1,\dots,\lambda_k)e_i.$$

For any fixed i, these maps $\varphi_i(R)$ define a natural transformation $L\to X$ of the functor $L=\mathrm{Hom}(A,-)$: Rings $\to$ Sets, where A denotes the Hopf algebra of L, to the functor X : Rings $\to$ Sets, $X(R)=R[x_1,\dots,x_n]$. Therefore, by the Yoneda Lemma, there exists an element $p_i=\sum_{j_1,\dots,j_k\ge0}p_{ij_1\cdots j_k}x_1^{j_1}\cdots x_k^{j_k}\in A[x_1,\dots,x_k]$ such that $\varphi_i(g)=X(g)(p_i)=\sum_{j_1,\dots,j_k\ge0}g(p_{ij_1\cdots j_k})x_1^{j_1}\cdots x_k^{j_k}$ for any ring R and any $g\in L(R)=\mathrm{Hom}(A,R)$.

Consider any elementary root unipotent $x_\beta(\xi)\in U(\sigma)$, $\beta=\beta_m$. The Chevalley commutator relations and the natural properties of the maximal torus imply that for any $g\in T\cdot EL$ and any index $(j_1,\dots,j_k)$ with $j_l=0$ for $l\ne m$, we have $g(p_{ij_1\cdots j_k})=0$ whenever $m_r(\beta_i)<m_r(\beta)$ (this amounts to $[g,x_\beta(\xi)]\in U^{m_r(\beta)}$) or $j_m=0$ (this amounts to the fact that all root factors of $[g,x_\beta(\xi)]$ have a multiple of ξ as their coefficient). Hence, for the same $i,j_1,\dots,j_k$, we have $g(p_{ij_1\cdots j_k})=0$

for any $g \in \Omega(R)$, where Ω is the big cell of L. Since Ω is a dense open subscheme of $\operatorname{Spec} A = L$, the same equalities also hold for all $g \in L(R)$; but they imply $[g, x_\beta(\xi)] \in U(\sigma)$. □

This lemma immediately implies the following refinement of Lemma 4.4.

Corollary 4.7. *Suppose $1 \le m \le m_r(\widetilde{\alpha})$ and $|\Sigma_r(m)| \ge 2$. Then the $\mathbb{Z}[L]$-submodules of V_m are precisely $V_m(I)$ for $I \trianglelefteq R$.*

It follows from Lemma 4.5 that the intersection of ladder subgroups is a ladder subgroup, corresponding to the intersection of ladders. The next statement implies that the ladder subgroups form a lattice with respect to the natural operations.

Corollary 4.8. *Consider a set of α_r-ladders σ^t $(t \in \mathcal{T})$ in R. Then $\langle U(\sigma^t) \,|\, t \in \mathcal{T}\rangle = U(\sigma)$, where σ is an α_r-ladder defined by $\sigma_m = \sum\limits_{t \in \mathcal{T}} \sigma^t_m + \sum\limits_{m_1+m_2=m} \sigma_{m_1}\sigma_{m_2}$ for $1 \le m \le m_r(\widetilde{\alpha})$.*

Proof. Obviously, we have $\langle U(\sigma^t) \,|\, t \in \mathcal{T}\rangle \le U(\sigma)$. On the other hand, EL normalizes $\langle U(\sigma^t) \,|\, t \in \mathcal{T}\rangle$ since it normalizes each $U(\sigma^t)$ for $t \in \mathcal{T}$. Then it follows from Lemmas 4.2 and 4.3 that $\langle U(\sigma^t) \,|\, t \in \mathcal{T}\rangle \ge X_\alpha(\sigma_m)$ for all $1 \le m \le m_r(\widetilde{\alpha})$ and $\alpha \in \Sigma_r(m)$, thus we have $\langle U(\sigma^t) \,|\, t \in \mathcal{T}\rangle \ge U(\sigma)$. □

Corollary 4.9. *For any α_r-ladders τ and ρ in R, one has $[U(\tau), U(\rho)] = U(\sigma)$, where σ is an α_r-ladder defined by $\sigma_m = \sum\limits_{m_1+m_2=m} \tau_{m_1}\rho_{m_2}$ for all $1 \le m \le m_r(\widetilde{\alpha})$. In particular, $[U^k, U^n] = U^{k+n}$ for all natural integers k, n.*

Proof. It is easy to see that σ is indeed a ladder. It also follows directly from the Chevalley commutator formula that $[U(\tau), U(\rho)] \le U(\sigma)$. Conversely, consider any $1 \le m_1, m_2 \le m_r(\widetilde{\alpha})$ such that $m_1 + m_2 \le m_r(\widetilde{\alpha})$ and any $\xi_1 \in \tau_{m_1}$, $\xi_2 \in \rho_{m_2}$. By Lemma 2.1, we have $\widetilde{\alpha}_{m_1+m_2} - \widetilde{\alpha}_{m_1} \in \Phi$ and hence $\widetilde{\alpha}_{m_1+m_2} - \widetilde{\alpha}_{m_1} \in \Sigma_r(m_2)$. Note that by the definition of a ladder, at least one of τ_{m_1}, ρ_{m_2} is an ideal. Then since both $U(\sigma)$ and $U(\rho)$ normalize $[U(\tau), U(\rho)]$, Lemma 4.1 implies that $x_{\widetilde{\alpha}_{m_1+m_2}}(\xi_1\xi_2) \in [U(\tau), U(\rho)]$. Since $[U(\tau), U(\rho)]$ is also normalized by EL, we get $[U(\tau), U(\rho)] \ge U(\sigma)$ applying Lemma 4.2. □

5 Proof of Theorem 3.1: the Generic Case

In this section, we prove that in all cases except $\Phi = G_2$ and $r = 1$, the ladder subgroups $U(\sigma)$ studied in the previous section exhaust all subgroups in U normalized by EL. It is the statement (i) of Theorem 3.1. Throughout this section, we suppose $P = P_r \ne P_1$ when $\Phi = G_2$.

Lemma 5.1. *For any $1 \le m < k \le m_r(\widetilde{\alpha})$, one has*

$$x_{\widetilde{\alpha}_m}(\xi),\, x_{\widetilde{\alpha}_k}(\theta) \in \big\langle x_{\widetilde{\alpha}_m}(\xi)\, x_{\widetilde{\alpha}_k}(\theta)\big\rangle^{EL} U^{k+1}.$$

Proof. Since $U^{k+1} \trianglelefteq U$ and U^{k+1} is EL-invariant, we identify all the unipotent root elements with their images in $V = U/U^{k+1}$ and prove that $x_{\widetilde{\alpha}_m}(\xi)$, $x_{\widetilde{\alpha}_k}(\theta) \in$

$\langle x_{\widetilde{\alpha}_m}(\xi)\, x_{\widetilde{\alpha}_k}(\theta)\rangle^{EL} = H \le V$. Note that $X_{\widetilde{\alpha}_k}$ is central in V. By Lemma 2.1, there exists a root $\beta \in \Delta_r$ such that only one of $\widetilde{\alpha}_m + \beta$, $\widetilde{\alpha}_k + \beta$ is in Φ. Consider two cases.

Case 1: $\widetilde{\alpha}_k + \beta \in \Phi$. Then $[x_{\widetilde{\alpha}_m}(\xi), X_\beta] = 1$ and hence $[x_{\widetilde{\alpha}_m}(\xi)\, x_{\widetilde{\alpha}_k}(\theta), X_\beta] = [x_{\widetilde{\alpha}_k}(\theta), X_\beta]$. Since $X_{\widetilde{\alpha}_k}$ is central in V, both commutators $[x_{\widetilde{\alpha}_k}(\xi), [x_{\widetilde{\alpha}_k}(\xi), X_\beta]]$ and $[x_{\widetilde{\alpha}_k}(\xi), [x_{\widetilde{\alpha}_k}(\xi), [x_{\widetilde{\alpha}_k}(\xi), X_\beta]]]$ are trivial. Then by Lemma 4.1 $X_{\widetilde{\alpha}_k+\beta}(\theta R) \le H$ and hence $X_{\widetilde{\alpha}_k}(\theta R) \le H$.

Case 2: $\widetilde{\alpha}_m + \beta \in \Phi$. Then $[x_{\widetilde{\alpha}_k}(\xi), X_\beta] = 1$ and hence $[x_{\widetilde{\alpha}_m}(\xi)\, x_{\widetilde{\alpha}_k}(\theta), X_\beta] = [x_{\widetilde{\alpha}_m}(\xi), X_\beta] \subseteq H$. The centrality of $X_{\widetilde{\alpha}_k}$ in V also implies

$$[x_{\widetilde{\alpha}_m}(\xi)\, x_{\widetilde{\alpha}_k}(\theta), [x_{\widetilde{\alpha}_m}(\xi), X_\beta]] = [x_{\widetilde{\alpha}_m}(\xi), [x_{\widetilde{\alpha}_m}(\xi), X_\beta]],$$

and further,

$$[x_{\widetilde{\alpha}_m}(\xi)\, x_{\widetilde{\alpha}_k}(\theta), [x_{\widetilde{\alpha}_m}(\xi)\, x_{\widetilde{\alpha}_k}(\theta), [x_{\widetilde{\alpha}_m}(\xi), X_\beta]]] = [x_{\widetilde{\alpha}_m}(\xi), [x_{\widetilde{\alpha}_m}(\xi), [x_{\widetilde{\alpha}_m}(\xi), X_\beta]]].$$

Then by Lemma 4.1, we have $X_{\widetilde{\alpha}_m+\beta}(\xi R) \le H$ and $X_{\widetilde{\alpha}_m}(\xi R) \le H$. □

The following lemma shows that any EL-normalized subgroup H in U is generated by elementary root unipotents.

Lemma 5.2. *Let $u = \prod x_\alpha(u_\alpha) \in U$, where the product is taken over all roots of Σ_r in a level-adapted order. Then for all $\alpha \in \Sigma_r$, one has $x_\alpha(u_\alpha) \in \langle u\rangle^{EL}$.*

Proof. Denote $\langle u\rangle^{EL}$ by H. Fix a level-adapted order $\le$ on Φ^+. We argue by the inverse induction on the $\le$-minimal root $\alpha \in \Sigma_r$ satisfying $u_\alpha \ne 0$. Denote this root by $r(u)$. If $r(u)$ is the maximal possible, i.e., $r(u) = \widetilde{\alpha}$, then $u = x_{\widetilde{\alpha}}(u_{\widetilde{\alpha}})$ and the claim is trivial. Suppose $r(u) < \widetilde{\alpha}$, $r(u) \in \Sigma_r(m)$. We will prove $x_{r(u)}(u_{r(u)}) \in H$. Since H is normalized by EL, by Lemmas 4.4 and 4.2, we may assume $r(u) = \widetilde{\alpha}_m$ without loss of generality. Then we have $x_{\widetilde{\alpha}_m}(u_{\widetilde{\alpha}_m})\, v \in H$ for some $v \in U^{m+1}$.

We will show that H contains $x_{\widetilde{\alpha}_m}(u_{\widetilde{\alpha}_m}) \cdot v'$, where $r(v') > r(v)$. Let $m_r(r(v)) = k$. If $r(v) \ne \widetilde{\alpha}_k$, by Lemma 2.1, there exists a root $\beta \in \Delta_r^+$ such that $r(v) + \beta \in \Sigma_r(k)$. The element $w = [x_\beta(1), x_{\widetilde{\alpha}_m}(u_{\widetilde{\alpha}_m})\, v] = x_{\widetilde{\alpha}_m}(u_{\widetilde{\alpha}_m})\, [x_\beta(1), v]\, x_{\widetilde{\alpha}_m}(-u_{\widetilde{\alpha}_m})$ is in H, and the Chevalley commutator formula implies that $r(w) = r(v) + \beta$ and $w_{r(v)+\beta} = cv_{r(v)}$, $c \in R^*$. By the induction hypothesis, $x_{r(v)+\beta}(w_{r(v)+\beta}) \in \langle w\rangle^{EL} \le H$, hence by Lemma 4.2, $x_{r(v)}(v_{r(v)}) \in H$. Then H contains $x_{\widetilde{\alpha}_m}(u_{\widetilde{\alpha}_m})v \cdot x_{r(v)}(-v_{r(v)}) = x_{\widetilde{\alpha}_m}(u_{\widetilde{\alpha}_m})\, v'$, where $r(v') > r(v)$.

Suppose now that $r(v) = \widetilde{\alpha}_k$, i.e., $v \in x_{\widetilde{\alpha}_k}(v_{\widetilde{\alpha}_k})\, U^{k+1}$. As U^{k+1} is invariant under the action of EL, one has $\langle x_{\widetilde{\alpha}_m}(u_{\widetilde{\alpha}_m})\, v\rangle^{EL} U^{k+1} = \langle x_{\widetilde{\alpha}_m}(u_{\widetilde{\alpha}_m})\, x_{\widetilde{\alpha}_k}(v_{\widetilde{\alpha}_k})\rangle^{EL} U^{k+1}$. Then by Lemma 5.1, $x_{\widetilde{\alpha}_m}(u_{\widetilde{\alpha}_m}) \cdot v' \in \langle x_{\widetilde{\alpha}_m}(u_{\widetilde{\alpha}_m})\, v\rangle^{EL} \le H$ for some $v' \in U^{k+1}$. Obviously, $m_r(r(v')) \ge k+1$, hence $r(v') > r(v)$. □

The following statement is a corollary and in a sense a refinement of Lemma 5.2.

Lemma 5.3. *Let $u = \prod x_\alpha(u_\alpha) \in U$, where the product is taken over all roots of Σ_r in a level-adapted order. Then for all $\alpha \in \Sigma_r(m)$ such that $|\Sigma_r(m)| \ge 2$, one* *has $x_\alpha(u_\alpha) \in \langle [u, EL] \rangle$. In particular, if P is not extraspecial, then $u \in \langle [u, EL] \rangle$ for all $u \in U$.*

Proof. Let $H = \langle [u, EL] \rangle$. It is clear that H is normalized by EL. We denote by $r(u)$ the minimal (with respect to the level-adapted order mentioned in the statement) root $\alpha \in \Sigma_r$ satisfying $u_\alpha \neq 0$. Suppose $r(u) \in \Sigma_r(m)$ and $|\Sigma_r(m)| > 1$. We will prove $x_{r(u)}(u_{r(u)}) \in H$. Since $|\Sigma_r(m)| > 1$, by Lemma 2.1, there exists $\beta \in \Delta_r$ such that $r(u) + \beta \in \Sigma_r(m)$. Moreover, if $r(u) \neq \widetilde{\alpha}_m$, then one can take $\beta \in \Delta_r^+$, and if $r(u) = \widetilde{\alpha}_m$, then obviously $\beta \in \Delta_r^-$. One can represent $[u, x_\beta(1)] = v$ as a product $\prod x_\alpha(v_\alpha)$, $\alpha \in \Sigma_r$. It is easy to see that in both cases, the commutator relations imply $r(v) \in \Sigma_r(m)$ and $v_{r(v)} = cu_{r(u)}$, where $c \in R^*$ by **NVB**. Then by Lemmas 5.2 and 4.2, we get $x_{r(u)}(u_{r(u)}) \in \langle v \rangle^{EL} \leq H$.

Furthermore, since we exclude the case $(\Phi, r) = (G_2, 1)$, by Lemma 2.1, we have $|\Sigma_r(m)| = 1$ only if $r(u) = \widetilde{\alpha}$. It means that the claim can be proved by the inverse induction on $r(u)$. Indeed, it has been proved already that $x_{r(u)}(u_{r(u)}) \in H$, and if $u = x_{r(u)}(u_{r(u)}) \cdot u'$, then for any $z \in EL$, we have $[u, z] = [x_{r(u)}(u_{r(u)}) \cdot u', z] = {}^{x_{r(u)}(u_{r(u)})}[u', z][x_{r(u)}(u_{r(u)}), z]$, which implies $\langle [u', EL] \rangle \leq H$. $\square$

It is actually proved by now that under the assumption $(\Phi, r) \neq (G_2, 1)$, any EL-normalized subgroup in U must be of the form $U(\sigma)$, but let us write it out formally for completeness.

Proof of Theorem 3.1(i). Let H be a subgroup of U normalized by EL. Set $\sigma(\alpha) = \{\xi \in R \mid x_\alpha(\xi) \in H\}$ for any $\alpha \in \Sigma_r$. It is clear that $\sigma(\alpha)$ is an additive subgroup of R; moreover, for any $1 \leq m \leq m_r(\widetilde{\alpha})$ and $\alpha \in \Sigma_r(m)$, by Lemma 4.2, $\sigma(\alpha) \trianglelefteq R$ whenever $|\Sigma_r(m)| \geq 2$, and $\sigma(\alpha) = \sigma(\alpha')$ for all $\alpha' \in \Sigma_r(m)$. Set $\sigma_m = \sigma(\alpha)$ for $\alpha \in \Sigma_r$ and $1 \leq m \leq m_r(\widetilde{\alpha})$. By Lemma 4.3, $\sigma = (\sigma_1, \ldots, \sigma_{m_r(\widetilde{\alpha})})$ is an α_r-ladder in R. By the definition of σ, we have $U(\sigma) \leq H$. By Lemma 5.2, $H \leq U(\sigma)$. $\square$

6 Proof of Theorem 3.1: the Case $\Phi = G_2$ and $P = P_1$

In this section, we finish the proof of Theorem 3.1. Let G be a Chevalley group of type G_2, and $P = P_1$ be a maximal parabolic subgroup in G corresponding to the short simple root $\alpha_r = \alpha_1$. Then $\Delta_r = \Delta_1 = \{\alpha_2, -\alpha_2\}$ and

$$\Sigma_r = \Sigma_1 = \{\alpha_1,\, \alpha_1 + \alpha_2,\, 2\alpha_1 + \alpha_2,\, 3\alpha_1 + \alpha_2,\, 3\alpha_1 + 2\alpha_2\}.$$

So we have $\widetilde{\alpha} = 3\alpha_1 + 2\alpha_2$ and $m_r(\widetilde{\alpha}) = 3$. One can see that, firstly, $\Sigma_r(2)$ consists of the unique element $2\alpha_1 + \alpha_2$, and secondly, there exists no element $\beta \in \Delta_r$ such that only one of $\widetilde{\alpha}_1 + \beta$, $\widetilde{\alpha}_3 + \beta$ is in Φ. These are the principal differences between the case $(\Phi, r) = (G_2, 1)$ and those in §5.

We begin with the following technical lemma. Our choice of signs of the structure constants of the Chevalley group follows [21].

Lemma 6.1. *Let $u = \prod x_\alpha(u_\alpha) \in U$, where the product is taken over all roots of Σ_r in a level-adapted order. Then $\langle u \rangle^{EL} \geq X_{2\alpha_1+\alpha_2}(I^2)\, X_{3\alpha_1+\alpha_2}(I^3)\, X_{3\alpha_1+2\alpha_2}(I^3)$, where I is the ideal generated by u_{α_1} and $u_{\alpha_1+\alpha_2}$.*

Proof. Denote the group $\langle u\rangle^{EL}$ by H. By Lemma 4.4, we have $H/U^2 \geq V_1(I)$, therefore for any $\xi \in I$, there exists $v \in H$ with $v_{\alpha_1} = \xi$.

Consider any $\xi_i \in I$ $(i = 1, 2, 3)$ and the corresponding elements $v_i \in H$. It is easy to see that $w = [v_1, [x_{\alpha_2}(1), v_2]] = x_{2\alpha_1+\alpha_2}(c\xi_1\xi_2)\, x_{3\alpha_1+\alpha_2}(\eta_1)\, x_{3\alpha_1+2\alpha_2}(\eta_2)$, where $c \in R^*$ by **NVB** and $\eta_1, \eta_2 \in R$. Commuting w with $x_{\alpha_2}(\theta)$ and $x_{-\alpha_2}(\theta)$, $\theta \in R$, we see that $X_{3\alpha_1+2\alpha_2}(\eta_2 R), X_{3\alpha_1+\alpha_2}(\eta_1 R) \leq H$, thus $x_{2\alpha_1+\alpha_2}(c\xi_1\xi_2) \in H$ and the inclusion $X_{2\alpha_1+\alpha_1}(I^2) \leq H$ is proved.

Analogously, $[[v_3, x_{\alpha_2}(1)], w] = x_{3\alpha_1+2\alpha_2}(c'\xi_1\xi_2\xi_3)$, where $c' \in R^*$, which implies $X_{3\alpha_1+2\alpha_2}(I^3) \leq H$. Then by Lemma 4.2, $X_{3\alpha_1+\alpha_2}(I^3) \leq H$. □

The following lemma provides examples of subgroups in U which are normalized by EL and not generated by elementary root unipotents. More precisely, we describe the smallest subgroups with this property.

Lemma 6.2. *For any $\xi, \eta \in R$,*

$$\begin{aligned} &\langle x_{\alpha_1}(\xi)\, x_{3\alpha_1+\alpha_2}(\eta)\rangle^{EL} \\ &= \big\langle x_{\alpha_1}(\xi\theta)\, x_{3\alpha_1+\alpha_2}(\eta\theta),\ x_{\alpha_1+\alpha_2}(-\xi\theta)\, x_{3\alpha_1+2\alpha_2}(\eta\theta), \\ &\quad X_{2\alpha_1+\alpha_2}(\xi^2 R)\, X_{3\alpha_1+\alpha_2}(\xi^3 R)\, X_{3\alpha_1+2\alpha_2}(\xi^3 R) \mid \theta \in R\big\rangle \\ &= \langle x_{\alpha_1+\alpha_2}(-\xi)\, x_{3\alpha_1+2\alpha_2}(\eta)\rangle^{EL}. \end{aligned}$$

Proof. Denote by H the group in the middle. Direct calculations show that

$$\begin{aligned} &[x_{-\alpha_2}(\theta), x_{\alpha_1+\alpha_2}(\xi)\, x_{3\alpha_1+2\alpha_2}(\eta)] \\ &= x_{\alpha_1}(-\theta\xi)\, x_{2\alpha_1+\alpha_2}(\theta\xi^2)\, x_{3\alpha_1+\alpha_2}(\theta\eta + \xi^3\theta^2)\, x_{3\alpha_1+\alpha_2}(-\xi^3\theta) \end{aligned}$$

and

$$\begin{aligned} &[x_{\alpha_2}(\theta), x_{\alpha_1}(-\xi)\, x_{3\alpha_1+\alpha_2}(\eta)] \\ &= x_{\alpha_1+\alpha_2}(\xi\theta)\, x_{2\alpha_1+\alpha_2}(-\xi^2\theta)\, x_{3\alpha_1+\alpha_2}(\xi^3\theta)\, x_{3\alpha_1+2\alpha_2}(\eta\theta - \xi^3\theta^2) \end{aligned}$$

for every $\theta \in R$. It follows from Lemma 6.1 that both $\langle x_{\alpha_1}(\xi)\, x_{3\alpha_1+\alpha_2}(\eta)\rangle^{EL}$ and $\langle x_{\alpha_1+\alpha_2}(-\xi)\, x_{3\alpha_1+2\alpha_2}(\eta)\rangle^{EL}$ contain $X_{2\alpha_1+\alpha_2}(\xi^2 R)\, X_{3\alpha_1+\alpha_2}(\xi^3 R)\, X_{3\alpha_1+2\alpha_2}(\xi^3 R)$, hence the only thing to check is that EL normalizes H. We have just seen that $[EL, x_{\alpha_1}(\xi)\, x_{3\alpha_1+\alpha_2}(\eta)]$ and $[EL, x_{\alpha_1+\alpha_2}(-\xi)\, x_{3\alpha_1+2\alpha_2}(\eta)]$ lie in H (the second generator of EL in both cases commutes with the corresponding product). We finish the proof by observing that EL acts trivially on $X_{2\alpha_1+\alpha_2}(\xi^2 R)$ and normalizes $X_{3\alpha_1+\alpha_2}(\xi^3 R)\, X_{3\alpha_1+2\alpha_2}(\xi^3 R)$. □

By Lemma 6.2, the subgroup $U(M, A)$ defined in §3 is normalized by EL. It also admits the following explicit description.

Lemma 6.3. *For any coherent pair (M, A), one has*

$$\begin{aligned} U(M, A) = \big\{u \in U \,\big|\, &\text{for any decomposition } u = \prod_{\alpha\in\Sigma_r} x_\alpha(u_\alpha), \text{ one has} \\ &(u_{\alpha_1}, u_{3\alpha_1+\alpha_2}), (-u_{\alpha_1+\alpha_2}, u_{3\alpha_1+2\alpha_2}) \in M,\ u_{2\alpha_1+\alpha_2} \in A\big\}. \end{aligned}$$

Proof. It follows from the definition of a coherent pair when acting in the same way as in the proof of Lemma 4.5. □

Proof of Theorem 3.1(ii). Let H be a subgroup of U normalized by EL. Set

$$M = \{(\xi,\eta)\,|\,x_{\alpha_1}(\xi)\,x_{3\alpha_1+\alpha_2}(\eta) \in H\} \quad \text{and} \quad A = \{\xi\,|\,x_{2\alpha_1+\alpha_2}(\xi) \in H\}.$$

Obviously, A is an additive subgroup of R. We also have by Lemma 6.2 that M is an R-submodule of $R \times R$, and $\pi_1(M)^2 \le A$. Moreover, it follows from the identity $[x_{2\alpha_1+\alpha_2}(\theta),\, x_{\alpha_1+\alpha_2}(\xi)\,x_{3\alpha_1+2\alpha_2}(\eta)] = x_{3\alpha_1+2\alpha_2}(3\theta\xi)$ that $\{0\} \times \pi_1(M)A \le M$. Hence, the pair (M,A) is coherent. We now show that $U(M,A) = H$.

Observe by Lemma 6.2 that $M = \{(\xi,\eta)\,|\,x_{\alpha_1+\alpha_2}(-\xi)\,x_{3\alpha_1+2\alpha_2}(\eta) \in H\}$. Consider an element $u = \prod x_\alpha(u_\alpha) \in H$ (the product is taken over all roots of Σ_r in a level-adapted order). It follows from the Chevalley commutator formula that

$$[x_{\alpha_2}(1), u] = x_{\alpha_1+\alpha_2}(-u_{\alpha_1})\,x_{3\alpha_1+2\alpha_2}(u_{3\alpha+\alpha_2})\,v,$$

where $v \in X_{2\alpha_1+\alpha_2}(u_{\alpha_1}^2 R)\,X_{3\alpha_1+\alpha_2}(u_{\alpha_1}^3 R)\,X_{3\alpha_1+2\alpha_2}(u_{\alpha_1}^3 R)$. By Lemma 6.1, we have $v \in H$, thus H contains $x_{\alpha_1+\alpha_2}(-u_{\alpha_1})\,x_{3\alpha_1+2\alpha_2}(u_{3\alpha_1+\alpha_2})$ and $(u_{\alpha_1}, u_{3\alpha_1+\alpha_2}) \in M$. Commuting u with $x_{-\alpha_2}(1)$, we also get $x_{\alpha_1}(-u_{\alpha_1+\alpha_2})\,x_{3\alpha_1+\alpha_2}(u_{3\alpha_1+2\alpha_2}) \in H$, and $(-u_{\alpha_1+\alpha_2}, u_{3\alpha_1+2\alpha_2}) \in M$. Since all other factors of u are in H, we have $x_{2\alpha_1+\alpha_2}(u_{2\alpha_1+\alpha_2}) \in H$ as well. Therefore, $u_{2\alpha_1+\alpha_2} \in A$ and finally $u \in U(M,A)$. □

7 Proof of Theorem 3.3

In this section, we prove the main theorem of the paper, Theorem 3.3, and Corollary 3.4. Henceforth, we assume **NRO**. We first prove some technical lemmas, which actually reduce the description of EL-subgroups in P to the description of such subgroups in L and in U separately. The following lemma is in a way analogous to the statement (iv) of Lemma 2.1, as well as Lemma 7.2 below is to Lemma 5.1.

Lemma 7.1. *Let $\widetilde{\beta}$ be the root of maximal height in Δ_r^i for some $i \in \{1,2\}$. Then for any $1 \le k \le m_r(\widetilde{\alpha})$, except $k = 1$ when $(\Phi, r) = (C_l, 1)$, there exists $\beta \in \Delta_r^-$ such that only one of $\widetilde{\beta} + \beta$, $\widetilde{\alpha}_k + \beta$ is in Φ.*

Proof. Case 1: Φ is simply laced. In this case, all the roots have the same length, say $\sqrt{2}$. Then $(\alpha, \alpha') \in \{-1, 0, 1\}$ for any $\alpha, \alpha' \in \Phi$, and the sum $\alpha + \alpha'$ is a root only if $(\alpha, \alpha') = -1$. Let $j = 3 - i$. If there exists $\beta \in \Delta_r^j$ such that $\widetilde{\alpha}_k + \beta \in \Phi$, we can take it. Else $\widetilde{\alpha}_k$ is orthogonal to Δ_r^j.

Now if $|\Sigma_r(k)| = 1$, then we choose β to be equal to any simple root in Δ_r^i which is non-orthogonal to $\widetilde{\beta}$. Otherwise there exists a simple root $\alpha_n \in \Delta_r^i$ such that $\widetilde{\alpha}_k - \alpha_n \in \Phi$ since $\widetilde{\alpha}_k$ is the unique $\preceq$-maximal root in $\Sigma_r(k)$. The maximality of $\widetilde{\alpha}_k$ also means $(\alpha_m, \widetilde{\alpha}_k) \ge 0$ for all $\alpha_m \in \Delta_r$, so $(\alpha_n, \widetilde{\alpha}_k) > 0$ implies $(\widetilde{\beta}, \widetilde{\alpha}_k) > 0$. Therefore, $\delta = \widetilde{\alpha}_k - \widetilde{\beta}$ is a root. Since $\delta \ne \widetilde{\alpha}_k$ and obviously $\delta \in \Sigma_r(k)$, there exists a root $\gamma \in \Delta_r^+$ such that $\delta + \gamma \in \Phi$. Moreover, $\gamma \in \Delta_r^i$, as both $\widetilde{\beta}$ and $\widetilde{\alpha}_k$ are orthogonal to Δ_r^j. Now we note that $\delta + \gamma \in \Phi$ means $(\delta, \gamma) < 0$, i.e., $(\widetilde{\alpha}_k, \gamma) < (\widetilde{\beta}, \gamma)$. But $(\widetilde{\alpha}_k, \gamma) \ge 0$ since $\widetilde{\alpha}_k$ is maximal, and hence $(\widetilde{\alpha}_k, \gamma) = 0$ and $(\widetilde{\beta}, \gamma) = 1$. Then we can take $\beta = -\gamma$.

Case 2: $\Phi = B_l$. If $i = 1$, then we can take $\beta = -\alpha_{r-1}$ in the case $k = 1$ ($\Delta_r^i = \emptyset$ when $r = 1$), and $\beta = -\alpha_1$ in the case $k = 2$. If $i = 2$, then we can take $\beta = -\alpha_{r+1}$ in the case $k = 1$ ($\Delta_r^i = \emptyset$ when $r = l$), and $\beta = -\alpha_{r+2}$ in the case $k = 2$.

Case 3: $\Phi = C_l$. If $i = 1$ ($\Delta_r^i = \emptyset$ when $r = 1$), then we can take $\beta = -\alpha_{r-1}$ in all cases. If $i = 2$, then we can take $\beta = -\alpha_1$ in the case $k = 1$ (recall that we have excluded the pair $(r, k) = (1, 1)$), and $\beta = -\alpha_{r+1}$ in the case $k = 2$.

Case 4: $\Phi = F_4$. Since **NRO** holds, we have only two possibilities: $r = 1$ and $r = 2$. If $r = 1$, then $\Delta_r^1 = \emptyset$, and in the case $i = 2$, we take $\beta = -\alpha_2$ when $k = 1$, and $\beta = -\alpha_4$ when $k = 2$. If $r = 2$, then $\Delta_r^2 = \emptyset$, and in the case $i = 1$, we take $\beta = -\alpha_2$ whenever $k = 1$ or $k = 2$. □

Lemma 7.2. *Let $\widetilde{\beta}$ be the root of maximal height in one of Δ_r^1, Δ_r^2. Then for any $1 \le k \le m_r(\widetilde{\alpha})$, one has $x_{\widetilde{\beta}}(\xi)$, $x_{\widetilde{\alpha}_k}(\theta) \in \langle x_{\widetilde{\beta}}(\xi)\, x_{\widetilde{\alpha}_k}(\theta)\rangle^{EL} U^{k+1}$.*

Proof. If only there exists a root $\beta \neq -\widetilde{\beta} \in \Delta_r$ such that only one of $\widetilde{\beta}+\beta$, $\widetilde{\alpha}_k+\beta$ is in Φ, the claim can be proved just in the same way as Lemma 5.1. By Lemma 7.1, such β exists in all cases except $\Phi = C_l$, $r = 1$ and $k = 1$. In the latter case, one can see that $\widetilde{\alpha}_k = \widetilde{\alpha}_1 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$ and $\widetilde{\beta} = 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$. We have $[x_{-\alpha_2}(1),\, x_{\widetilde{\beta}}(\xi)\, x_{\widetilde{\alpha}_k}(\theta)] = x_{\widetilde{\beta}-\alpha_2}(c_1\xi)\, x_{\widetilde{\beta}-2\alpha_2}(c_2\xi)\, x_{\widetilde{\alpha}_k-\alpha_2}(c_3\theta)$, where c_1, c_2, c_3 are some structure constants of the Chevalley group. Further,

$$[x_{-\alpha_2}(1),\, x_{\widetilde{\beta}-\alpha_2}(c_1\xi)\, x_{\widetilde{\beta}-2\alpha_2}(c_2\xi)\, x_{\widetilde{\alpha}_k-\alpha_2}(c_3\theta)] = x_{\widetilde{\beta}-2\alpha_2}(c_1c_4\xi),$$

where c_4 is also a structure constant. Therefore, $x_{\widetilde{\beta}-2\alpha_2}(c\xi) \in \langle x_{\widetilde{\beta}}(\xi)\, x_{\widetilde{\alpha}_k}(\theta)\rangle^{EL}$, where $c \in R^*$ by **NVB**. Then by Lemma 4.1 we have $x_{\widetilde{\beta}}(\xi) \in \langle x_{\widetilde{\beta}}(\xi)\, x_{\widetilde{\alpha}_k}(\theta)\rangle^{EL}$, and the claim becomes obvious. □

The following lemma shows that any EL-normalized subgroup H of P contains a large enough EL-normalized subgroup of L.

Lemma 7.3. *Let H be a subgroup of P normalized by EL. Denote by H_L the image of H under the natural projection $P \to P/U \cong L$. Then $H \geq [H_L, EL]$.*

Proof. It follows from Lemma 3.2 that $[H_L, EL] = EL^1(R, I_1) \times EL^2(R, I_2)$ for some $I_1, I_2 \trianglelefteq R$. For any $\alpha \in \Delta_r^i$ $(i = 1, 2)$, one has $X_\alpha(I_i)^{EL^i} = EL^i(R, I_i)$ by Lemma 4.1. Fix $i \in \{1, 2\}$ such that Δ_r^i is not empty. Let $\widetilde{\beta}$ denote the maximal root of Δ_r^i. Then it is enough to prove $x_{\widetilde{\beta}}(\xi) \in H$ for any $\xi \in I_i$.

Since $x_{\widetilde{\beta}}(\xi) \in H_L$, there exists $u \in U$ such that $x_{\widetilde{\beta}}(\xi)u \in H$. We can write $u = \prod x_\alpha(u_\alpha)$, where the product is taken over all $\alpha \in \Sigma_r$ in some level-adapted order $\le$. We denote by $r(u)$ the minimal (with respect to $\le$) root in Σ_r such that $u_\alpha \neq 0$. We will show that H contains $x_{\widetilde{\beta}}(\xi)u'$, where $r(u') > r(u)$. Let $m_r(r(u)) = k$. If $r(u) \neq \widetilde{\alpha}_k$, then by Lemma 2.1, there exists a root $\beta \in \Delta_r^+$ such that $r(u) + \beta \in \Sigma_r(k)$. The element $w = [x_\beta(1), x_{\widetilde{\beta}}(\xi)u] = x_{\widetilde{\beta}}(\xi)[x_\beta(1), u]x_{\widetilde{\beta}}(-\xi) = \prod x_\alpha(w_\alpha)$ is in $H \cap U$, and the Chevalley commutator formula implies that $r(w) = r(u) + \beta$ and $w_{r(u)+\beta} = cu_{r(u)}$, $c \in R^*$. Since H is normalized by EL, by Lemma 5.2 together with Lemma 4.2, we get $x_{r(u)}(u_{r(u)}) \in H$. Then H contains $x_{\widetilde{\beta}}(\xi)u \cdot x_{r(u)}(-u_{r(u)}) = x_{\widetilde{\beta}}(\xi)u'$, where $r(u') > r(u)$.

Suppose now that $r(u) = \widetilde{\alpha}_k$, i.e., $u \in x_{\widetilde{\alpha}_k}(u_{\widetilde{\alpha}_k})U^{k+1}$. As U^{k+1} is invariant under the action of EL, one has $\langle x_{\widetilde{\beta}}(\xi)u\rangle^{EL}U^{k+1} = \langle x_{\widetilde{\beta}}(\xi)\, x_{\widetilde{\alpha}_k}(u_{\widetilde{\alpha}_k})\rangle^{EL}U^{k+1}$. Then by Lemma 7.2, one has $x_{\widetilde{\beta}}(\xi) \cdot u' \in \langle x_{\widetilde{\beta}}(\xi)u\rangle^{EL} \leq H$ for some $u' \in U^{k+1}$. Obviously, $m_r(r(u')) \geq k+1$, hence $r(u') > r(u)$. □

Proof of Theorem 3.3. Consider an element $zu \in H$, where $z \in H_L$ and $u \in U$. One has $[z, EL] \subseteq H$ by Lemma 7.3. For any $h \in EL$, one has $[zu, h] = {}^z[u, h][z, h]$, then H contains ${}^z[u, h] = zuhu^{-1}h^{-1}z^{-1} = {}^{zu}[h, u^{-1}]$, and therefore $[h, u^{-1}] \in H$. If P is not extraspecial, then $u \in H$ by Lemma 5.3, hence we get $H_L \leq H$, and together with Lemma 4.6, this finishes the proof. If P is extraspecial, then Lemma 5.3 implies $u \in U(\sigma)X_{\widetilde{\alpha}} = U(\sigma_1, R)$, and hence we get the inclusion $H \leq EL^*(I_1, I_2) \ltimes U(\sigma)X_{\widetilde{\alpha}}$.

The converse statement for P not extraspecial follows directly from Lemma 4.6 and the definition of $EL^*(I_1, I_2)$. If P is extraspecial, one also need to recall that EL acts trivially on $X_{\widetilde{\alpha}}$. □

Proof of Corollary 3.4. Let $H \leq P$ be normalized by EP. Then U normalizes $H \cap U = U(\sigma)$, and hence by Corollary 4.9, we have $\sigma_1 \subseteq \sigma_2 \subseteq \cdots \subseteq \sigma_{m_r(\widetilde{\alpha})}$. By Theorem 3.3, we have $H \geq EL^1(R, I_1) \times EL^2(R, I_2)$. Further, it follows from the basic properties of root systems that $\alpha_1 + \cdots + \alpha_l$ is a root. Therefore, by Lemma 2.1, for both $i = 1$ and $i = 2$ (unless Δ_r^i is empty), one can find roots $\alpha \in \Sigma_r(1)$ and $\beta \in \Delta_r^i$ such that $\alpha + \beta$ is a root. By Lemma 4.1, we have $X_\beta(I_i)^{X_\alpha} \geq X_{\beta+\alpha}(I_i)$, hence $H \cap U \geq X_{\beta+\alpha}(I_i)$, which implies $I_i \subseteq \sigma_1$. It remains to prove $H_L \leq E^*(\Phi, R, \sigma_1)$.

Since $I_1 + I_2 \subseteq \sigma_1$, the image of H_L under the reduction homomorphism ρ_{σ_1} : $G(\Phi, R) \to G(\Phi, R/\sigma_1)$ centralizes $EL(R/\sigma_1) = E(\Delta_r, R/\sigma_1)$. By Lemma 2.2, the centralizer of $EL(R/\sigma_1) = E(\Delta_r, R/\sigma_1)$ in $L(R/\sigma_1) = G(\Delta_r, R/\sigma_1)$ coincides with the set of R/σ_1-valued points of the scheme-theoretic centre of the reductive algebraic group L and therefore is contained in the torus $T(\Phi, R/\sigma_1)$ (see [9, Exp. XXII 4.1.7]). In particular, any element $t \in \rho_{\sigma_1}(H_L)$ satisfies $tx_\beta(\eta)t^{-1} = x_\beta(\chi(\beta)\eta)$ for all $\beta \in \Phi$ and $\eta \in R/\sigma_1$, where $\chi : \Lambda \to (R/\sigma_1)^*$ is an R/σ_1-character of the weight lattice Λ of G. The centralizer property implies that $\chi(\beta) = 1$ for all $\beta \in \Delta_r$. On the other hand, we know that $[H_L, U] \leq U(\sigma_1, \dots, \sigma_{m_r(\widetilde{\alpha})})$, and hence $[t, x_{\alpha_r}(1)] = x_{\alpha_r}(\chi(\alpha_r) - 1)$ is in $U(0, \sigma_2/\sigma_1, \dots, \sigma_{m_r(\widetilde{\alpha})}/\sigma_1)$. By Lemma 4.5, this implies that $\chi(\alpha_r) = 1$ as well. But then χ is identically 1 on Φ, or t centralizes $E(\Phi, R/\sigma_1)$. Again by Lemma 2.2 (or by [3]), this is the same as to say that t is in the centre of $G(\Phi, R/\sigma_1)$. Thus, H_L is contained in the inverse image under ρ_{σ_1} of the centre of $G(\Phi, R/\sigma_1)$, that is, in $E^*(\Phi, R, \sigma_1)$.

Suppose now that H is an EL-normalized subgroup of P satisfying $I_1 + I_2 \subseteq \sigma_1 \subseteq \cdots \subseteq \sigma_{m_r(\widetilde{\alpha})}$ and $H_L \leq E^*(\Phi, R, \sigma_1)$. We will prove

$$[H, U] \leq \big(EL^1(R, I_1) \times EL^2(R, I_2)\big) \ltimes U(\sigma) \leq H.$$

It follows from the Chevalley commutator relations and the inclusions $\sigma_1 \subseteq \cdots \subseteq \sigma_{m_r(\widetilde{\alpha})}$ that $[U(\sigma), U] \leq U(\sigma)$ if P is not extraspecial, and that $[U(\sigma)X_{\widetilde{\alpha}}, U] \leq U(\sigma)$ if P is extraspecial. Further, since $H_L \leq E^*(\Phi, R, \sigma_1)$, we have $[H_L, U] \leq E(R, \Phi, \sigma_1)$. On the other hand, $[H_L, U] \leq U$, and it is clear that $U \cap E(\Phi, R, \sigma_1) = U(\sigma_1, \dots, \sigma_1)$. This proves $[H_L, U] \leq U(\sigma_1, \dots, \sigma_1) \leq U(\sigma)$. □

Acknowledgements. I wish to express my hearty thanks to Nikolai Vavilov who introduced me to the subject and made many helpful suggestions concerning the layout of the paper, and to Victor Petrov for fruitful discussions. I also gratefully acknowledge the hospitality of the Bielefeld University, where an essential part of the present paper was written, and of Anthony Bak, who arranged the visit.

References

[1] E. Abe, Chevalley groups over local rings, *Tôhoku Math. J.* **21** (1969) 474–494.
[2] E. Abe, Normal subgroups of Chevalley groups over commutative rings, *Contemp. Math.* **83** (1989) 1–17.
[3] E. Abe, J.F. Hurley, Centers of Chevalley groups over commutative rings, *Comm. Algebra* **16** (1988) 57–74.
[4] E. Abe, K. Suzuki, On normal subgroups of Chevalley groups over commutative rings, *Tôhoku Math. J.* **28** (1976) 185–198.
[5] H. Azad, M. Barry, G. Seitz, On the structure of parabolic subgroups, *Comm. Algebra* **18** (1990) 551–562.
[6] N. Bourbaki, *Groupes et algèbres de Lie*, Chapitres 4–6, Hermann, Paris, 1968.
[7] D.L. Costa, G.E. Keller, Radix redux: normal subgroups of symplectic groups, *J. Reine Angew. Math.* **427** (1992) 51–105.
[8] D.L. Costa, G.E. Keller, On the normal subgroups of $G_2(A)$, *Trans. Amer. Math. Soc.* **351** (1999) 5051–5088.
[9] M. Demazure, A. Grothendieck, *Schémas en groupes*, Lecture Notes in Mathematics, 151–153, Springer-Verlag, Berlin-Heidelberg-New York, 1970.
[10] V. Kazakevich, A. Stavrova, Subgroups normalized by the elementary Levi subgroup, *J. Math. Sci.* **134** (2006) 2549–2557.
[11] C. Parker, G. Röhrle, The restriction of minuscule representations to parabolic subgroups, *Math. Proc. Camb. Phil. Soc.* **135** (2003) 59–79.
[12] R. Richardson, G. Röhrle, R. Steinberg, Parabolic subgroups with abelian unipotent radical, *Invent. Math.* **110** (1992) 649–671.
[13] G. Röhrle, On the structure of parabolic subgroups in algebraic groups, *J. Algebra* **157** (1993) 80–115.
[14] G. Röhrle, On normal abelian subgroups in parabolic groups, *Ann. Inst. Fourier* **48** (1998) 1455–1482.
[15] A.V. Stepanov, On the distribution of subgroups normalized by a given subgroup, *J. Soviet Math.* **64** (1993) 769–776.
[16] K. Suzuki, On parabolic subgroups of Chevalley groups over local rings, *Tôhoku Math. J.* **28** (1976) 57–66.
[17] L.N. Vaserstein, On normal subgroups of Chevalley groups over commutative rings, *Tôhoku Math. J.* **38** (1986) 219–230.
[18] N.A. Vavilov, Parabolic subgroups of Chevalley groups over a semilocal ring, *J. Soviet Math.* **37** (1987) 942–952.
[19] N.A. Vavilov, Structure of Chevalley groups over commutative rings, in: *Proc. Conf. Non-associative Algebras and Related Topics* (Hiroshima, 1990), World Sci. Publ., Singapore, 1991, pp. 219–335.
[20] N.A. Vavilov, M.R. Gavrilovich, An A_2-proof of structure theorems for Chevalley groups of types E_6 and E_7, *St. Petersburg Math. J.* **16** (2005) 649–672.
[21] N. Vavilov, E. Plotkin, Chevalley groups over commutative rings I: Elementary calculations, *Acta Appl. Math.* **45** (1996) 73–113.