CONFORMAL INVARIANCE OF SPIN CORRELATIONS IN THE PLANAR ISING MODEL

DMITRY CHELKAK[A,C], CLÉMENT HONGLER[B], AND KONSTANTIN IZYUROV[C]

Abstract. We rigorously prove existence and conformal invariance of scaling limits of magnetization and multi-point spin correlations in the critical Ising model on an arbitrary simply connected planar domain. This solves a number of conjectures coming from physical and mathematical literatures. The proof is based on convergence results for discrete holomorphic spinor observables.

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[A] St.Petersburg Department of Steklov Mathematical Institute. Fontanka 27, 191023 St.Petersburg, Russia, *E-mail address:* dchelkak@pdmi.ras.ru.

[B] Department of Mathematics, Columbia University. 2990 Broadway, New York, NY 10027, USA, *E-mail address:* hongler@math.columbia.edu.

[C] Chebyshev Laboratory, Department of Mathematics and Mechanics, Saint-Petersburg State University, 14th Line, 29b, 199178 Saint-Petersburg, Russia. *Current address:* Department of Mathematics and Statistics, University of Helsinki, P.O. box 68, 00014 Helsinki, Finland, *E-mail address:* konstantin.izyurov@helsinki.fi.

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1. Introduction

The Ising model plays a central role in equilibrium statistical mechanics, being a standard example of an order-disorder phase transition in dimensions two and above. Besides purely mathematical interest, it has found successful applications to several fields in theoretical physics and computer sciences.

The phase transition in the Ising model in two dimensions has been a subject of extensive study, both in mathematics and physics literature. The critical temperature value on the square lattice was derived by Kramers and Wannier [KrWa41]. Onsager [Ons44] computed the free energy and the critical exponents of the model. Later on, many exact computations were performed by McCoy and Wu [McWu73].

Further, a gradual understanding of the Ising model at criticality led to the conjecture by Belavin, Polyakov and Zamolodchikov that its scaling limit (as well as scaling limits of other critical models) should be conformally invariant and described by Conformal Field Theory [BPZ84a, BPZ84b]. Loosely speaking, this conjecture can be formulated as follows: for any conformal map $\varphi:\Omega\to\Omega'$,

$$(\text{scaling limit of the model on }\Omega')=\varphi(\text{scaling limit of the model on }\Omega).$$

In particular, if Ω_δ are discrete approximations to a continuous planar domain Ω, then various quantities (expectations, probabilities etc.) in the model, under a proper normalization, have conformally invariant or covariant limits as the mesh size δ tends to zero. Moreover, Conformal Field Theory predicts exact formulae for these limits.

In the *full-plane* case, many results were obtained after the seminal work of Onsager and Kaufman in late 1940's. The Onsager's formula for the spontaneous magnetization was proven by Yang [Yan52]. The diagonal and horizontal spin-spin correlations were explicitly computed by Wu [McWu73]. A number of remarkable results were obtained for the massive limits, see [WMTB76, SMJ80, PaTr83] and references therein. Palmer [Pal07] justified the CFT predictions at criticality by taking the zero-mass limit. At criticality, the full-plane energy correlation functions (that is, the correlations of n pairs of neighboring spins) were computed on periodic isoradial graphs by Boutillier and de Tilière [BoDT10, BoDT11], and the 2n-point full-plane spin correlation functions were treated by Dubédat, combining exact bosonization techniques [Dub11b] and results on monomer correlations in the dimer model [Dub11a].

However, the group of conformal self-maps of the full plane is only finite dimensional. Hence, in order for the conformal invariance conjecture to acquire its full strength, it is important to consider *general planar domains with boundary.* In this setting, mathematical proofs of conformal invariance and CFT predictions at criticality have remained out of reach until recently. Smirnov [Smi06] has rigorously established conformal covariance of the *fermionic observables* in the Ising model on the square grid. Later, this result has been proven to be universal in the family of isoradial graphs [ChSm12], and led to the proof of convergence of the interfaces in the Ising model to Schramm's SLE_3 curves [CDHKS12]. At the same time, the scaling limit of *energy correlations* for bounded domains has been rigorously treated in [HoSm10, Hon10], confirming the CFT predictions for the energy field. Nevertheless, the corresponding question about the *spin correlations* remained open.

In this paper, we rigorously prove existence and conformal covariance of scaling limits of all multi-point spin correlation functions in any simply connected planar domain with + boundary conditions. Our main result (see Theorem 1.3) reads as follows. Let $\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1}\dots\sigma_{a_n}]$ denote the correlation of spins at the sites $a_1,\dots,a_n$ with + boundary conditions in a discrete domain Ω_δ. Then

$$\delta^{-\frac{n}{8}}\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1}\dots\sigma_{a_n}] \to \mathcal{C}^n\cdot\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_\Omega$$

as Ω_δ approximates Ω and the mesh size δ tends to zero. Here $\mathcal{C}$ is an explicit lattice-dependent constant, and $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_\Omega$ is an explicit conformally covariant tensor of degree $\frac{1}{8}$ with respect to each of the variables, see (1.3). For example, in the case of magnetization (expectation of just one spin), one gets

$$\delta^{-\frac{1}{8}}\mathbb{E}^+_{\Omega_\delta}[\sigma_a] \to \mathcal{C}\cdot\mathrm{rad}^{-\frac{1}{8}}(a,\Omega),$$

where $\mathrm{rad}(a,\Omega)$ denotes the conformal radius of Ω as seen from a, in other words, $\mathrm{rad}(a,\Omega)=|\varphi'(0)|$, where φ is a conformal map from the disc $\{z\in\mathbb{C}:|z|<1\}$ to Ω mapping the origin to a. A sketch of the proof of this result can also be found in the ICMP2012 proceedings [Hon12].

We establish a similar convergence for two-point function in the case of free boundary conditions (see Theorem 1.1). The convergence of the corresponding

n-point correlations (which by symmetry are only non-zero for even n) can be obtained by our methods as well, but for simplicity we do not include this in the present paper. Moreover, our previous results [ChIz11] immediately allow one to treat alternating $+/-$ boundary conditions (see Corollary 1.4). The technique we use also applies to *mixed correlations*, involving spin, energy, disorder and boundary change operators, and extends to *multiply connected domains*, which will be worked out in a subsequent paper.

The explicit formulae for the the scaling limits of $\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{a_1}\dots,\sigma_{a_n}]$ were derived by Conformal Field Theory methods in a number of papers originating in the seminal work [BPZ84a]. In [Car84], it was explained how to handle the half-plane case by CFT means, in particular, the two-point correlations were treated. This result was later extended to $n=3$ [BuGu87], and extrapolated to larger n [BuGu93]. In our approach the answers come in an explicit, but different form, being defined via solutions to a special interpolation problem, so one should additionally check that they coincide with the known CFT predictions. In this paper, we do this check in two situations: for small values $n=1,2,3$ and in the special case when all $a_1,\dots,a_n$ are on the same vertical line in the half-plane (see further discussion in Section 2.7 and the Appendix), which gives all the properties of the continuous correlations we need in the proof.

Our method is based on the extracting information from some discrete holomorphic observables in bounded domains by means of discrete complex analysis – the approach that was firstly implemented in [Smi06, Smi10] for basic fermionic observables. More precisely, we use the spinor version of those which was introduced in [ChIz11]. Fermionic observables per se essentially go back to the Kaufman-Onsager considerations and can be written as a product $\langle\psi_z\rangle=\langle\sigma_z\mu_z\rangle$ of spin and disorder operators in the notation of [KaCe71]. Their spinor versions $\langle\psi_z\sigma_{a_1}\mu_{a_2}\dots\mu_{a_n}\rangle$ can be found in the works of Kyoto's school [SMJ77, SMJ79a, SMJ79b, SMJ80]. However, rigorous proofs of convergence results require a delicate analysis of some Riemann-type boundary value problems for discrete holomorphic functions developed more recently [Smi06, Smi10, ChSm12].

Simultaneously and independently of our work, Dubédat announced analogous results for $2n$-point spin correlations in bounded domains Ω via the exact bosonization approach [Dub11b], and Camia, Garban and Newman obtained some results [CGN12] about the properly renormalized spin field seen as a random distribution (generalized function) on Ω.

1.1. **Main results.** The Ising model on a graph $\mathcal{G}$ is a random assignment of ±1 spins to the vertices of $\mathcal{G}$. In our paper we prefer a dual setup and consider the model on the *faces* $\mathcal{V}=\mathcal{V}^\circ_{\Omega_\delta}$ of lattice approximations Ω_δ, $\delta\to0$, to a bounded simply connected planar domain $\Omega\subset\mathbb{C}$. The probability of a spin configuration $\sigma\in\{\pm1\}^{\mathcal{V}}$ is proportional to

$$e^{-\beta\mathbf{H}(\sigma)},$$

where $\beta>0$ is the inverse temperature and $\mathbf{H}(\sigma):=-\sum_{x\sim y}\sigma_x\sigma_y$ is the energy of the configuration. More precisely, we will work with discrete planar domains Ω_δ which are subsets of the *square grids* rotated by 45° of diagonal mesh sizes 2δ (thus, the distance between adjacent spins is $\sqrt{2}\delta$, see Figure 1).

From now on we only consider the model at its *critical point*, which for the square grid corresponds to the parameter value $\beta_{\mathrm{c}}=\frac{1}{2}\ln\left(\sqrt{2}+1\right)$. We also introduce a (lattice-dependent) constant that will appear in the statements of our theorems:

$$\mathcal{C} = 2^{\frac{1}{6}} e^{-\frac{3}{2}\zeta'(-1)}, \tag{1.1}$$

where ζ' denotes the derivative of Riemann's zeta function. We first state the convergence theorem for the two-points functions, both with + (the spins on the boundary of Ω_δ are set to +) and free (no restrictions are set for boundary spins) boundary conditions:

Theorem 1.1. *Let Ω be a bounded simply connected domain and Ω_δ be discretizations of Ω by the square grids of diagonal mesh size 2δ. Then, for any $\epsilon > 0$, we have*

$$\delta^{-\frac{1}{4}} \mathbb{E}^{+}_{\Omega_\delta}[\sigma_a \sigma_b] \;\to\; \mathcal{C}^2 \cdot \langle \sigma_a \sigma_b \rangle^{+}_{\Omega} \quad \text{and} \quad \delta^{-\frac{1}{4}} \mathbb{E}^{\mathrm{free}}_{\Omega_\delta}[\sigma_a \sigma_b] \;\to\; \mathcal{C}^2 \cdot \langle \sigma_a \sigma_b \rangle^{\mathrm{free}}_{\Omega},$$

as $\delta \to 0$, uniformly over all $a, b \in \Omega$ at distance at least ϵ from $\partial\Omega$ and from each other, where $\langle \sigma_a \sigma_b \rangle^{+}_{\Omega}$ and $\langle \sigma_a \sigma_b \rangle^{\mathrm{free}}_{\Omega}$ are explicit functions given by (1.4), and the constant $\mathcal{C}$ is given by (1.1).

Remark 1.2. (i) Here and below we follow the traditional notation developed by physicists and denote the limits of spin correlations by the symbols $\langle \sigma_{a_1} \dots \sigma_{a_n} \rangle^{\mathfrak{b}}_{\Omega}$ which should be understood as *functions* of the points $a_1, \dots, a_n \in \Omega$ which may also depend on Ω and boundary conditions $\mathfrak{b}$ as parameters.

(ii) The functions $\langle \sigma_a \sigma_b \rangle^{+}_{\Omega}$ and $\langle \sigma_a \sigma_b \rangle^{\mathrm{free}}_{\Omega}$ are conformal covariants of degree $\frac{1}{8}$ with respect to a and b: for any conformal map $\varphi : \Omega \to \Omega'$, one has

$$\langle \sigma_a \sigma_b \rangle_{\Omega} = \langle \sigma_{\varphi(a)} \sigma_{\varphi(b)} \rangle_{\Omega'} \cdot |\varphi'(a)|^{\frac{1}{8}} |\varphi'(b)|^{\frac{1}{8}}. \tag{1.2}$$

Moreover, we have a similar result for the multi-point correlations:

Theorem 1.3. *Let Ω be a bounded simply connected domain and Ω_δ be discretizations of Ω by the square grids of diagonal mesh size 2δ. Then, for any $\epsilon > 0$ and any $n = 1, 2, \dots$, we have*

$$\delta^{-\frac{n}{2}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{a_1} \dots \sigma_{a_n}] \quad \to \quad \mathcal{C}^n \cdot \langle \sigma_{a_1} \dots \sigma_{a_n} \rangle^{+}_{\Omega}$$

as $\delta \to 0$, uniformly over all $a_1, \dots, a_n \in \Omega$ at distance at least ϵ from $\partial\Omega$ and from each other, and the functions $\langle \sigma_{a_1} \dots \sigma_{a_n} \rangle^{+}_{\Omega}$ have the following covariance under conformal mappings $\varphi : \Omega \to \Omega'$:

$$\langle \sigma_{a_1} \dots \sigma_{a_n} \rangle^{+}_{\Omega} = \langle \sigma_{\varphi(a_1)} \dots \sigma_{\varphi(a_n)} \rangle^{+}_{\Omega'} \cdot \textstyle\prod_{k=1}^{n} |\varphi'(a_k)|^{\frac{1}{8}}. \tag{1.3}$$

Due to the conformal covariance property, it is sufficient to compute the continuous correlation functions in the upper half-plane $\mathbb{H}$: indeed, applying (1.2), (1.3) with $\varphi : \Omega \to \mathbb{H}$, one obtains those functions in Ω. For the magnetization and two-point functions, we have

$$\begin{aligned}
\langle \sigma_a \rangle^{+}_{\mathbb{H}} &= \frac{2^{\frac{1}{4}}}{(2\Im\mathfrak{m}\, a)^{\frac{1}{8}}}, & \langle \sigma_a \rangle^{+}_{\Omega} &= 2^{\frac{1}{4}} \mathrm{rad}(a, \Omega)^{-\frac{1}{8}}, \\
\langle \sigma_a \sigma_b \rangle^{+}_{\mathbb{H}} &= \frac{(u_{ab} + u_{ab}^{-1})^{\frac{1}{2}}}{(2\Im\mathfrak{m}\, a)^{\frac{1}{8}} (2\Im\mathfrak{m}\, b)^{\frac{1}{8}}}, & \langle \sigma_a \sigma_b \rangle^{+}_{\Omega} &= \frac{\langle \sigma_a \rangle^{+}_{\Omega} \langle \sigma_b \rangle^{+}_{\Omega}}{(1 - \exp(-2\mathrm{d}^{\mathrm{hyp}}_{\Omega}(a, b)))^{\frac{1}{4}}}, \\
\langle \sigma_a \sigma_b \rangle^{\mathrm{free}}_{\mathbb{H}} &= \frac{(u_{ab}^{-1} - u_{ab})^{\frac{1}{2}}}{(2\Im\mathfrak{m}\, a)^{\frac{1}{8}} (2\Im\mathfrak{m}\, b)^{\frac{1}{8}}}, & \langle \sigma_a \sigma_b \rangle^{\mathrm{free}}_{\Omega} &= \langle \sigma_a \sigma_b \rangle^{+}_{\Omega} \cdot \exp(-\tfrac{1}{2}\mathrm{d}^{\mathrm{hyp}}_{\Omega}(a, b)),
\end{aligned} \tag{1.4}$$

where $u_{ab} := |(b-a)/(b-\overline{a})|^{\frac{1}{2}}$, $\mathrm{rad}(a,\Omega)$ denotes the conformal radius of Ω seen from a and $\mathrm{d}_\Omega^{\mathrm{hyp}}(a,b)$ is the hyperbolic distance between a and b in Ω. For the discussion of the explicit form of $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_{\mathbb{H}}^+$ for $n\geq 3$, we refer the reader to the Appendix.

Using the results of [ChIz11], one immediately arrives at the following generalization. Let $\mathfrak{b}_\delta = \{b_1^\delta,\dots,b_{2m}^\delta\}$ be a collection of points on $\partial\Omega_\delta$ appearing in counterclockwise order and approximating a continuous collection $\mathfrak{b}=\{b_1,\dots,b_{2m}\}\subset\partial\Omega$. Denote by $\mathbb{E}_{\Omega_\delta}^{\mathfrak{b}_\delta}$ an expectation for the Ising model with + boundary conditions on the counterclockwise arcs $[b_{2j-1}^\delta, b_{2j}^\delta]$ and $-$ boundary condition on the complementary arcs $[b_{2j}^\delta, b_{2j+1}^\delta]$, $j=1,\dots,m$, where we set $b_{2m+1}^\delta := b_1^\delta$.

Corollary 1.4. *Suppose that the approximation Ω_δ of Ω is regular near the points of $\mathfrak{b}$ in the sense of* [ChIz11, Definition 3.14]. *As $\delta\to 0$, one has*

$$\delta^{-\frac{n}{2}}\mathbb{E}_{\Omega_\delta}^{\mathfrak{b}_\delta}[\sigma_{a_1}\dots\sigma_{a_n}] \to \mathcal{C}^n\cdot\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_\Omega^{\mathfrak{b}},$$

where $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_\Omega^{\mathfrak{b}}$ satisfies the conformal covariance property (1.3).

Proof. Write $\mathbb{E}_{\Omega_\delta}^{\mathfrak{b}_\delta}[\sigma_{a_1}\dots\sigma_{a_n}] = (\mathbb{E}_{\Omega_\delta}^{\mathfrak{b}_\delta}[\sigma_{a_1}\dots\sigma_{a_n}]/\mathbb{E}_{\Omega_\delta}^{+}[\sigma_{a_1}\dots\sigma_{a_n}])\cdot\mathbb{E}_{\Omega_\delta}^{+}[\sigma_{a_1}\dots\sigma_{a_n}]$. By [ChIz11, Corollary 5.10], the first term converges to an explicit conformally *invariant* limit. Thus the result follows from Theorem 1.3. □

1.2. **Key steps in the proof.** In this section we list the key results that allow us to prove Theorems 1.1 and 1.3. The first small step deals with the normalizing factors. It is a celebrated result of Wu [McWu73] that in the unique infinite-volume limit of the critical planar Ising model (i.e., in the case $\Omega=\mathbb{C}$), one has the following asymptotics:

$$\mathbb{E}_{\mathbb{C}_\delta}[\sigma_{0_\delta}\sigma_{1_\delta}] \sim \mathcal{C}^2\cdot\delta^{\frac{1}{4}},\quad \delta\to 0, \tag{1.5}$$

where $\mathbb{C}_\delta$ denotes the square grid $\delta(1+i)\mathbb{Z}^2$, while 0_δ and 1_δ stand for proper approximations of the points $0,1\in\mathbb{C}$ (keep in mind that our square lattice is rotated by 45°, so this is the *diagonal* spin-spin correlation). Instead of deriving the correct normalization of spin correlations in bounded domains directly, we relate it to the behavior of the normalizing factor

$$\varrho(\delta) := \mathbb{E}_{\mathbb{C}_\delta}[\sigma_{0_\delta}\sigma_{1_\delta}].$$

Namely, we prove that, as $\delta\to 0$,

$$\begin{aligned}(\varrho(\delta))^{-\frac{n}{2}}\cdot\mathbb{E}_{\Omega_\delta}^{+}[\sigma_{a_1}\dots\sigma_{a_n}] &\to \langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_\Omega^+,\\ (\varrho(\delta))^{-1}\cdot\mathbb{E}_{\Omega_\delta}^{\mathrm{free}}[\sigma_a\sigma_b] &\to \langle\sigma_a\sigma_b\rangle_\Omega^{\mathrm{free}},\end{aligned}$$

which, combined with (1.5), readily gives Theorems 1.1 and 1.3. We point out that apart from this reduction, we never use (1.5) in this paper. On the other hand, our methods also allow one to give a new proof of the explicit formula for the diagonal spin-spin correlations in the full-plane case as well as to derive an explicit formulae for the magnetization in the half-plane, see the forthcoming work [ChHo13].

The following theorem, concerning *discrete logarithmic derivatives* of the spin correlations with + boundary conditions, is a cornerstone for the whole paper:

Theorem 1.5. *Let Ω be a bounded simply connected domain and Ω_δ be discretizations of Ω by the square grids $\delta(1+i)\mathbb{Z}^2$. Then, for any $\epsilon>0$ and any $n=1,2\dots$,* *we have*

$$\frac{1}{2\delta}\left(\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1+2\delta}\sigma_{a_2}\dots\sigma_{a_n}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1}\dots\sigma_{a_n}]}-1\right) \quad\to\quad \mathfrak{Re}\mathcal{A}_\Omega(a_1,\dots,a_n), \tag{1.6}$$

$$\frac{1}{2\delta}\left(\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1+2i\delta}\sigma_{a_2}\dots\sigma_{a_n}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1}\dots\sigma_{a_n}]}-1\right) \quad\to\quad -\mathfrak{Im}\mathcal{A}_\Omega(a_1,\dots,a_n) \tag{1.7}$$

as $\delta\to 0$, uniformly over all faces $a_1,\dots a_n\in\Omega_\delta$ at distance at least ϵ from $\partial\Omega$ and from each other. The function $\mathcal{A}_\Omega(a_1,\dots,a_n)$ is defined explicitly via solution to some special interpolation problem (see further details in Section 2.5) and has the following covariance property under conformal mappings $\varphi:\Omega\to\Omega'$:

$$\mathcal{A}_\Omega(a_1,\dots,a_n)=\mathcal{A}_{\Omega'}(\varphi(a_1),\dots,\varphi(a_n))\cdot\varphi'(a_1)+\frac{1}{8}\frac{\varphi''(a_1)}{\varphi'(a_1)}. \tag{1.8}$$

Proof. The proof is based on the convergence results for the discrete spinor observables. A rather delicate analysis is needed since we are interested in the values of observables near their singular points. See further details in Sections 2 and 3. □

Now one can reconstruct the quantities $\log\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle^+_\Omega$ by integrating their derivatives $\mathcal{A}_\Omega(a_1,\dots,a_n)$ inside Ω. In particular, the constants in the integration can be chosen so that the explicit formulae (1.4) given above for $n=1,2$ hold true. In Section 2.8 we also discuss how to choose those multiplicative constants coherently for all Ω and n in order to construct the continuous correlation functions. Note that the conformal covariance degree $\frac{1}{8}$ in (1.3) is a direct consequence of the covariance rule (1.8) for $\mathcal{A}_\Omega(a_1,\dots,a_n)$ (see Remark 2.21).

Theorem 1.5 allows us to prove the following weaker form of the convergence result for the spin correlations:

Corollary 1.6. *Under conditions of Theorem 1.5, for any $n\geq 1$ there exist some normalizing factors $\varrho_n(\delta,\Omega_\delta)$ that might depend on Ω_δ but not on the positions of the points $a_1,\dots,a_n$ such that*

$$\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1}\dots\sigma_{a_n}] \;\sim\; \varrho_n(\delta,\Omega_\delta)\cdot\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle^+_\Omega$$

as $\delta\to 0$, uniformly over all faces $a_1,\dots a_n\in\Omega_\delta$ at distance at least ϵ from $\partial\Omega$ and from each other.

Proof. See Section 2.8. □

We now focus on the special case $n=2$. The next theorem is a crucial tool which allows us to compare the normalizing factors $\varrho_2(\delta,\Omega_\delta)$ with the full-plane case. We denote by $\mathbb{E}^{\mathrm{free}}_{\Omega^\bullet_\delta}$ the expectation for the critical Ising model defined on the *vertices* of Ω_δ (with free boundary conditions).

Theorem 1.7. *Let Ω be a bounded simply connected domain and Ω_δ be discretizations of Ω by the square grids $\delta(1+i)\mathbb{Z}^2$. Then, for any $\epsilon>0$, we have*

$$\frac{\mathbb{E}^{\mathrm{free}}_{\Omega^\bullet_\delta}[\sigma_{a+\delta}\sigma_{b+\delta}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]} \underset{\delta\to 0}{\longrightarrow} \mathcal{B}_\Omega(a,b),$$

uniformly over all faces a,b at distance at least ϵ from $\partial\Omega$ and from each other, where $\mathcal{B}_\Omega(a,b)$ is a conformal invariant of (Ω,a,b) which can be explicitly written via two-point functions (1.4) as $\mathcal{B}_\Omega(a,b)=\langle\sigma_a\sigma_b\rangle^{\mathrm{free}}_\Omega/\langle\sigma_a\sigma_b\rangle^+_\Omega$.

Proof. The proof is based on the convergence results for the discrete spinor observables and the Kramers-Wannier duality, see further details in Sections 2.4, 2.6. □

Sketch of the proof of Theorem 1.1. Having the results of Corollary 1.6 and Theorem 1.7, we only need to prove that $\varrho_2(\delta,\Omega_\delta)\sim\varrho(\delta)$ as $\delta\to 0$. The classical FKG inequality gives

$$\mathbb{E}^{\mathrm{free}}_{\Omega^\bullet_\delta}[\sigma_{a+\delta}\sigma_{b+\delta}]\le\mathbb{E}_{\mathbb{C}_\delta}[\sigma_a\sigma_b]\le\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]$$

and it is easy to see that $\mathcal{B}_\Omega(a,b)\to 1$ as b approaches to a. Since the normalizing factors $\varrho_2(\delta,\Omega_\delta)$ do not depend on the positions of $a,b\in\Omega$, we conclude that $\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]\sim\mathbb{E}_{\mathbb{C}_\delta}[\sigma_a\sigma_b]$ in the double limit when $\delta\to 0$ and (later on) $b\to a$, thus relating $\varrho_2(\delta,\Omega_\delta)$ with the full-plane normalization, see details in Section 2.9. □

Sketch of the proof of Theorem 1.3. Once the asymptotics $\varrho_2(\delta,\Omega_\delta)\sim\varrho(\delta)$, $\delta\to 0$, is established, we derive asymptotics of all other $\varrho_n(\delta,\Omega_\delta)$, $n\neq 2$, using the following observation: as the point a_1 approaches $\partial\Omega$, the continuous correlation functions, when coherently normalized, satisfy

$$\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle^+_\Omega\sim\langle\sigma_{a_1}\rangle^+_\Omega\langle\sigma_{a_2}\dots\sigma_{a_n}\rangle^+_\Omega \tag{1.9}$$

and the same decorrelation result $\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1}\dots\sigma_{a_n}]\sim\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1}]\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_2}\dots\sigma_{a_n}]$ holds true in the double limit $\delta\to 0$ and $a_1\to\partial\Omega$ (see details in Section 2.10). This implies the recurrent formula $\varrho_{n+1}(\delta,\Omega_\delta)\sim\varrho_1(\delta,\Omega_\delta)\varrho_n(\delta,\Omega_\delta)$ for $n=1,2,\dots$ and, further, $\varrho_n(\delta,\Omega_\delta)\sim(\varrho(\delta))^{n/2}$ for all n. □

1.3. **Organization of the paper.** Section 2 contains all the main ideas. Details, especially those involving hard discrete complex analysis techniques, are mostly postponed to Section 3. The readers not interested in these details may restrict themselves to Section 2 only.

We fix the notation in Section 2.1. The main tool of this paper, the discrete holomorphic spinor observables, is introduced and discussed in Sections 2.2 and 2.3. In Section 2.4, we prove that the ratios of spin correlations that appear in Theorems 1.5 and 1.7 can be expressed in terms of these observables. In Sections 2.5 and 2.6, we discuss the continuous counterparts of the discrete observables and state the convergence Theorems 2.15, 2.17 and 2.19, that easily imply Theorems 1.5 and 1.7. We derive the formulae (1.4) in Section 2.7, and prove Corollary 1.6 in Section 2.8. We complete the proofs of Theorems 1.1 and 1.3 in Sections 2.9 and 2.10, respectively.

Section 3 is devoted to the proof of Theorems 2.15, 2.17 and 2.19. We discuss the discrete properties of our observables and their full-plane analogue in Sections 3.1–3.3, and finish the proof of the main convergence theorems in Sections 3.4 and 3.5. In the Appendix, we explain how to compute explicitly the continuous spinor observables in the half-plane. We also check that the correlation functions coincide with the CFT predictions at least when all points are on the same vertical line in the upper half-plane, and prove decorrelation identity (1.9).

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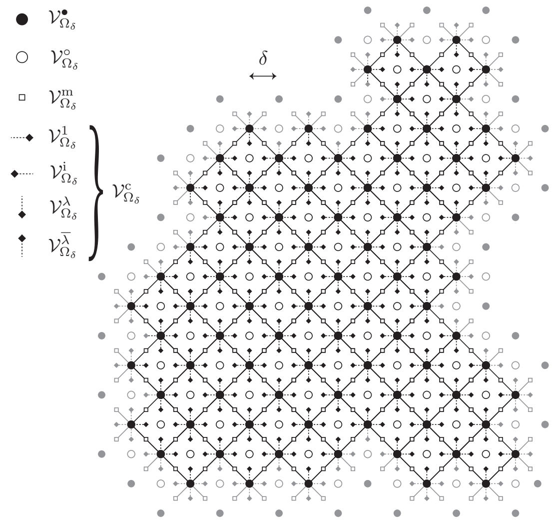

FIGURE 1. An example of a discrete domain Ω_δ and notation for the sets of vertices ($\mathcal{V}^{\bullet}_{\Omega_\delta}$), faces ($\mathcal{V}^{\circ}_{\Omega_\delta}$), edge midpoints ($\mathcal{V}^{\bullet}_{\Omega_\delta}$) and (four types) of corners ($\mathcal{V}^{\mathrm{c}}_{\Omega_\delta} = \mathcal{V}^{1}_{\Omega_\delta} \cup \mathcal{V}^{\mathrm{i}}_{\Omega_\delta} \cup \mathcal{V}^{\lambda}_{\Omega_\delta} \cup \mathcal{V}^{\overline{\lambda}}_{\Omega_\delta}$). The mesh size δ is a half-diagonal of a square face, thus the distance between adjacent spins is $\sqrt{2}\delta$. The inner vertices, faces, edges and corners are colored black, while the boundary ones are colored gray.

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2. Holomorphic Spinors and Correlation Functions

2.1. **Notation.** We start by fixing the notation which is used throughout the paper.

2.1.1. *Graph notation.* Recall that we work on the square grid rotated by 45°

$$\mathbb{C}_\delta := \{\delta\,(1+i)\,(m+in) : m, n \in \mathbb{Z}\}\,.$$

The mesh size δ is hence the size of a half-diagonal of a square face. We often will identify the vertices of $\mathbb{C}_\delta$ with the corresponding complex numbers, the faces of $\mathbb{C}_\delta$ with their centers, the edges of $\mathbb{C}_\delta$ with their midpoints, etc.

We call *discrete domain of mesh size* δ a union of grid faces (see also Figure 1 for notation given below). We say that Ω_δ is *simply connected* if the corresponding polygonal domain is simply connected.

- We denote by $\mathrm{Int}\mathcal{V}^{\circ}_{\Omega_\delta}$ the set of all *faces* belonging to Ω_δ (which are identified with their centers), by $\mathrm{Int}\mathcal{V}^{\bullet}_{\Omega_\delta}$ the set of all *vertices* incident to these faces, and by $\mathrm{Int}\mathcal{V}^{\mathrm{m}}_{\Omega_\delta}$ the set of all *edges* incident to $\mathcal{V}^{\circ}_{\Omega_\delta}$, which are identified with their midpoints (or the vertices of a medial lattice).

In order to simplify the presentation, we also assume that Ω_δ *has no fiords of a single face width*, i.e., all the edges joining vertices from $\mathrm{Int}\mathcal{V}^{\bullet}_{\Omega_\delta}$ belong to $\mathrm{Int}\mathcal{V}^{\mathrm{m}}_{\Omega_\delta}$. This technical assumption can be easily relaxed, if necessary.

- We denote by $\partial\mathcal{V}^{\circ}_{\Omega_\delta}$, $\partial\mathcal{V}^{\bullet}_{\Omega_\delta}$ and $\partial\mathcal{V}^{\mathrm{m}}_{\Omega_\delta}$ the sets of *boundary faces, vertices and edges*, i.e. those faces, vertices and edges which are incident but do not belong to $\mathrm{Int}\mathcal{V}^{\circ}_{\Omega_\delta}$, $\mathrm{Int}\mathcal{V}^{\bullet}_{\Omega_\delta}$ and $\mathrm{Int}\mathcal{V}^{\mathrm{m}}_{\Omega_\delta}$, respectively (see Figure 1).
- We set $\mathcal{V}^{\circ}_{\Omega_\delta} := \mathrm{Int}\mathcal{V}^{\circ}_{\Omega_\delta} \cup \partial\mathcal{V}^{\circ}_{\Omega_\delta}$ etc.

Below we also need to work with four *corners* of a given square face separately (see the crucial Definition 2.1 below). For a given vertex $v \in \mathbb{C}_\delta$, we identify the nearby corners with the points $v \pm \frac{1}{2}\delta$ and $v \pm \frac{1}{2}\delta i$ on the complex plane.

- We denote by $\mathcal{V}^{\mathrm{c}}_{\Omega_\delta}$ the set of all corners incident to the vertices from $\mathrm{Int}\mathcal{V}^{\bullet}_{\Omega_\delta}$. We also set $\mathcal{V}^{\mathrm{cm}}_{\Omega_\delta} := \mathcal{V}^{\mathrm{c}}_{\Omega_\delta} \cup \mathcal{V}^{\mathrm{m}}_{\Omega_\delta}$.
- We use the notation $x \sim y$ if each of x, y is either a vertex, a face, an edge or a corner, and they are adjacent or incident to each other.

2.1.2. *Double covers.* In this paper we often deal with holomorphic functions (both discrete and continuous) which are defined on a double cover of a planar domain Ω and changes the sign between sheets (i.e., have the multiplicative monodromy -1 around branching points). We call such functions *holomorphic spinors*. The following notation will be used below:

- For a planar domain Ω and $a \in \Omega$, we denote by $[\Omega, a]$ the double cover of $\Omega \setminus \{a\}$ branching around a. All such double covers are naturally viewed as subdomains of $[\mathbb{C}, a]$. We often identify points on a double cover with their base points (so each base point is identified with the two points above it).
- For several marked points $a_1, \dots, a_n \in \Omega$, we denote by $[\Omega, a_1, \dots, a_n]$ the double cover of $\Omega \setminus \{a_1, \dots, a_n\}$ which branches around each of $a_1, \dots, a_n$.
- We will often compare spinors defined on $[\Omega, a_1, \dots, a_n]$ with those defined on $[\mathbb{C}, a_1]$ (e.g., with $1/\sqrt{z - a_1}$ or $\sqrt{z - a_1}$) near the common branching point a_1. We write such equations meaning that they are valid in a small neighborhood of a_1, with the natural correspondence of sheets.

We will also consider double covers of discrete domains. In this case the following slightly modified notation will be convenient:

- For a discrete domain Ω_δ and a *face* $a \in \mathrm{Int}\mathcal{V}^{\circ}_{\Omega_\delta}$, we set $[\Omega_\delta, a^{\rightarrow}] := [\Omega_\delta, a] \setminus \{a + \frac{\delta}{2}\}$, excluding both points over the corner $a + \frac{\delta}{2}$ from the natural double cover branching at a. Similarly, we set $[\Omega_\delta, a_1^{\rightarrow}, a_2, \dots a_n] := [\Omega_\delta, a_1, \dots, a_n] \setminus \{a_1 + \frac{\delta}{2}\}$, if several faces $a_1, \dots, a_n \in \mathrm{Int}\mathcal{V}^{\circ}_{\Omega_\delta}$ are marked.

2.1.3. *Contours.* Recall that we consider the critical Ising model on the *faces* of Ω_δ. In order to define the main tool of the paper – discrete holomorphic spinors, we need some additional notation related to the contour representation of the model known as the *low-temperature expansion*, see [Pal07, Chapter 1].

- We denote by $\mathcal{C}_{\Omega_\delta}$ the family of all collections of closed contours on Ω_δ, i.e. the family of subsets of edges $\omega \subset \mathcal{V}^{\mathrm{m}}_{\Omega_\delta}$ such that every vertex $v \in \mathcal{V}^{\bullet}_{\Omega_\delta}$ belongs to an even number of edges in ω.

The set $\mathcal{C}_{\Omega_\delta}$ is in a natural 1-to-1 correspondence with the spin configurations on Ω_δ with + boundary conditions: trace an edge between any two adjacent faces with different spins. Under this mapping, the probability of a collection of interfaces $\omega \subset \Omega_\delta$ becomes proportional to $\alpha_{\mathrm{c}}^{\#\mathrm{edges}(\omega)}$, where $\alpha_{\mathrm{c}} = \exp(-2\beta_{\mathrm{c}}) = \sqrt{2}-1$.

Below we also introduce families of contour collections which, besides a number of closed loops, contain a single path running from one fixed corner x to another corner or an edge midpoint y:

- For $x, y \in \mathcal{V}^{\mathrm{cm}}_{\Omega_\delta}$, let $\pi_{x,y} = x \sim v_1 \sim \ldots \sim v_n \sim y$ be some simple lattice path with $v_1, \ldots, v_n \in \mathcal{V}^{\bullet}_{\Omega_\delta}$. We set $\mathcal{C}_{\Omega_\delta}(x,y) := \{\omega \oplus \pi_{x,y}, \omega \in \mathcal{C}_{\Omega_\delta}\}$, where $\oplus$ denotes the XOR, or symmetric difference. It is easy to see that $\mathcal{C}_{\Omega_\delta}(x,y)$ does not depend on the particular choice of $\pi_{x,y}$. Note that, for any $\gamma \in \mathcal{C}_{\Omega_\delta}(x,y)$, there exists a (non-unique) decomposition of γ into a collection of disjoint, simple loops and a path $\mathrm{p}(\gamma) \subset \gamma$ running from x to y. By a decomposition we mean that each edge in $\gamma \in \mathcal{C}_{\Omega_\delta}(x,y)$ belongs to exactly one loop (or to $\mathrm{p}(\gamma)$) and is visited only once, and that there are no transversal intersections or self-intersections (see Figure 2).

2.2. **Construction of discrete spinor observables.** Now we are ready to introduce discrete spinor observables. The following definition generalizes the construction given in [ChIz11] to the case when a "source point" is inside Ω_δ.

Definition 2.1. Let Ω_δ be a discrete domain and $a_1, \ldots, a_n \in \mathrm{Int}\mathcal{V}^{\circ}_{\Omega_\delta}$ be inner faces. For a corner $z \in \mathcal{V}^{\mathrm{c}}_{[\Omega_\delta, \vec{a_1}, \ldots, a_n]}$ (below we also extend this definition to edge midpoints, see Remark 2.2(iii)), we define

$$F_{[\Omega_\delta, a_1, \ldots, a_n]}(z) := \frac{1}{\mathcal{Z}^{+}_{\Omega_\delta}[\sigma_{a_1} \ldots \sigma_{a_n}]} \sum_{\gamma \in \mathcal{C}_{\Omega_\delta}(a_1 + \frac{\delta}{2}, z)} \alpha_{\mathrm{c}}^{\#\mathrm{edges}(\gamma)} \cdot \phi_{a_1, \ldots, a_n}(\gamma, z), \tag{2.1}$$

where

- $\#\mathrm{edges}(\gamma)$ is the number of full edges contained in γ and $\alpha_{\mathrm{c}} = \sqrt{2}-1$;
- the complex phase $\phi_{a_1, \ldots, a_n}(\gamma, z)$ is defined by (see also Figure 2)

$$\phi_{a_1, \ldots, a_n}(\gamma, z) := e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma))} \cdot (-1)^{\#\mathrm{loops}_{a_1, \ldots, a_n}(\gamma \setminus \mathrm{p}(\gamma))} \cdot \mathrm{sheet}_{a_1, \ldots, a_n}(\mathrm{p}(\gamma), z),$$

  where, for a decomposition of γ mentioned in Section 2.1.3,
  - $\mathrm{wind}(\mathrm{p}(\gamma))$ is the total winding (increment of the argument) of the path $\mathrm{p}(\gamma)$ when going from $a_1 + \frac{\delta}{2}$ to z,
  - $\#\mathrm{loops}_{a_1, \ldots, a_n}(\gamma \setminus \mathrm{p}(\gamma))$ is the number of loops in $\gamma \setminus \mathrm{p}(\gamma)$ that contain an odd number of marked points $a_1, \ldots, a_n$ (equivalently, that do not lift to the double cover $[\Omega_\delta, a_1, \ldots, a_n]$ as closed loops),
  - the last factor $\mathrm{sheet}_{a_1, \ldots, a_n}(\mathrm{p}(\gamma), z)$ is equal to $+1$ if z is on the same sheet of $[\Omega_\delta, a_1, \ldots, a_n]$ as the end of the lift of $\mathrm{p}(\gamma)$, and to -1 otherwise (more precisely, we fix one of the two points lying over the "source" $a_1 + \frac{\delta}{2}$ once forever and identify all other $z \in [\Omega_\delta, a_1, \ldots, a_n]$ with paths running from *this* $a_1 + \frac{\delta}{2}$ to z modulo homotopy and an appropriate index 2 subgroup of the fundamental group);
- the normalizing factor $\mathcal{Z}^{+}_{\Omega_\delta}[\sigma_{a_1} \ldots \sigma_{a_n}]$ is defined by

$$\mathcal{Z}^{+}_{\Omega_\delta}[\sigma_{a_1} \ldots \sigma_{a_n}] := \sum_{\omega \in \mathcal{C}_{\Omega_\delta}} \alpha_{\mathrm{c}}^{\#\mathrm{edges}(\omega)} (-1)^{\#\mathrm{loops}_{a_1, \ldots, a_n}(\omega)}. \tag{2.2}$$

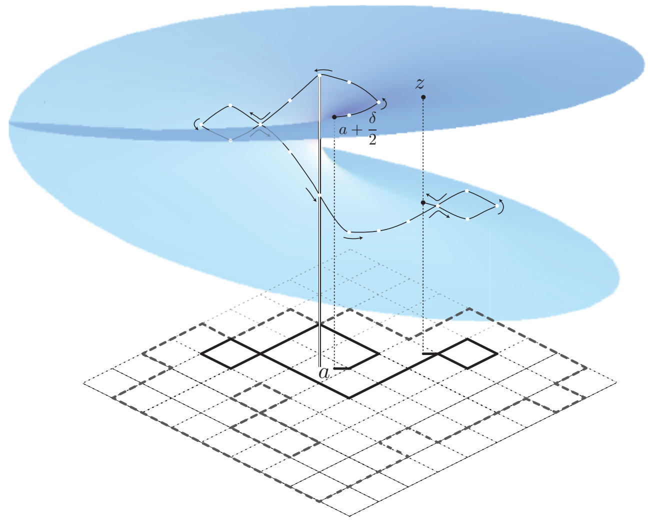

FIGURE 2. A contour collection $\gamma \in \mathcal{C}_{\Omega_\delta}(a+\frac{\delta}{2},z)$ decomposed into non-intersecting loops (dashed) and a path $\mathrm{p}(\gamma)$. Running from $a+\frac{\delta}{2}$ to the projection of z, this path makes a 3π turn counterclockwise, thus $e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma))}=i$. There is a single loop in γ surrounding a, hence $(-1)^{\mathrm{loops}_a(\gamma\setminus\mathrm{p}(\gamma))}=-1$. Being lifted to the double cover $[\Omega_\delta,a]$, this path ends on the other sheet, thus $\mathrm{sheet}_a(\mathrm{p}(\gamma),z)=-1$, and $\phi_a(\gamma,z)=i$.

Remark 2.2. (i) For any $\omega\in\mathcal{C}_{\Omega_\delta}$, the sign $(-1)^{\#\mathrm{loops}_{a_1,\dots,a_n}(\omega)}$ coincides with the product of spins $\sigma_{a_1}\dots\sigma_{a_n}$ in the corresponding Ising model configuration with + boundary values. Since $\alpha_{\mathrm{c}}^{\#\mathrm{edges}(\omega)}$ is just the Ising weight of ω, one concludes that

$$\mathcal{Z}^+_{\Omega_\delta}[\sigma_{a_1}\dots\sigma_{a_n}]=\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1}\dots\sigma_{a_n}]\cdot\mathcal{Z}^+_{\Omega_\delta}>0,$$

where $\mathcal{Z}^+_{\Omega_\delta}=\sum_{\omega\in\mathcal{C}_{\Omega_\delta}}\alpha_{\mathrm{c}}^{\#\mathrm{edges}}$ is the partition function of the model.

(ii) It is easy to check that the complex phase $\phi_{a_1,\dots,a_n}(\gamma,z)$ is independent of the choice of a decomposition of γ into a path $\mathrm{p}(\gamma)$ and a collection of loops, e.g., see discussion in [ChIz11]. Note that there are four types of corners: lying to the right of a nearby vertex v, below v, to the left of v, and upper v. For each of these groups, the total turning of the path $\mathrm{p}(\gamma)$ is defined uniquely modulo 2π. Therefore, the discrete spinors introduced above always have purely real values at the first group corners, are collinear to $\lambda:=e^{\frac{\pi}{4}i}$ for the second group, etc. This motivates the following notation:

- we partition the set $\mathcal{V}^{\mathrm{c}}_{\mathbb{C}_\delta}$ of all corners into four subsets $\mathcal{V}^1_{\Omega_\delta}$, $\mathcal{V}^\lambda_{\Omega_\delta}$, $\mathcal{V}^i_{\Omega_\delta}$ and $\mathcal{V}^{\overline{\lambda}}_{\Omega_\delta}$ depending on the position of a nearby vertex $v\in\mathcal{V}^\bullet_{\mathbb{C}_\delta}$ (to the left, upper, to the right, below) with respect to the corner.

(iii) *We extend Definition 2.1 to edge midpoints* $z\in\mathcal{V}^{\mathrm{m}}_{[\Omega_\delta,a_1^{\rightarrow},\dots,a_n]}$ *by adding the factor* $(\cos\frac{\pi}{8})^{-1}$ *to the formula (2.1)*, with $\#\mathrm{edges}(\gamma)$ being the number of full edges contained in γ and the complex phase $\phi_{a_1,\dots,a_n}(\gamma,z)$ being defined as above.

Note that each edge midpoint z can be reached by a path $\mathrm{p}(\gamma)$ from two opposite sides. Thus, in this case the argument of the spinor value $F_{[\Omega_\delta,a_1,\dots,a_n]}(z)$ is unfixed.
(iv) The definition of $F_{[\Omega_\delta,a_1,\dots,a_n]}$ is invariant under permutations of $a_2,\dots,a_n$. The reader should always keep in mind, however, that the point a_1 plays a special role. The same applies to $\mathcal{A}_\Omega(a_1,\dots,a_n)$ and other related notation below.

2.3. **S-holomorphicity and boundary conditions.** A version of discrete holomorphicity, the notion of *s-holomorphicity* was introduced in [Smi06] together with the nonbranching version of discrete holomorphic observables, as a tool to study the critical Ising model on the square lattice. The properties of such functions were further investigated in [ChSm12] for a more general class of graphs. On the square grid, s-holomorphic functions may be thought of as (more classical) discrete holomorphic functions whose real part is defined on $\mathcal{V}^1_{\Omega_\delta}$ and imaginary part on $\mathcal{V}^i_{\Omega_\delta}$, extended in a particular way to $\mathcal{V}^\lambda_{\Omega_\delta}$, $\mathcal{V}^{\overline{\lambda}}_{\Omega_\delta}$, and further to $\mathcal{V}^{\mathrm{m}}_{\Omega_\delta}$ (see more details in Section 3.1). Our definitions resemble those in [Smi06].

Definition 2.3. With each corner $x\in\mathcal{V}^\tau_{\mathbb{C}_\delta}$ (with $\tau\in\left\{1,i,\lambda,\overline{\lambda}\right\}$), we associate the line $\ell(x):=\tau\mathbb{R}$ in the complex plane, and denote by $\mathsf{P}_{\ell(x)}$ the projection onto that line, defined by

$$\mathsf{P}_{\ell(x)}[w]:=\tfrac{1}{2}\left(w+\tau^2\overline{w}\right),\quad w\in\mathbb{C}.$$

We say that a function $F:\mathcal{V}^{\mathrm{cm}}_{\Omega_\delta}\to\mathbb{C}$ is *s-holomorphic in* Ω_δ if for every $x\in\mathcal{V}^{\mathrm{c}}_{\Omega_\delta}$ and $z\in\mathcal{V}^{\mathrm{m}}_{\Omega_\delta}$ that are adjacent, one has

$$F(x)=\mathsf{P}_{\ell(x)}\left[F(z)\right].$$

For functions defined on double covers, we introduce the notion of s-holomorphicity exactly in the same manner.

The following proposition contains the crucial properties of $F_{[\Omega_\delta,a_1,\dots,a_n]}$ that will allow us to analyze their scaling limits. For $z\in\partial\mathcal{V}^{\mathrm{m}}_{\Omega_\delta}$, let $\nu_{\mathrm{out}}(z)$ denote the *"outer normal to the boundary at z"*: the edge whose midpoint is z, oriented towards the exterior of the domain and viewed as a complex number.

Proposition 2.4. *The function $F_{[\Omega_\delta,a_1,\dots,a_n]}$ is s-holomorphic and has (multiplicative) monodromy -1 around each of the marked points $a_1,\dots,a_n$, thus being a discrete s-holomorphic spinor on $\mathcal{V}^{\mathrm{cm}}_{[\Omega_\delta,a_1^\to,\dots,a_n]}$. Also,*

$$\mathfrak{Im}\left[F_{[\Omega_\delta,a_1,\dots,a_n]}(z)\sqrt{\nu_{\mathrm{out}}(z)}\right]=0\quad\text{for all}\quad z\in\partial\mathcal{V}^{\mathrm{m}}_{[\Omega_\delta,a_1^\to,\dots,a_n]}.\tag{2.3}$$

Proof. We give a proof (based on the standard XOR bijection, cf. [ChSm12]) in Section 3.1. □

Remark 2.5. The boundary conditions (2.3) are a priori not robust enough to pass to the scaling limit: even if the limiting domain Ω has a smooth boundary, the discrete normal $\nu_{\mathrm{out}}(z)$ can possibly admit only the values $e^{\pm\frac{\pi i}{4}}$ and $e^{\pm\frac{3\pi i}{4}}$, and so does not (pointwise) converge to its continuous counterpart. These conditions become much nicer, if one finds a way "to integrate" the square of $F_{[\Omega_\delta,a_1,\dots,a_n]}$: the real part of this integral satisfies Dirichlet boundary conditions on $\partial\Omega$ due to (2.3). This approach is not that straightforward, since the square of a discrete holomorphic function is, in general, not discrete holomorphic, and so does not have a well-defined discrete primitive. However, it was noted in [Smi06] that one can naturally define the real part of the integral, using the *s-holomorphicity* of discrete observables, which is a stronger property than the usual discrete holomorphicity. Moreover, a technique developed in [ChSm12] allows one to treat this real part essentially as if it were a harmonic function, see further details in Section 3.3.

2.4. **From discrete spinors to ratios of correlations.** The following lemma expresses ratios of spin correlations in terms of the spinor observables introduced in Section 2.2, providing a crucial ingredient for the proof of Theorems 1.5 and 1.7.

Lemma 2.6. *For any $n = 1, 2, \dots,$ we have*

$$\frac{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a_1+2\delta}\sigma_{a_2}\dots\sigma_{a_n}\right]}{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a_1}\dots\sigma_{a_n}\right]} \quad = \quad F_{[\Omega_\delta,a_1,\dots,a_n]}\left(a_1+\tfrac{3\delta}{2}\right), \tag{2.4}$$

where we take the corner $a_1+\frac{3\delta}{2}$ on the same sheet as the "source point" $a_1+\frac{\delta}{2}$. Moreover, in the case of just two marked points, we also have

$$\frac{\mathbb{E}^{\mathrm{free}}_{\Omega^\bullet_\delta}\left[\sigma_{a+\delta}\sigma_{b+\delta}\right]}{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_b\right]} \quad = \quad \pm iF_{[\Omega_\delta,a,b]}\left(b+\tfrac{\delta}{2}\right), \tag{2.5}$$

where $\mathbb{E}^{\mathrm{free}}_{\Omega^\bullet_\delta}$ denotes the expectation for the critical Ising model defined on the vertices of Ω_δ (with free boundary conditions outside the set $\mathcal{V}^\bullet_{\Omega_\delta}$) and the sign $\pm$ depends on the sheet where the corner $b+\frac{\delta}{2}$ is taken.

Proof. Recall that

$$F_{[\Omega_\delta,a_1,\dots,a_n]}\left(a_1+\tfrac{3\delta}{2}\right) \quad = \quad \frac{\sum_{\gamma\in\mathcal{C}_{\Omega_\delta}\left(a_1+\frac{\delta}{2},a_1+\frac{3\delta}{2}\right)}\alpha_{\mathrm{c}}^{\#\mathrm{edges}(\gamma)}\phi_{a_1,\dots,a_n}\left(\gamma,a_1+\frac{3\delta}{2}\right)}{\mathcal{Z}^{+}_{\Omega_\delta}\left[\sigma_{a_1}\dots\sigma_{a_n}\right]},$$

while, taking into account Remark 2.2(i),

$$\frac{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a_1+2\delta}\sigma_{a_2}\dots\sigma_{a_n}\right]}{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a_1}\dots\sigma_{a_n}\right]} \quad = \quad \frac{\sum_{\omega\in\mathcal{C}_{\Omega_\delta}}\alpha_{\mathrm{c}}^{\#\mathrm{edges}(\omega)}(-1)^{\#\mathrm{loops}_{a_1+2\delta,\dots,a_n}(\omega)}}{\mathcal{Z}^{+}_{\Omega_\delta}\left[\sigma_{a_1}\dots\sigma_{a_n}\right]}.$$

There is a simple bijection between the sets $\mathcal{C}_{\Omega_\delta}\left(a_1+\frac{\delta}{2},a_1+\frac{3\delta}{2}\right)$ and $\mathcal{C}_{\Omega_\delta}$: removing the two corner-edges $(a_1+\frac{\delta}{2},a_1+\delta)$ and $(a_1+\delta,a_1+\frac{3\delta}{2})$ from a given γ, we obtain a collection of closed loops $\omega(\gamma)\in\mathcal{C}_{\Omega_\delta}$ and vice versa. So, it suffices to show that

$$\phi_{a_1\dots,a_n}\left(\gamma,a_1+\tfrac{3\delta}{2}\right) = (-1)^{\#\mathrm{loops}_{a_1+2\delta,a_2,\dots,a_n}(\omega(\gamma))}.$$

Let us pick any loop in γ and remove it. The left-hand side (respectively, the right-hand side) has changed the sign if and only if there was an odd number of points $a_1,\dots,a_n$ (respectively, $a_1+2\delta,a_2,\dots,a_n$) inside the loop. However, no loop in γ separates a_1 from $a_1+2\delta$ (such a loop would intersect $\mathrm{p}(\gamma)$), so the two sides can only change sign simultaneously. Thus it is sufficient to consider the case when γ is just a single non-self-intersecting path $\mathrm{p}(\gamma)$ running from $a_1+\frac{\delta}{2}$ to $a_1+\frac{3\delta}{2}$, which is treated by the following observations: $\mathrm{sheet}_{a_1\dots,a_n}(\mathrm{p}(\gamma),a_1+\frac{3\delta}{2})=-1$ if and only if there is an odd number of points $a_1,\dots,a_n$ inside the loop $\omega(\mathrm{p}(\gamma))$, and $\mathrm{wind}(\mathrm{p}(\gamma))=2\pi \mod 4\pi$ if an only if $\omega(\mathrm{p}(\gamma))$ separates a_1 from $a_1+2\delta$.

For (2.5), the Kramers-Wannier duality (e.g., see [Pal07, Chapter 1]) implies

$$\frac{\mathbb{E}^{\mathrm{free}}_{\Omega^\bullet_\delta}\left[\sigma_{a+\delta}\sigma_{b+\delta}\right]}{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\sigma_b\right]} = \frac{\sum_{\omega\in\mathcal{C}_{\Omega_\delta}(a+\frac{\delta}{2},b+\frac{\delta}{2})}\alpha_{\mathrm{c}}^{\#\mathrm{edges}(\omega)}}{\mathcal{Z}^{+}_{\Omega_\delta}[\sigma_a\sigma_b]},$$

hence it is sufficient to prove that the (purely imaginary) number $\phi_{a,b}\left(\gamma, b+\frac{\delta}{2}\right)$ does not depend on γ. We have $\#\mathrm{loops}_{a,b}(\gamma\setminus \mathrm{p}(\gamma))=0$, since any loop in γ either surrounds both a,b or none of them (otherwise it would intersect the path $\mathrm{p}(\gamma)$ joining $a+\frac{\delta}{2}$ and $b+\frac{\delta}{2}$). Further, let $\pi^\circ_{ba}=b\sim v_1\sim\dots\sim v_m\sim a$ be some simple lattice path with $v_j\in\mathcal{V}^\circ_{\Omega_\delta}$. Then, $\mathrm{p}(\gamma)\cup(b+\frac{\delta}{2},b)\cup\pi^\circ_{ab}\cup(a,a+\frac{\delta}{2})$ is a loop, which we denote by $l(\gamma)$. Let $n(\gamma)$ be the number of self-intersections of $l(\gamma)$ (in order words, the number of intersections of $\mathrm{p}(\gamma)$ with π°_{ba}). Since $\exp[-\frac{i}{2}\mathrm{wind}(l(\gamma))]=(-1)^{n(\gamma)+1}$ (this is true for any closed loop), we see that

$$\exp[-\tfrac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma))]=(-1)^{n(\gamma)+1}\cdot\exp[\tfrac{i}{2}\mathrm{wind}((b+\tfrac{\delta}{2},b)\cup\pi^\circ_{ba}\cup(a,a+\tfrac{\delta}{2}))],$$

where the second factor does not depend on γ. But we may also view π°_{ba} as a cut defining a sheet of the double cover $[\Omega,a,b]$, meaning that $\mathrm{sheet}_{a,b}(p(\gamma),b+\frac{\delta}{2})=(-1)^{n(\gamma)}$, which proves the desired result. $\square$

Remark 2.7. (i) Similarly to (2.4), one can check that

$$\frac{\mathbb{E}^+_{\Omega_\delta}\left[\sigma_{a_1+(1\pm i)\delta}\sigma_{a_2}\dots\sigma_{a_n}\right]}{\mathbb{E}^+_{\Omega_\delta}\left[\sigma_{a_1}\dots\sigma_{a_n}\right]} \quad=\quad e^{\pm\frac{\pi i}{4}}\,F_{[\Omega_\delta,a_1,\dots,a_n]}\left(a_1+\left(1\pm\tfrac{i}{2}\right)\delta\right). \qquad (2.6)$$

The proof boils down to the identity

$$e^{\pm\frac{\pi i}{4}}\phi_{a_1,\dots,a_n}\left(\gamma,a_1+\left(1\pm\tfrac{i}{2}\right)\delta\right)=(-1)^{\#\mathrm{loops}_{a_1+\delta(1\pm i),a_2,\dots,a_n}(\omega(\gamma))}$$

for the natural bijection $\gamma\mapsto\omega(\gamma)$ removing the two corner-edges $(a_1+\frac{\delta}{2},a_1+\delta)$ and $(a_1+\delta,a_1+(1\pm\frac{i}{2})\delta)$ from a given $\gamma\in\mathcal{C}_{\Omega_\delta}\left(a_1+\frac{\delta}{2},a_1+(1\pm\frac{i}{2})\delta\right)$, which we leave to the reader.
(ii) The identity (2.5) can be extended to the case of $2n$ marked points, see [ChIz11, Proposition 5.6], thus allowing one to treat $2n$-point correlation functions with free boundary conditions.

2.5. **Continuous spinors.** In this Section, we introduce the continuous counterparts of the discrete spinor observables defined in Section 2.2: the continuous holomorphic spinors $f_{[\Omega,a_1,\dots,a_n]}$. We define them as solutions to the conformally covariant Riemann boundary value problem (2.7)–(2.9), which is a continuous analogue of the corresponding discrete boundary value problem (see Remark 2.10 below).

Definition 2.8. Let Ω be a bounded simply connected domain with smooth boundary, and $a_1,\dots,a_n\in\Omega$. We define $f_{[\Omega,a_1,\dots,a_n]}$ to be the (unique) holomorphic spinor on $[\Omega,a_1,\dots,a_n]$, branching around each of $a_1,\dots,a_n$ and satisfying the following conditions:

$$\Im\mathfrak{m}\left[f_{[\Omega,a_1,\dots,a_n]}(z)\sqrt{\nu_{\mathrm{out}}(z)}\right] \quad=\quad 0,\quad z\in\partial\Omega; \qquad (2.7)$$

$$\lim_{z\to a_1}\sqrt{z-a_1}\cdot f_{[\Omega,a_1,\dots,a_n]}(z) \quad=\quad 1; \qquad (2.8)$$

$$\lim_{z\to a_k}\sqrt{z-a_k}\cdot f_{[\Omega,a_1,\dots,a_n]}(z) \quad\in\quad i\mathbb{R},\quad k=2,\dots,n, \qquad (2.9)$$

where $\nu_{\mathrm{out}}(z)$ denotes the outer normal to the boundary of Ω at z.

Remark 2.9. (i) Note that a solution to the boundary value problem (2.7)–(2.9) is unique, if it exists. Indeed, if f_1,f_2 denote two different solutions, then the spinor f_1-f_2 satisfies (2.7) and (2.9), while $\lim_{z\to a_1}\sqrt{z-a_1}\cdot(f_1(z)-f_2(z))=0$.

Applying the Cauchy residue theorem to the *single-valued* function $(f_1(z)-f_2(z))^2$, one arrives at

$$0 \le i^{-1}\int_{\partial\Omega}(f_1(z)-f_2(z))^2dz = 2\pi\sum_{k=2}^{n}\lim_{z\to a_k}(z-a_k)(f_1(z)-f_2(z))^2 \le 0, \quad (2.10)$$

where the first inequality easily follows from (2.7) and the second from (2.9).

(ii) If $\varphi:\Omega\to\Omega'$ is a conformal mapping, then one has

$$f_{[\Omega,a_1\dots,a_n]}(z) = f_{[\Omega',\varphi(a_1)\dots,\varphi(a_n)]}(\varphi(z))\cdot(\varphi'(z))^{1/2}. \quad (2.11)$$

Indeed, it is straightforward to check that the right-hand side solves the boundary-value problem (2.7) – (2.9). We use this covariance property as a *definition* of the continuous spinor in an arbitrary simply connected domain Ω, using a conformal map to some smooth bounded Ω'.

(iii) In Section 2.7 and the Appendix we give explicit solutions to (2.7)–(2.9) in the upper half-plane, thus proving the existence of $f_{[\Omega,a_1\dots,a_n]}(z)$.

Remark 2.10. The first condition (2.7) in Definition 2.8 is a natural counterpart of (2.3). The third condition (2.9) comes from the following observation: a discrete primitive $H_{[\Omega,a_1,\dots,a_n]}$ of the "discrete differential form" $\mathfrak{Re}\,[F^2_{[\Omega_\delta,a_1,\dots,a_n]}dz]$ (which may be defined due to the s-holomorphicity property of the discrete observable, see Remark 2.5) remains bounded from below near the branching points $a_2,\dots,a_n$ as $\delta\to 0$. Thus, we impose the same condition for the scaling limits, which means that $\mathfrak{Re}\,[\int f^2_{[\Omega,a_1,\dots,a_n]}dz]$ should behave like $c_k\log|z-a_k|$ for some *negative* c_k as $z\to a_k$, $k=2,\dots,n$, implying (2.9). The second condition (2.8) which fixes the behavior of $f_{[\Omega,a_1,\dots,a_n]}$ near the "source point" a_1 is the most delicate one and will be clarified later on (see Section 3.2, particularly Lemma 3.5). Note that it is sufficient to assume that $f_{[\Omega,a_1,\dots,a_n]}$ does not blow up faster than $1/\sqrt{z-a_1}$ at a_1. Indeed, in this case the argument similar to (2.10) shows that $\lim_{z\to a_1}(z-a_1)(f_{[\Omega,a_1,\dots,a_n]}(z))^2>0$ and the rest is just a proper choice of the normalization.

Let us now introduce the quantities that play a central role in our computations of scaling limits of spin correlations, appearing as the limits of discrete logarithmic derivatives in Theorem 1.5.

Definition 2.11. We define the complex number $\mathcal{A}_\Omega(a_1,\dots,a_n)=\mathcal{A}_{[\Omega,a_1,\dots,a_n]}$ as the coefficient in the expansion

$$f_{[\Omega,a_1,\dots,a_n]} = \frac{1}{\sqrt{z-a_1}} + 2\mathcal{A}_{[\Omega,a_1,\dots,a_n]}\sqrt{z-a_1} + O(|z-a_1|^{3/2}) \quad (2.12)$$

of $f_{[\Omega,a_1,\dots,a_n]}$ near the point a_1. In the special case $n=2$, we also define the quantity $\mathcal{B}_\Omega(a,b)=\mathcal{B}_{[\Omega,a,b]}>0$ as the coefficient in the expansion of $f_{[\Omega,a,b]}$ near b:

$$f_{[\Omega,a,b]} = \pm\frac{i\mathcal{B}_{[\Omega,a,b]}}{\sqrt{z-b}} + O(|z-b|^{1/2}), \quad (2.13)$$

where the sign $\pm$ depends on the sheet of $[\Omega,a,b]$.

Remark 2.12. Note that the covariance rule (1.8) for $\mathcal{A}_{[\Omega,a_1,\dots,a_n]}$ directly follows from the conformal covariance (2.11) of spinor observables: if $\varphi:\Omega\to\Omega'$ is a conformal mapping, then one has

$$\begin{aligned} f_{[\Omega,a_1,\dots,a_n]}(z) &= (\varphi'(z))^{1/2} f_{[\Omega',\varphi(a_1),\dots,\varphi(a_n)]}(\varphi(z)) \\ &= \left[\frac{\varphi'(z)}{\varphi(z)-\varphi(a_1)}\right]^{\frac{1}{2}} \cdot \left[1+2\mathcal{A}_\varphi\cdot(\varphi(z)-\varphi(a_1))+\dots\right] \\ &= \left[\frac{1+\frac{\varphi''(a_1)}{\varphi'(a_1)}(z-a_1)+\dots}{(z-a_1)(1+\frac{\varphi''(a_1)}{2\varphi'(a_1)}(z-a_1)+\dots)}\right]^{\frac{1}{2}} \cdot \left[1+2\mathcal{A}_\varphi\cdot\varphi'(a_1)(z-a_1)+\dots\right] \\ &= \frac{1}{\sqrt{z-a_1}}\left[1+2\left(\mathcal{A}_\varphi\cdot\varphi'(a_1)+\frac{1}{8}\frac{\varphi''(a_1)}{\varphi'(a_1)}\right)(z-a_1)+\dots\right], \end{aligned}$$

where $\mathcal{A}_\varphi = \mathcal{A}_{[\Omega',\varphi(a_1),\dots,\varphi(a_n)]}$. Similar arguments show that the coefficient $\mathcal{B}_{[\Omega,a,b]}$ is conformally invariant, i.e.,

$$\mathcal{B}_\Omega(a,b) = \mathcal{B}_{\Omega'}(\varphi(a),\varphi(b)). \tag{2.14}$$

Thus, it is sufficient to find those quantities for some canonical domain, e.g., for the upper half-plane $\mathbb{H}$. This is done in Section 2.7 for $n=1,2$ and in the Appendix for $n\geq 3$.

2.6. **Convergence of spinors.** The main purpose of this section is to derive Theorem 1.5 and Theorem 1.7 from convergence results for discrete spinor observables $F_{[\Omega_\delta,a_1,\dots,a_n]}$ which are formulated in Theorems 2.15, 2.17 and 2.19 below. The proofs of those are given in Section 3, which is the most technical part of our paper. We use the following conventions concerning convergence of discrete s-holomorphic functions:

- we say that a family of discrete domains $(\Omega_\delta)_\delta$ approximates a continuous domain $\Omega\subset\mathbb{C}$ as $\delta\to 0$, if $\partial\Omega_\delta$ converges to $\partial\Omega$ in the Hausdorff sense (note that our proofs can be easily generalized for the Carathéodory convergence of planar domains which is weaker than the Hausdorff one used in this paper for simplicity);
- we say that an s-holomorphic function (or a spinor) F_δ defined in $\Omega_\delta^{\mathrm{cm}}$ (or its double cover) tends to a holomorphic function (or a spinor) f as $\delta\to 0$, if the "mid-edge values" $F_\delta\big|_{\Omega_\delta^{\mathrm{m}}}$ approximate the values of f, while the "corner values" $F\big|_{\Omega_\delta^\tau}$, $\tau\in\{1,\lambda,i,\overline{\lambda}\}$, tend to the projections of f onto the corresponding lines $\tau\mathbb{R}$ (see Definition 2.3);
- we say that a convergence of discrete functions $F_\delta(z)=F_\delta(z;a_1,a_2,\dots)$ to $f(z)=f(z;a_1,a_2,\dots)$ is uniform on some compact set, iff the differences $|F_\delta(z;a_1,a_2,\dots)-f(z;a_1,a_2,\dots)|$ are uniformly small as $\delta\to 0$, when we interpret lattice vertices (or mid-edges, corners, etc) $z,a_1,a_2,\dots$ as the corresponding complex points when we plug them into f.

The crucial ingredient of our proofs is the interplay between (a) the values of discrete spinor observables near their branching points, which are related to the ratios of spin correlations by Lemma 2.6, and (b) the mid-range behavior of these observables, which can be further related to the asymptotics expansions of their scaling limits (2.12), (2.13).

As a main tool to relate (a) and (b), we use a *full-plane version* $F_{[\mathbb{C}_\delta,a]}$ *of the spinor observable* (since we are interested in local considerations, it is sufficient to stick to the case of one marked point). Though it could be constructed as an infinite-volume limit of the finite-domain observables, we prefer a more explicit strategy, which is outlined after the following lemma claiming the existence of $F_{[\mathbb{C}_\delta,a]}$.

Lemma 2.13. *For $a\in\mathcal{V}^\circ_{\mathbb{C}_\delta}$, there exists a (unique) s-holomorphic spinor $F_{[\mathbb{C}_\delta,a]}:\mathcal{V}^{\mathrm{cm}}_{[\mathbb{C}_\delta,a^\rightarrow]}\to\mathbb{C}$ such that $F_{[\mathbb{C}_\delta,a]}(a+\frac{3\delta}{2})=1$ and $F_{[\mathbb{C}_\delta,a]}(z)=o(1)$ as $z\to\infty$. Moreover,*

$$\frac{1}{\vartheta(\delta)}F_{[\mathbb{C}_\delta,a]}(z)\underset{\delta\to 0}{\longrightarrow}\frac{1}{\sqrt{z-a}}=:f_{[\mathbb{C},a]}(z), \tag{2.15}$$

uniformly on compact subsets of $\mathbb{C}\setminus\{a\}$, where $\vartheta(\delta)$ is defined as

$$\vartheta(\delta):=F_{[\mathbb{C}_\delta,a]}\left(a+\tfrac{3\delta}{2}+2\delta\lfloor\tfrac{1}{2\delta}\rfloor\right). \tag{2.16}$$

Proof. The detailed proof is given in Section 3.2. First, we define the (real) values of $F_{[\mathbb{C}_\delta,a]}$ on $\mathcal{V}^1_{\mathbb{C}_\delta}$ as the discrete harmonic measure of the tip point $a+\frac{3\delta}{2}$ in the slit discrete plane $\mathcal{V}^1_{\mathbb{C}_\delta}\setminus\{x+a:x\le 0\}$. This definition is motivated by the following observation: in the continuous setup, the function $\mathfrak{Re}\,[1/\sqrt{z-a}\,]$ can be viewed as the properly normalized harmonic measure of the (small neighborhood of) tip a in the slit plane $\mathbb{C}\setminus\{x+a:x\le 0\}$. Second, we extend $F_{[\mathbb{C}_\delta,a]}$ to $\mathcal{V}^i_{\mathbb{C}_\delta}$ by harmonic conjugation, then by symmetry to another sheet of $[\mathbb{C}_\delta,a]$, and eventually as an s-holomorphic function to $\mathcal{V}^\lambda_{\mathbb{C}_\delta},\mathcal{V}^{\overline{\lambda}}_{\mathbb{C}_\delta}$ and $\mathcal{V}^{\mathrm{m}}_{\mathbb{C}_\delta}$. The convergence (2.15) follows from known results on convergence of the harmonic measures (e.g., see [ChSm11]). □

Remark 2.14. The normalizing factor $\vartheta(\delta)$ is essentially the value of $F_{[\mathbb{C}_\delta,a]}$ at $a+1$. In Section 3.2 we show that

$$\mathrm{C}_-\sqrt{\delta}\le\vartheta\,(\delta)\le\mathrm{C}_+\sqrt{\delta}, \tag{2.17}$$

where $\mathrm{C}_\pm>0$ are some absolute constants. Note that one can compute the limit $\lim_{\delta\to0}\vartheta(\delta)/\sqrt{\delta}$ using the recent work of Dubédat [Dub11a], but we do not need this sharp result.

We further use the normalizing factors $\vartheta(\delta)$ introduced above in order to formulate the following convergence theorem for discrete spinor observables away from $\partial\Omega$ and $a_1,\dots,a_n$:

Theorem 2.15. *Let discrete simply connected domains Ω_δ approximate a bounded simply connected domain Ω as $\delta\to0$. Then, for any $\epsilon>0$ and any $n=1,2,\dots$, we have*

$$\frac{1}{\vartheta\,(\delta)}F_{[\Omega_\delta,a_1,\dots,a_n]}(z)\underset{\delta\to 0}{\longrightarrow}f_{[\Omega,a_1,\dots,a_n]}(z),$$

uniformly over all $a_1,\dots,a_n\in\mathcal{V}^\circ_{\Omega_\delta}$ and $z\in\mathcal{V}^{\mathrm{m}}_{\Omega_\delta}$ which are at the distance at least ϵ from $\partial\Omega$ and from each other.

Proof. See Section 3.4. □

Since we are interested in the coefficient $\mathcal{A}_{[\Omega,a_1,\dots,a_n]}$ in front of the term $\sqrt{z-a_1}$ in (2.12), along with the discrete analogue $F_{[\mathbb{C}_\delta,a]}$ of the function $1/\sqrt{z-a}$ given by Lemma 2.13, we also need a discrete counterpart $G_{[\mathbb{C}_\delta,a]}$ of the function $\sqrt{z-a}$. We construct $G_{[\mathbb{C}_\delta,a]}$ by "discrete integration" of $F_{[\mathbb{C}_\delta,a]}$, just mimicking the continuous setup. It is sufficient to define $G_{[\mathbb{C}_\delta,a]}$ on $\mathcal{V}^1_{\mathbb{C}_\delta}$ only, as Lemma 2.6 deals with the (real) values of discrete observables at the point $a_1+\frac{3\delta}{2}\in\mathcal{V}^1_{\mathbb{C}_\delta}$.

Lemma 2.16. *For $a \in \mathcal{V}^\circ_{\mathbb{C}_\delta}$, there exists a (unique) discrete harmonic spinor $G_{[\mathbb{C}_\delta,a]} : \mathcal{V}^1_{[\mathbb{C}_\delta,a]} \to \mathbb{R}$ such that $G_{[\mathbb{C}_\delta,a]}$ vanishes on the half-line $\{x+a : x \le 0\}$, $G_{[\mathbb{C}_\delta,a]}(a+\frac{3\delta}{2}) = \delta$, and $G_{[\mathbb{C}_\delta,a]}(z) = O(|z-a|^{1/2})$ as $z \to \infty$. Moreover,*

$$\frac{1}{\vartheta(\delta)} G_{[\mathbb{C}_\delta,a]}(z) \quad \underset{\delta\to 0}{\longrightarrow} \quad \mathfrak{Re}\sqrt{z-a} \;\; =: g_{[\mathbb{C},a]}(z), \tag{2.18}$$

uniformly on compact subsets of $\mathbb{C} \setminus \{a\}$.

Proof. For $z \in \mathcal{V}^1_{[\mathbb{C}_\delta,a]}$, we set $G_{[\mathbb{C}_\delta,a]}(z) := \delta \cdot \sum_{j=0}^{\infty} F_{[\mathbb{C}_\delta,a]}(z-2j\delta)$. Certainly, one should check the convergence of this series and the harmonicity on the half-line $\{x+a : x \ge 0\}$, see further details in Section 3.2. □

In the continuum limit, the leading term in the expansion of $f_{[\Omega,a_1,\dots,a_n]} - f_{[\mathbb{C},a_1]}$ near a_1 is given by $2\mathcal{A}_{[\Omega,a_1,\dots,a_n]}\sqrt{z-a_1}$. It is hence plausible to believe that the same holds true for the discrete spinors, and one has

$$\left(F_{[\Omega_\delta,a_1,\dots,a_n]} - F_{[\mathbb{C}_\delta,a_1]}\right)\left(a_1+\tfrac{3\delta}{2}\right) \;\approx\; 2\mathfrak{Re}\,\mathcal{A}_{[\Omega_\delta,a_1,\dots,a_n]} \cdot G_{[\mathbb{C}_\delta,a_1]}\left(a_1+\tfrac{3\delta}{2}\right)$$

up to higher-order terms (the real part appears due to discrete complex analysis subtleties, as real and imaginary parts of s-holomorphic functions are defined on different lattices, and $a_1 + \frac{3\delta}{2} \in \mathcal{V}^1_{\Omega_\delta}$). We justify this heuristics in

Theorem 2.17. *Under conditions of Theorem 2.15, we have*

$$F_{[\Omega_\delta,a_1,\dots,a_n]}\left(a_1+\tfrac{3\delta}{2}\right) - 1 - 2\mathfrak{Re}\,\mathcal{A}_{[\Omega,a_1,\dots,a_n]} \cdot \delta \quad = \quad o\,(\delta) \tag{2.19}$$

as $\delta \to 0$, uniformly over all $a_1,\dots,a_n \in \mathcal{V}^\circ_{\Omega_\delta}$ which are at the distance at least ϵ from $\partial\Omega$ and from each other.

Proof. See Section 3.5. □

Remark 2.18. In the proof of Theorem 2.17 we show that $F_{[\Omega_\delta,a_1,\dots,a_n]}$ and $F_{[\mathbb{C}_\delta,a_1]}$ are δ-close to each other at all points around a_1, in particular at $a_1 + (1 \pm \frac{i}{2})\delta$. Together with (2.6), this implies

$$\frac{\mathbb{E}^+_{\Omega_\delta}\left[\sigma_{a_1+(1\pm i)\delta}\sigma_{a_2}\dots\sigma_{a_n}\right]}{\mathbb{E}^+_{\Omega_\delta}\left[\sigma_{a_1}\dots\sigma_{a_n}\right]} = 1 + O(\delta)$$

as $\delta \to 0$, uniformly over all $a_1,\dots,a_n \in \mathcal{V}^\circ_{\Omega_\delta}$ which are at the distance at least ϵ from $\partial\Omega$ and from each other.

The similar analysis for the quantity $\mathcal{B}_{[\Omega,a,b]}$ which is defined by the expansion (2.13), is even simpler since we need to match the first-order coefficients instead of the second-order ones. The result is given by

Theorem 2.19. *For $n=2$, under conditions of Theorem 2.15, we have*

$$F_{[\Omega_\delta,a,b]}\left(b+\tfrac{\delta}{2}\right) \pm i\mathcal{B}_{[\Omega,a,b]} \quad = \quad o\,(1) \tag{2.20}$$

as $\delta \to 0$, uniformly over all $a,b \in \mathcal{V}^\circ_{\Omega_\delta}$ which are at the distance at least ϵ from $\partial\Omega$ and from each other, where the sign $\pm$ depends on the sheet of $[\Omega,a,b]$.

Proof. See Section 3.5. □

Proof of Theorems 1.5 and 1.7. Due to Lemma 2.6, asymptotics (1.6) is a reformulation of (2.19), while Theorem 1.7 is equivalent to Theorem 2.19 (the sign $\pm$ is fixed due to the positivity of spin-spin correlations). To check (1.7), we rotate our domain Ω around a by 90° clockwise. According to conformal covariance rule (1.8), the coefficient $\mathcal{A}$ multiplies by i, so the desired result follows from $\mathfrak{Re}\,[i\mathcal{A}] = -\mathfrak{Im}\,\mathcal{A}$. $\square$

2.7. **Computations for one and two points.** Now we illustrate the results obtained in the previous section constructing the continuous spinors solving the boundary value problem (2.7)–(2.9) for two simplest cases $n=1$ and $n=2$. We use the upper half-plane $\mathbb{H}$ as a canonical domain. Since it is unbounded, one should impose the additional regularity condition at ∞ ensuring that $f_{[\mathbb{H},a_1,\dots,a_n]} = O(|z|^{-1})$ as $z\to\infty$, so that after a conformal mapping to a bounded domain with smooth boundary the spinor remains bounded.

2.7.1. *The case* $n=1$. For a single marked point $a\in\mathbb{H}$ we have

$$f_{[\mathbb{H},a]}(z) = \frac{(2i\mathfrak{Im}\,a)^{\frac12}}{\sqrt{(z-a)(z-\overline{a})}}, \tag{2.21}$$

since this spinor clearly satisfies both conditions (2.7) and (2.8). Writing down the asymptotic expansion (2.12) at a, one sees that

$$\mathcal{A}_{\mathbb{H}}(a) = -\frac{1}{8i\mathfrak{Im}\,a}.$$

Further, we *define* the quantity $\log\langle\sigma_a\rangle^+_{\mathbb{H}}$ as a primitive of the differential form $\mathfrak{Re}[\mathcal{A}_{\mathbb{H}}(a)da] = -dy/8y$, where $a=x+iy$:

$$\langle\sigma_a\rangle^+_{\mathbb{H}} = \mathrm{const}\cdot(\mathfrak{Im}\,a)^{-\frac18},\quad a\in\mathbb{H}.$$

The proper choice of the multiplicative constant which is discussed below leads us to the definition $\langle\sigma_a\rangle^+_{\mathbb{H}} := 2^{\frac14}(2\mathfrak{Im}\,a)^{-\frac18}$.

2.7.2. *The case* $n=2$. Similarly, for two marked points $a,b\in\mathbb{H}$, one can check that the spinor

$$f_{[\mathbb{H},a,b]}(z) = \frac{(2i\mathfrak{Im}\,a)^{\frac12}}{|b-\overline{a}|+|b-a|}\cdot\frac{[(\overline{b}-\overline{a})(\overline{b}-a)]^{\frac12}(z-b)+[(b-a)(b-\overline{a})]^{\frac12}(z-\overline{b})}{[(z-a)(z-\overline{a})(z-b)(z-\overline{b})]^{1/2}}, \tag{2.22}$$

satisfies (2.7)–(2.9). Looking at the expansion (2.13) at b, one obtains

$$\mathcal{B}_{\mathbb{H}}(a,b) = \frac{(4\mathfrak{Im}\,a\,\mathfrak{Im}\,b)^{\frac12}}{|b-\overline{a}|+|b-a|} = \frac{\langle\sigma_a\sigma_b\rangle^{\mathrm{free}}_{\mathbb{H}}}{\langle\sigma_a\sigma_b\rangle^+_{\mathbb{H}}}.$$

Further, expansions near a give

$$\mathcal{A}_{\mathbb{H}}(a,b) = -\frac{1}{8i\mathfrak{Im}\,a} + \frac{|b-\overline{a}|-|b-a|}{4(|b-\overline{a}|+|b-a|)}\left(\frac{1}{b-a}-\frac{1}{\overline{b}-a}\right).$$

Direct computations show that one can define a primitive $\int^a\mathfrak{Re}\,[\mathcal{A}_{\mathbb{H}}(z;b)dz]$ for $a\in\mathbb{H}$ (here we treat b as a parameter) that coincides with the function $\log\langle\sigma_a\sigma_b\rangle^+_{\mathbb{H}}$ given by (1.4). Note that $\langle\sigma_a\sigma_b\rangle^+_{\mathbb{H}}$ is uniquely defined (up to a multiplicative constant) by its logarithmic derivative with respect to a and the symmetry $\langle\sigma_a\sigma_b\rangle^+_{\mathbb{H}} = \langle\sigma_b\sigma_a\rangle^+_{\mathbb{H}}$. The multiplicative constants in the formulae (1.4) for $n=1,2$ are chosen so that $\langle\sigma_a\sigma_b\rangle^+_{\mathbb{H}} \sim \langle\sigma_a\rangle^+_{\mathbb{H}}\langle\sigma_b\rangle^+_{\mathbb{H}}$ if, say, $a\to\partial\mathbb{H}$, resembling decorrelation properties of the critical Ising model which are given in Section 2.10. Below we show that, for all n, the similar integration procedure allows one to *define* real-valued symmetric functions $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_{\mathbb{H}}^+$ so that

$$\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_{\Omega}^+ \sim \langle\sigma_{a_1}\rangle_{\Omega}^+\langle\sigma_{a_2}\dots\sigma_{a_n}\rangle_{\Omega}^+ \quad \text{as} \quad a_1\to\partial\Omega. \tag{2.23}$$

2.8. **Proof of Corollary 1.6 and definition of the functions $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_{\Omega}^+$.** The main purpose of this section is to define the functions $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_{\Omega}^+$ starting with their logarithmic derivatives with respect to each of a_k and (eventual) decorrelation properties (2.23). To do so we firstly need to know that the corresponding differential form (2.24) is exact, and we derive this fact from the convergence of the ratios of spin correlations given in Proposition 2.20. Note that (2.25) is essentially equivalent to the claim of Corollary 1.6.

Let us denote $a_k = x_k+iy_k$, and consider a differential form $\mathcal{L}_{\Omega,n}$ on the manifold $\widetilde{\Omega}^n := \{(a_1,\dots a_n)\in\Omega^n : a_j\neq a_k, j<k\}$ of all n-tuples of pairwise distinct points in Ω, defined as follows:

$$\begin{aligned}\mathcal{L}_{\Omega,n} &:= \sum_{k=1}^{n} \mathfrak{Re}\,\left[\mathcal{A}_\Omega(a_k,a_1,\dots\widehat{a}_k\dots,a_n)da_k\right] \\ &= \sum_{k=1}^{n}\left[\mathfrak{Re}\,\mathcal{A}_\Omega(a_k,a_1,\dots\widehat{a}_k\dots,a_n)dx_k - \mathfrak{Im}\,\mathcal{A}_\Omega(a_k,a_1,\dots\widehat{a}_k\dots,a_n)dy_k\right]\end{aligned} \tag{2.24}$$

(we use the standard notation $\widehat{a}_k$ for the omitted argument). The following proposition is a straightforward corollary of Theorem 1.5 obtained via integration with respect to positions of points. In particular, this provides a *proof of Corollary 1.6.*

Proposition 2.20. *The form $\mathcal{L}_{\Omega,n}$ is exact. If $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_{\Omega}^+ := \exp[\int\mathcal{L}_{\Omega,n}]$ denotes the exponential of its primitive, then, as Ω_δ approximates Ω, one has*

$$\frac{\mathbb{E}_{\Omega_\delta}^+[\sigma_{b_1}\dots\sigma_{b_n}]}{\mathbb{E}_{\Omega_\delta}^+[\sigma_{a_1}\dots\sigma_{a_n}]} \underset{\delta\to0}{\longrightarrow} \frac{\langle\sigma_{b_1}\dots\sigma_{b_n}\rangle_{\Omega}^+}{\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_{\Omega}^+} \tag{2.25}$$

for all n-tuples $(a_k)_1^n, (b_k)_1^n\in\widetilde{\Omega}^n$.

Remark. The proof given below ensures that this convergence is uniform, if all a_k are at distance at least ϵ from $\partial\Omega$ and each other, and the same holds true for b_k.

Proof. Color the faces of Ω_δ black and white, in a chessboard fashion. By Remark 2.18, the ratio of spin-spin correlations at two adjacent spins tends to 1, uniformly away from the boundary, so we can assume that all a_k, b_k are colored white. Let $a_1'\in\Omega$ be such that $[a_1,a_1']$ is a horizontal segment contained in Ω and disjoint with $a_2,\dots,a_n$. Denote by $a_1=v_1\sim\dots\sim v_{m_\delta}=a_1'$ a straight horizontal lattice path approximating $[a_1,a_1']$. Then, by Theorem 1.5, one has

$$\begin{aligned}\log\frac{\mathbb{E}_{\Omega_\delta}^+\left[\sigma_{v_{j+1}}\sigma_{a_2}\dots\sigma_{a_n}\right]}{\mathbb{E}_{\Omega_\delta}^+\left[\sigma_{v_j}\sigma_{a_2}\dots\sigma_{a_n}\right]} &= \left(\frac{\mathbb{E}_{\Omega_\delta}^+\left[\sigma_{v_{j+1}}\sigma_{a_2}\dots\sigma_{a_n}\right]}{\mathbb{E}_{\Omega_\delta}^+\left[\sigma_{v_j}\sigma_{a_2}\dots\sigma_{a_n}\right]}-1\right)(1+o(1)) \\ &= 2\delta\cdot\left[\mathfrak{Re}\,\mathcal{A}_\Omega(v_j;a_2,\dots,a_n)+o(1)\right]\end{aligned}$$

as $\delta \to 0$, where the $o(1)$ terms are unform in j. Consequently,

$$\log \frac{\mathbb{E}^+_{\Omega_\delta}\left[\sigma_{a_1'}\sigma_{a_2}\dots\sigma_{a_n}\right]}{\mathbb{E}^+_{\Omega_\delta}\left[\sigma_{a_1}\sigma_{a_2}\dots\sigma_{a_n}\right]} = 2\delta\cdot\sum_{j=1}^{m_\delta}\mathfrak{Re}\,\mathcal{A}_\Omega(v_j,a_2,\dots,a_n)+o(1)$$

$$\underset{\delta\to 0}{\longrightarrow}\int_{[a_1,a_1']}\mathfrak{Re}\,\mathcal{A}_\Omega(x_1+iy_1,a_2,\dots,a_n)dx_1.$$

A similar formula with $-\mathfrak{Im}\,\mathcal{A}_\Omega(x_1+iy_1,a_2,\dots,a_n)dy_1$ in the right-hand side applies to the case when $[a_1,a_1']$ is a vertical segment. Moreover, one can move other points $a_2,\dots,a_n$ along horizontal and vertical segments as well. Therefore,

$$\log\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_{b_1}\dots\sigma_{b_n}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1}\dots\sigma_{a_n}]}\ \underset{\delta\to 0}{\longrightarrow}\ \int_\gamma \mathcal{L}_{\Omega,n},$$

where γ is any path in $\widetilde{\Omega}^n$ that connects n-tuples $(a_k)_1^n$ and $(b_k)_1^n$ and consists of segments (in $\widetilde{\Omega}^n$) with all but one coordinates fixed. Since the left-hand side does not depend on the choice of the path, the form $\mathcal{L}_{\Omega,n}$ is exact. □

Remark 2.21. Note that $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle^+_\Omega$ is a symmetric function of $a_1,\dots,a_n$ (as so is $\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1}\dots\sigma_{a_n}]$). Integrating the covariance rule (1.8) for $\mathcal{A}_\Omega$, one immediately obtains

$$\begin{aligned}\log\frac{\langle\sigma_{b_1}\sigma_{a_2}\dots\sigma_{a_n}\rangle^+_\Omega}{\langle\sigma_{a_1}\sigma_{a_2}\dots\sigma_{a_n}\rangle^+_\Omega} &= \int_{a_1}^{b_1}\mathfrak{Re}\,[\mathcal{A}_\Omega(z,a_2,\dots,a_n)dz]\\ = \int_{a_1}^{b_1}\mathfrak{Re}\,\left[\mathcal{A}_{\Omega'}(\varphi(z);\varphi(a_2),\dots,\varphi(a_n))\varphi'(z)dz\right] &+ \frac{1}{8}\int_{a_1}^{b_1}\mathfrak{Re}\,\left[\frac{\varphi''(z)}{\varphi'(z)}dz\right]\\ = \log\frac{\langle\sigma_{\varphi(b_1)}\sigma_{\varphi(a_2)}\dots\sigma_{\varphi(a_n)}\rangle^+_{\Omega'}}{\langle\sigma_{\varphi(a_1)}\sigma_{\varphi(a_2)}\dots\sigma_{\varphi(a_n)}\rangle^+_{\Omega'}} &+ \frac{1}{8}\log\frac{|\varphi'(b_1)|}{|\varphi'(a_1)|}.\end{aligned}$$

Iterating, one arrives at the following covariance rule for the ratios of correlations which is a weaker form of (1.3):

$$\frac{\langle\sigma_{b_1}\dots\sigma_{b_n}\rangle^+_\Omega}{\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle^+_\Omega} = \frac{\langle\sigma_{\varphi(b_1)}\dots\sigma_{\varphi(b_n)}\rangle^+_{\Omega'}}{\langle\sigma_{\varphi(a_1)}\dots\sigma_{\varphi(a_n)}\rangle^+_{\Omega'}}\cdot\prod_{k=1}^n\frac{|\varphi'(b_k)|^{\frac18}}{|\varphi'(a_k)|^{\frac18}}. \tag{2.26}$$

Proposition 2.20 defines the continuous correlation functions $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle^+_\Omega$ up to multiplicative constants, which may depend on Ω and n, since the primitive of $\mathcal{L}_{\Omega,n}$ is defined up to an additive constant. A natural way to choose these constants coherently for all domains and any number of points is suggested by the following lemma. Denote by

$$\mathcal{D}_\Omega(a,b) := \frac{|a-b|}{\operatorname{dist}(\{a,b\},\partial\Omega)} \tag{2.27}$$

the quantity that measures how deeply in the bulk of Ω the points a,b are, and let

$$\langle\sigma_a\sigma_b\rangle^+_{\mathbb{C}} := |a-b|^{-\frac14}.$$

Lemma 2.22. *There exists a unique way to choose the multiplicative normalization of $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle^+_\Omega$ so that for all domains Ω and for all $n=1,2,\dots$ the following holds:*

$$\frac{\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle^+_\Omega}{\langle\sigma_{a_1}\rangle^+_\Omega\langle\sigma_{a_2}\dots\sigma_{a_n}\rangle^+_\Omega}\to 1\quad\text{as}\quad a_1\to\partial\Omega\quad\text{and}\quad\frac{\langle\sigma_a\sigma_b\rangle^+_\Omega}{\langle\sigma_a\sigma_b\rangle^+_{\mathbb{C}}}\to 1\quad\text{as}\quad\mathcal{D}_\Omega(a,b)\to 0. \tag{2.28}$$

Being normalized in this way, the correlation functions $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_\Omega^+$ *satisfy the conformal covariance rule (1.3).*

Proof. See Section A.3. □

The properties (2.28) are motivated by the corresponding properties of the discrete correlations, see Lemmas 2.24 and 2.26 below. If one had an independent proof of those lemmas staying in the discrete setup, one would get (2.28) for free. However, our proof goes in the other direction: we use (2.28) to prove Lemmas 2.24 and 2.26. Thus, we have to derive Lemma 2.22 directly from the explicit description of continuous correlations, which we do in the Appendix.

Summarizing, we *define* the continuous correlation functions $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_\Omega^+$ to be the exponentials of the primitives of $\mathcal{L}_{\Omega,n}$, normalized as in Lemma 2.22. As it was discussed in Section 2.7, in the particular cases $n=1$ and $n=2$, this definition reproduces explicit formulae (1.4). The case $n\geq 3$ is discussed in the Appendix.

2.9. **From ratios of correlations to Theorem 1.1.** This section is devoted to the proof of Theorem 1.1. Our goal is to relate the normalizing factors $\rho_2(\delta,\Omega_\delta)$ from Corollary 1.6 (which, in principle, might depend on Ω), with the full-plane normalization $\rho(\delta)$. The proof is based on Theorem 1.7, which claims the convergence of the ratios of free and + spin-spin correlations to an explicit limit $\mathcal{B}_\Omega(a,b)=\langle\sigma_a\sigma_b\rangle_\Omega^{\mathrm{free}}/\langle\sigma_a\sigma_b\rangle_\Omega^+$. We also use classical FKG (e.g., see [Gri06, Chapter 2]) and GHS (see [GHS70]) inequalities for the Ising model. The small additional ingredient is given by

Remark 2.23. The following is fulfilled:

$$\mathcal{B}_\Omega(a,b)\to 1 \ \text{ as } \ \mathcal{D}_\Omega(a,b)\to 0, \quad \text{and} \quad \mathcal{B}_\Omega(a,b)\to 0 \ \text{ as } \ a\to\partial\Omega, \tag{2.29}$$

where the quantity $\mathcal{D}_\Omega(a,b)$ is given by (2.27). This follows readily from the conformal invariance of $\mathcal{B}_\Omega(a,b)$ and the explicit formulae (1.4) in the half-plane. Also, one has

$$\frac{\langle\sigma_a\sigma_b\rangle_\Omega^+}{\langle\sigma_a\sigma_b\rangle_\mathbb{C}^+}\to 1 \ \text{ as } \ \mathcal{D}_\Omega(a,b)\to 0. \tag{2.30}$$

Indeed, let φ be a conformal map from Ω to $\mathbb{H}$ such that $\varphi(a)=i$. Due to standard estimates, one has $\varphi(b)\to i$ and $|\varphi'(b)|\to|\varphi'(a)|$ as $\mathcal{D}_\Omega(a,b)\to 0$. Therefore,

$$\frac{\langle\sigma_a\sigma_b\rangle_\Omega^+}{\langle\sigma_a\sigma_b\rangle_\mathbb{C}^+}=\frac{\langle\sigma_{\varphi(a)}\sigma_{\varphi(b)}\rangle_\mathbb{H}^+|\varphi'(a)|^{\frac18}|\varphi'(b)|^{\frac18}}{\langle\sigma_a\sigma_b\rangle_\mathbb{C}^+}=\frac{|\varphi(a)-\varphi(b)|^{-\frac14}|\varphi'(a)|^{\frac14}}{|a-b|^{-\frac14}}\,(1+o(1))\to 1.$$

Lemma 2.24. *For any* $\eta>0$ *there exists an* $\epsilon>0$ *such that the following holds: if* $\mathcal{D}_\Omega(a,b)<\epsilon$ *and* Ω_δ *approximates* Ω*, then*

$$1-\eta\leq\frac{\mathbb{E}_{\mathbb{C}_\delta}[\sigma_a\sigma_b]}{\mathbb{E}_{\Omega_\delta}^+[\sigma_a\sigma_b]}\leq 1$$

provided that δ *is small enough.*

Proof. By FKG inequality, $\mathbb{E}_{\Lambda_\delta}^{\mathrm{free}}[\sigma_a\sigma_b]\leq\mathbb{E}_{\mathbb{C}_\delta}[\sigma_a\sigma_b]\leq\mathbb{E}_{\Omega_\delta}^+[\sigma_a\sigma_b]$ for any domain Λ_δ containing a,b, hence the right-hand side readily follows. For the left-hand side, choose $\Lambda_\delta=\Omega_\delta^\bullet-\delta$. We have, by Theorem 1.7,

$$\frac{\mathbb{E}_{\Lambda_\delta}^{\mathrm{free}}[\sigma_a\sigma_b]}{\mathbb{E}_{\Omega_\delta}^+[\sigma_a\sigma_b]}=\frac{\mathbb{E}_{\Omega_\delta^\bullet}^{\mathrm{free}}[\sigma_{a+\delta}\sigma_{b+\delta}]}{\mathbb{E}_{\Omega_\delta}^+[\sigma_a\sigma_b]}\underset{\delta\to0}{\longrightarrow}\mathcal{B}_\Omega(a,b).$$

Due to (2.29), one can choose ϵ so that $\mathcal{B}_\Omega(a,b) \ge 1-\frac{\eta}{2}$, which gives the result. □

Proof of Theorem 1.1. Fix $\eta > 0$. For shortness, below we will write $a+\epsilon$ for its lattice approximations $a+2\delta\lfloor\frac{\epsilon}{2\delta}\rfloor$, and $a+1$ for its lattice approximation $a+2\delta\lfloor\frac{1}{2\delta}\rfloor$. Denote

$$R^{\Omega_\delta}_{a,b} := (\varrho(\delta))^{-1}\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]. \tag{2.31}$$

Recall that $\varrho(\delta) = \mathbb{E}_{\mathbb{C}_\delta}[\sigma_a\sigma_{a+1}]$ by definition, so one can write

$$R^{\Omega_\delta}_{a,b} = \frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_{a+\epsilon}]}\cdot\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_{a+\epsilon}]}{\mathbb{E}_{\mathbb{C}_\delta}[\sigma_a\sigma_{a+\epsilon}]}\cdot\frac{\mathbb{E}_{\mathbb{C}_\delta}[\sigma_a\sigma_{a+\epsilon}]}{\mathbb{E}_{\mathbb{C}_\delta}[\sigma_a\sigma_{a+1}]}.$$

By Lemma 2.24, we can find a small $\epsilon > 0$ and a large domain Λ_δ containing a, b, $a+\epsilon$ and $a+1$, such that

$$1-\eta \;\le\; \frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_{a+\epsilon}]}{\mathbb{E}_{\mathbb{C}_\delta}[\sigma_a\sigma_{a+\epsilon}]} \;\le\; 1 \quad\text{and}\quad 1-\eta \;\le\; \frac{\mathbb{E}_{\mathbb{C}_\delta}[\sigma_a\sigma_{a+\epsilon}]}{\mathbb{E}_{\mathbb{C}_\delta}[\sigma_a\sigma_{a+1}]}\cdot\frac{\mathbb{E}^+_{\Lambda_\delta}[\sigma_a\sigma_{a+1}]}{\mathbb{E}^+_{\Lambda_\delta}[\sigma_a\sigma_{a+\epsilon}]} \;\le\; 1+\eta,$$

provided that δ is small enough. Consequently,

$$(1-\eta)^2\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_{a+\epsilon}]}\cdot\frac{\mathbb{E}^+_{\Lambda_\delta}[\sigma_a\sigma_{a+\epsilon}]}{\mathbb{E}^+_{\Lambda_\delta}[\sigma_a\sigma_{a+1}]} \;\le\; R^{\Omega_\delta}_{a,b} \;\le\; (1+\eta)\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_{a+\epsilon}]}\cdot\frac{\mathbb{E}^+_{\Lambda_\delta}[\sigma_a\sigma_{a+\epsilon}]}{\mathbb{E}^+_{\Lambda_\delta}[\sigma_a\sigma_{a+1}]},$$

and, by convergence of the *ratios* of spin correlations proven in Proposition 2.20,

$$(1-\eta)^3\frac{\langle\sigma_a\sigma_b\rangle^+_\Omega}{\langle\sigma_a\sigma_{a+\epsilon}\rangle^+_\Omega}\cdot\frac{\langle\sigma_a\sigma_{a+\epsilon}\rangle^+_\Lambda}{\langle\sigma_a\sigma_{a+1}\rangle^+_\Lambda} \;\le\; R^{\Omega_\delta}_{a,b} \;\le\; (1+\eta)^2\frac{\langle\sigma_a\sigma_b\rangle^+_\Omega}{\langle\sigma_a\sigma_{a+\epsilon}\rangle^+_\Omega}\cdot\frac{\langle\sigma_a\sigma_{a+\epsilon}\rangle^+_\Lambda}{\langle\sigma_a\sigma_{a+1}\rangle^+_\Lambda}$$

for δ small enough. Since η can be chosen arbitrary small, and the bounds do not depend on δ, it only remains to show that we can make the factor

$$\frac{\langle\sigma_a\sigma_{a+\epsilon}\rangle^+_\Lambda}{\langle\sigma_a\sigma_{a+\epsilon}\rangle^+_\Omega\langle\sigma_a\sigma_{a+1}\rangle^+_\Lambda}$$

as close to 1 as we wish by choosing ϵ small enough and Λ large enough. However, this follows readily from (2.30) if we multiply this factor by $\langle\sigma_a\sigma_{a+1}\rangle_{\mathbb{C}} = 1$. Thus, $\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b] \sim \varrho(\delta)\langle\sigma_a\sigma_b\rangle^+_\Omega$ as $\delta\to 0$. To derive the asymptotics of two-point correlations for free boundary conditions as $\delta \to 0$, note that Theorem 1.7 implies

$$\mathbb{E}^{\mathrm{free}}_{\Omega^\bullet_\delta-\delta}[\sigma_a\sigma_b] \sim \mathcal{B}_\Omega(a,b)\cdot\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b] \sim \varrho(\delta)\cdot\mathcal{B}_\Omega(a,b)\langle\sigma_a\sigma_b\rangle^+_\Omega = \varrho(\delta)\langle\sigma_a\sigma_b\rangle^{\mathrm{free}}_\Omega.$$

The fact that we have $\Omega^\bullet_\delta - \delta$ instead of Ω_δ plays no role, since they both approximate the same continuous domain Ω and the convergence of $(\varrho(\delta))^{-1}\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]$ is independent of the particular choice of lattice approximations. □

Remark 2.25. As a simple byproduct of our analysis, we obtain the rotational invariance of the full-plane correlations recently proven by Pinson [Pin12]: by FKG inequality, for any (large) domain Ω_δ, one has $\mathbb{E}^{\mathrm{free}}_{\Omega_\delta}[\sigma_a\sigma_b] \le \mathbb{E}_{\mathbb{C}_\delta}[\sigma_a\sigma_b] \le \mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]$ and, due to Theorem 1.1 and (2.29), both sides have the same asymptotics when Ω_δ exhausts $\mathbb{C}_\delta$. Then, (2.30) gives the desired result:

$$(\rho(\delta))^{-1}\mathbb{E}_{\mathbb{C}_\delta}[\sigma_a\sigma_b] \to \langle\sigma_a\sigma_b\rangle_{\mathbb{C}} = |a-b|^{-\frac{1}{4}} \quad\text{as}\quad \delta\to 0.$$

2.10. **Decorrelation near the boundary and the proof of Theorem 1.3.** This section is devoted to the proof of Theorem 1.3. Note that it was already proven above in the special case $n = 2$, as a part of Theorem 1.1. Our goal is to relate the normalizing factors in the Corollary 1.6 with $\varrho(\delta)$. Below we rely upon decorrelation identities (2.28).

Lemma 2.26. *Given a domain Ω with marked points $a_2, \ldots, a_n$, $n \geq 2$, and a number $\eta > 0$, there exists $\epsilon > 0$ such that the following holds: if $a_1 \in \Omega$ is ϵ-close to the boundary, Ω_δ approximates Ω and δ is small enough, then*

$$1 - \eta \leq \frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1}]\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_2} \ldots \sigma_{a_n}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1} \ldots \sigma_{a_n}]} \leq 1.$$

Proof. The upper bound follows readily from FKG inequality. For the lower one, consider first the case $n = 2$. A celebrated application (e.g., see [DeMo10]) of the GHS inequality [GHS70] reads $\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b] - \mathbb{E}^+_{\Omega_\delta}[\sigma_a]\mathbb{E}^+_{\Omega_\delta}[\sigma_b] \leq \mathbb{E}^{\mathrm{free}}_{\Omega_\delta}[\sigma_a\sigma_b]$, or equivalently,

$$1 - \frac{\mathbb{E}^{\mathrm{free}}_{\Omega_\delta}[\sigma_a\sigma_b]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]} \leq \frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_a]\mathbb{E}^+_{\Omega_\delta}[\sigma_b]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]}.$$

As $\delta \to 0$, the left-hand side converges to $1 - \mathcal{B}_\Omega(a; b)$, so (2.29) implies the claim.

To prove the result for $n \geq 3$, assume that we have already proved Theorem 1.3 for all $n' < n$ (the precise description of our induction scheme is given in the proof of Theorem 1.3 below, see (2.32)). Let γ be a crosscut (simple path) in Ω separating a_1 from $a_2, \ldots, a_n$, and let Ω' and Ω'' be the corresponding connected components. Note that FKG inequality implies

$$\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1}]\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_2} \ldots \sigma_{a_n}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1} \ldots \sigma_{a_n}]} \cdot \frac{\mathbb{E}^+_{\Omega'_\delta}[\sigma_{a_1}]\mathbb{E}^+_{\Omega''_\delta}[\sigma_{a_2} \ldots \sigma_{a_n}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1}]\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_2} \ldots \sigma_{a_n}]} \geq 1$$

as $\mathbb{E}^+_{\Omega'_\delta}[\sigma_{a_1}]\mathbb{E}^+_{\Omega''_\delta}[\sigma_{a_2} \ldots \sigma_{a_n}]$ is equal to the correlation of $\sigma_{a_1}, \ldots, \sigma_{a_n}$ in Ω_δ conditioned on the event that all spins neighboring γ are $+$. By the induction assumption, the second factor converges to

$$\frac{\langle \sigma_{a_1} \rangle^+_{\Omega'}}{\langle \sigma_{a_1} \rangle^+_{\Omega}} \cdot \frac{\langle \sigma_{a_2} \ldots \sigma_{a_n} \rangle^+_{\Omega''}}{\langle \sigma_{a_2} \ldots \sigma_{a_n} \rangle^+_{\Omega}}$$

as $\delta \to 0$, hence it is sufficient to show that we can make this quantity arbitrary close to 1 by choosing a_1 and γ appropriately. We first choose a crosscut γ in such a way that Ω'' would be Carathéodory close to Ω as seen from $a_2, \ldots, a_n$ and then put a_1 deeply inside Ω', so that Ω' would be Carathéodory close to Ω as seen from a_1. If two domains are Carathéodory close, then the conformal maps from these domains to $\mathbb{H}$ mapping the marked point to i (say, with positive derivative there) are uniformly close on compacts together with their derivatives. Thus, the lemma follows from continuity of the half-plane functions $\langle \sigma_{a_1} \ldots \sigma_{a_n} \rangle^+_{\mathbb{H}}$ with respect to positions of $a_1, \ldots, a_n$. □

Proof of Theorem 1.3. Recall that the special case $n = 2$ is already done as a part of Theorem 1.1. We proceed by induction which (together with the proof of Lemma 2.26 given above) starts as follows:

$$T_2 \& L_2 \Rightarrow T_1 \Rightarrow L_3 \Rightarrow T_3 \Rightarrow L_4 \Rightarrow T_4 \Rightarrow \ldots \qquad (2.32)$$

(where T_j and L_j mean the particular cases of Theorem 1.3 and Lemma 2.26).

Let $n=1$ and $\eta>0$ be fixed. Similarly to (2.31), denote

$$R_a^{\Omega_\delta} := (\varrho(\delta))^{-\frac{1}{2}}\mathbb{E}^+_{\Omega_\delta}[\sigma_a].$$

For any $b\in\Omega$, one can write

$$(R_a^{\Omega_\delta})^2 = \frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_a]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_b]}\cdot\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]}{\varrho(\delta)}\cdot\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_a]\,\mathbb{E}^+_{\Omega_\delta}[\sigma_b]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]}.$$

By Lemma 2.26, if we choose b close enough to the boundary, then

$$1-\eta\le\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_a]\,\mathbb{E}^+_{\Omega_\delta}[\sigma_b]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]}\le 1,$$

provided that δ is small enough. Due to Proposition 2.20 and Theorem 1.1, one has

$$\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_a]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_b]}\underset{\delta\to0}{\longrightarrow}\frac{\langle\sigma_a\rangle^+_\Omega}{\langle\sigma_b\rangle^+_\Omega}\quad\text{and}\quad\frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_a\sigma_b]}{\varrho(\delta)}\underset{\delta\to0}{\longrightarrow}\langle\sigma_a\sigma_b\rangle^+_\Omega.$$

Consequently,

$$(1-\eta)^3\left(\langle\sigma_a\rangle^+_\Omega\right)^2\cdot\frac{\langle\sigma_a\sigma_b\rangle^+_\Omega}{\langle\sigma_a\rangle^+_\Omega\langle\sigma_b\rangle^+_\Omega}\le(R_a^{\Omega_\delta})^2\le(1+\eta)^2\left(\langle\sigma_a\rangle^+_\Omega\right)^2\cdot\frac{\langle\sigma_a\sigma_b\rangle^+_\Omega}{\langle\sigma_a\rangle^+_\Omega\langle\sigma_b\rangle^+_\Omega}$$

provided that δ is small enough. Since η can be chosen arbitrary small, and we can make the term $\langle\sigma_a\sigma_b\rangle^+_\Omega/\langle\sigma_a\rangle^+_\Omega\langle\sigma_b\rangle^+_\Omega$ arbitrary close to 1 choosing b sufficiently close to $\partial\Omega$ (recall that this particular case of (2.28) follows from explicit computations in the half-plane given in Section 2.7), we complete the proof for $n=1$ by remark that positivity of magnetization fixes the sign of $R_a^{\Omega_\delta}$.

To get the convergence of $R^{\Omega_\delta}_{a_1,\dots,a_n}:=\varrho(\delta)^{-\frac{n}{2}}\mathbb{E}^+_{\Omega_\delta}[\sigma_{a_1},\dots\sigma_{a_n}]$ for $n\ge3$, proceed by induction and write it as

$$R^{\Omega_\delta}_{a_1,\dots,a_n}=\frac{\mathbb{E}^+_\Omega[\sigma_{a_1}\sigma_{a_2}\dots\sigma_{a_n}]}{\mathbb{E}^+_\Omega[\sigma_b\sigma_{a_2}\dots\sigma_{a_n}]}\cdot\frac{\mathbb{E}^+_\Omega[\sigma_b\sigma_{a_2}\dots\sigma_{a_n}]}{\mathbb{E}^+_\Omega[\sigma_b]\mathbb{E}^+_\Omega[\sigma_{a_2}\dots\sigma_{a_n}]}\cdot R_b^{\Omega_\delta}R^{\Omega_\delta}_{a_2,\dots,a_n}.$$

The proof is finished similarly to the case $n=1$: take b close to the boundary, estimate the second ratio in the right-hand side by Lemma 2.26, then use convergence of the first ratio and $R_b^{\Omega_\delta}R^{\Omega_\delta}_{a_2,\dots,a_n}$ as $\delta\to0$ (this follows from Proposition 2.20 and the induction hypothesis, respectively) and asymptotics (2.28) for the continuous correlation functions: $\langle\sigma_b\rangle^+_\Omega\langle\sigma_{a_2}\dots\sigma_{a_n}\rangle^+_\Omega/\langle\sigma_b\sigma_{a_2}\dots\sigma_{a_n}\rangle^+_\Omega\to1$ as $b\to\partial\Omega$. □

3. Proofs of the main convergence Theorems 2.15, 2.17 and 2.19 for discrete spinors

3.1. **S-holomorphicity of discrete observables.**

The notion of s-holomorphicity was essentially introduced by Smirnov in [Smi06] and used for the study of the planar critical Ising model in [Smi06, Smi10, ChSm12, Hon10, HoSm10, ChIz11]. Our definitions follow [HoSm10] and are equivalent to those of [Smi10, ChSm12, ChIz11] after the multiplication by $\sqrt{i}$, see also Section 3.3 below.

Recall that, for $\tau\in\{1,i,\lambda,\overline{\lambda}\}$, we associate the line $\ell(x)=\tau\mathbb{R}$ in the complex plane with each corner $x\in\mathcal{V}^\tau_{\mathbb{C}_\delta}$, and denote by $\mathsf{P}_{\ell(x)}$ the projection onto that line:

$$\mathsf{P}_{\ell(x)}[w]=\tfrac{1}{2}\left(w+\tau^2\overline{w}\right),\quad w\in\mathbb{C}.$$

Definition 2.3 says that a function $F:\mathcal{V}^{\mathrm{cm}}_{\Omega_\delta}\to\mathbb{C}$ is *s-holomorphic in* Ω_δ (we use the same definitions working on double covers) if for every $x\in\mathcal{V}^{\mathrm{c}}_{\Omega_\delta}$ and $z\in\mathcal{V}^{\mathrm{m}}_{\Omega_\delta}$ that are adjacent, one has

$$F(x)=\mathsf{P}_{\ell(x)}\left[F(z)\right].$$

Remark 3.1. The set $\mathcal{V}^{1,i}_{\mathbb{C}_\delta}=\mathcal{V}^{1}_{\mathbb{C}_\delta}\cup\mathcal{V}^{i}_{\mathbb{C}_\delta}$ may be viewed as a square lattice, divided into $\mathcal{V}^{1}_{\mathbb{C}_\delta}$ and $\mathcal{V}^{i}_{\mathbb{C}_\delta}$ in a chessboard fashion. By definition, the restriction of an s-holomorphic function F to $\mathcal{V}^{1,i}_{\mathbb{C}_\delta}$ is real on $\mathcal{V}^{1}_{\mathbb{C}_\delta}$ and purely imaginary on $\mathcal{V}^{i}_{\mathbb{C}_\delta}$. It is not difficult to check ([Smi10]) that this restriction is in fact *discrete holomorphic* in the most usual sense, that is, for any $x\in\mathcal{V}^{1,i}_{\mathbb{C}_\delta}$ one has

$$F(x+i\delta)-F(x+\delta)=i\cdot\left[F(x+(1+i)\delta)-F(x)\right]. \tag{3.1}$$

The converse is also true: given discrete holomorphic function $F:\mathcal{V}^{1,i}_{\mathbb{C}_\delta}\to\mathbb{R}\cup i\mathbb{R}$, one can first extend it to $\mathcal{V}^{\mathrm{m}}_{\mathbb{C}_\delta}$ by the formula $F(z):=F(z+i\frac{\delta}{2})+F(z-i\frac{\delta}{2})$ and then to $\mathcal{V}^{\lambda,\overline{\lambda}}_{\mathbb{C}_\delta}$ by $F(x):=\mathsf{P}_{\ell(x)}\left[F\left(x\pm\frac{\delta}{2}\right)\right]$ (due to (3.1), these projections coincide).

We now check the s-holomorphicity of discrete spinor observables, essentially mimicking [ChSm12, HoSm10, ChIz11]. For shortness, below we use the notation

$$a_1^{\rightarrow}:=a_1+\tfrac{\delta}{2}\,\in\,\mathcal{V}^{i}_{\mathbb{C}_\delta}.$$

Proof of Proposition 2.4. Let $z\in\mathcal{V}^{\mathrm{m}}_{[\Omega_\delta,a_1,\dots,a_n]}$ be a medial vertex and x be one of four nearby corners so that $|x-z|=\frac{\delta}{2}$. We should check that

$$\mathsf{P}_{\ell(x)}\left[F_{[\Omega_\delta,a_1,\dots,a_n]}(z)\right]\;=\;F_{[\Omega_\delta,a_1,\dots,a_n]}(x), \tag{3.2}$$

where the values of $F_{[\Omega_\delta,a_1,\dots,a_n]}$ are defined as sums over the sets $\mathcal{C}_{\Omega_\delta}(a_1^{\rightarrow},z)$ and $\mathcal{C}_{\Omega_\delta}(a_1^{\rightarrow},x)$, respectively. There is a simple bijection $\omega_{zx}:\gamma_z\mapsto\gamma_x$ between these two sets provided by taking XOR (symmetric difference) of a configuration with two half edges (zv) and (vx), where v denotes the vertex which is adjacent to both x and z. Hence, it is sufficient to check that for any $\gamma_z\in\mathcal{C}_{\Omega_\delta}(a_1^{\rightarrow},z)$ one has

$$(\cos\tfrac{\pi}{8})^{-1}\cdot\mathsf{P}_{\ell(x)}\left[\alpha_c^{\#\mathrm{edges}(\gamma_z)}\cdot\phi_{a_1,\dots,a_n}(\gamma_z,z)\right]=\alpha_c^{\#\mathrm{edges}(\gamma_x)}\cdot\phi_{a_1,\dots,a_n}(\gamma_x,x) \tag{3.3}$$

(the additional factor $(\cos\frac{\pi}{8})^{-1}$ comes from our defintion of the discrete observable on medial vertices, see Remark 2.2). There are two cases: either (zv) is contained in γ, which leads us to

$$\#\mathrm{edges}(\gamma_x)=\#\mathrm{edges}(\gamma_z),\quad \exp[-\tfrac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma_x))]=e^{\pm\frac{i\pi}{8}}\exp[-\tfrac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma_z))],$$

or not, which leads to

$$\#\mathrm{edges}(\gamma_x)=\#\mathrm{edges}(\gamma_z)+1,\quad \exp[-\tfrac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma_x))]=e^{\pm\frac{3i\pi}{8}}\exp[-\tfrac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma_z))].$$

Let us also note that in both cases

$$(-1)^{\#\mathrm{loops}_{a_1,\dots,a_n}(\gamma_x)}\,\mathrm{sheet}_{a_1,\dots,a_n}(\gamma_x,x)=(-1)^{\#\mathrm{loops}_{a_1,\dots,a_n}(\gamma_z)}\,\mathrm{sheet}_{a_1,\dots,a_n}(\gamma_z,z)$$

since if ω_{zx} destroys a loop in γ_z that changed the sheet of $[\Omega_\delta,a_1,\dots,a_n]$ (leading to the change of the first factor), then this loop becomes a part of $\mathrm{p}(\gamma_x)$, so the second factor changes simultaneously. Thus, one can factor out $\alpha_c^{\#\mathrm{edges}(\gamma_z)}\cdot\phi_{a_1,\dots,a_n}(\gamma_z,z)$ from both sides of (3.3). In the first case (3.3) readily follows, while in the second it becomes equivalent to

$$(\cos\tfrac{\pi}{8})^{-1}\cos\tfrac{3\pi}{8}=\sqrt{2}-1=\alpha_{\mathrm{c}}\,.$$

Thus, $F_{[\Omega,a_1,\dots,a_n]}$ is s-holomorphic. It has multiplicative monodromy -1 around each of the marked points $a_1,\dots,a_n$ due to the factor $\mathrm{sheet}_{a_1,\dots,a_n}(\gamma,z)$ in (2.1). In order to prove that $F_{[\Omega,a_1,\dots,a_n]}$ obeys boundary conditions (2.3), it is sufficient to note that $\mathrm{wind}(\mathrm{p}(\gamma)) = \nu_{\mathrm{out}}(z) \mod 2\pi$, if z is on the boundary. $\square$

The spinor $F_{[\Omega,a_1,\dots,a_n]}$ is not defined at the corner $a_1^{\to}$ and is not s-holomorphic there. The next lemma shows, in particular, that its values at the nearby medial vertices $a_1+\frac{1\pm i}{2}\delta$ have different projections onto the imaginary line $i\mathbb{R}=\ell(a_1^{\to})$, so one cannot extend $F_{[\Omega,a_1,\dots,a_n]}$ to $a_1^{\to}$ in an s-holomorphic way.

Lemma 3.2. *For medial vertices $a_1+\frac{1\pm i}{2}\delta$ taken on the same sheet of the double cover $[\Omega_\delta,a_1,\dots,a_n]$ as the "source" $a_1^{\to}$, one has*

$$\mathsf{P}_{i\mathbb{R}}\left[F_{[\Omega_\delta,a_1,\dots,a_n]}\left(a_1+\tfrac{1\pm i}{2}\delta\right)\right]=\mp i.$$

Proof. We consider the medial vertex $z:=a_1+\frac{1+i}{2}\delta$, the opposite case is similar. Given a configuration $\gamma\in\mathcal{C}_\delta(a_1^{\to},z)$ and applying, as above, the XOR bijection with two half-edges $(a_1^{\to},a_1+\delta)$ and $(a_1+\delta,z)$, we obtain a configuration $\omega(\gamma)\in\mathcal{C}_\delta$. Since the normalizing factor $\mathcal{Z}^{+}_{\Omega_\delta}[a_1,\dots,a_n]$ is a sum over $\mathcal{C}_\delta$ (see (2.2)), it is sufficient to show that, for any γ,

$$(\cos\tfrac{\pi}{8})^{-1}\cdot\mathsf{P}_{i\mathbb{R}}\left[\alpha_{\mathrm{c}}^{\#\mathrm{edges}(\gamma)}\phi_{a_1,\dots,a_n}(\gamma,z)\right]=-i\cdot\alpha_{\mathrm{c}}^{\#\mathrm{edges}(\omega(\gamma))}(-1)^{\#\mathrm{loops}_{a_1,\dots,a_n}(\omega(\gamma))}.$$

Consider two cases, as in the proof of Proposition 2.4 above. If $(a_1+\delta,z)\in\gamma$ (respectively, $(a_1+\delta,z)\notin\gamma$), then

$$\#\mathrm{edges}(\omega(\gamma))=\#\mathrm{edges}(\gamma)\quad(\text{respectively,}\quad\#\mathrm{edges}(\omega(\gamma))=\#\mathrm{edges}(\gamma)+1)\,.$$

We may disregard all loops in $\omega(\gamma)$ that do not contain the edge $(a_1+\delta,a_1+i\delta)$, as they contribute the same sign to both sides. Further, if $(a_1+\delta,z)\in\gamma$, then $\mathrm{wind}(\mathrm{p}(\gamma))=\frac{3\pi}{4}\mod 4\pi$, otherwise $\mathrm{wind}(\mathrm{p}(\gamma))=\frac{7\pi}{4}\mod 2\pi$. Hence the lemma boils down to the following elementary identities:

$$(\cos\tfrac{\pi}{8})^{-1}\cdot\mathsf{P}_{i\mathbb{R}}[e^{-\frac{3\pi i}{8}}]=-i\quad\text{and}\quad(\cos\tfrac{\pi}{8})^{-1}\cdot\mathsf{P}_{i\mathbb{R}}[e^{-\frac{7\pi i}{8}}]=-i\alpha_c.\qquad\square$$

3.2. **The full-plane discrete spinor $F_{[\mathbb{C}_\delta,a]}$ and its discrete primitive $G_{[\mathbb{C}_\delta,a]}$.** This subsection is mainly devoted to the construction of the full-plane analogue of discrete spinor observables, as announced in Lemma 2.13. After this, we also prove Lemma 2.16 and the double-sided estimate (2.17) of the normalizing factor $\vartheta(\delta)$.

3.2.1. *Discrete harmonic measure in the slit plane.* We start with an important technical ingredient – the discrete Beurling estimate with optimal exponent $\frac{1}{2}$. On the square lattice, it was obtained by Kesten [Kes87], and then generalized by Lawler and Limic [LaLi04]. Given a face a, let $\mathbb{X}_\delta\subset\mathcal{V}^1_{\mathbb{C}_\delta}$ denote the slit discrete plane:

$$\mathrm{Int}\,\mathbb{X}_\delta:=\mathcal{V}^1_{\mathbb{C}_\delta}\setminus L_a,\qquad L_a:=\{x+a^{\to}+\delta : x<0\}$$

(recall that $a^{\to}+\delta=a+\frac{3\delta}{2}\in\mathcal{V}^1_{\mathbb{C}_\delta}$).

Lemma 3.3. *For all $z\in\mathbb{X}_\delta$, $A\subset\mathbb{X}_\delta$, and some absolute constant $C>0$, the following estimates are fulfilled:*

$$\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\to}+\delta\}}(z)\le C\delta^{\frac12}|z-a|^{-\frac12},\tag{3.4}$$

$$\mathrm{hm}^{\mathbb{X}_\delta}_{A}(a^{\to}+\delta)\le C\delta^{\frac12}(\mathrm{dist}(a;A))^{-\frac12},\tag{3.5}$$

where $\mathrm{hm}_A^{\mathbb{X}_\delta}(z)$ *denotes the discrete harmonic measure of a set* A *in* $\mathbb{X}_\delta$ *viewed from* z*, i.e., the probability for the simple random walk on* $\mathcal{V}^1_{\mathbb{C}_\delta}$ *(considered as a shifted square grid* $(2\delta\mathbb{Z})^2$*) started at* z *to reach* A *before it hits the boundary of* $\mathbb{X}_\delta$*.*

Proof. This easily follows from [LaLi04] and simple reversibility arguments for random walks. □

Below we also need some additional estimates for the discrete harmonic measure $\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^\to+\delta\}}$ and its discrete derivatives.

Lemma 3.4. *(i) For* $z\in\mathbb{X}_\delta$ *such that* $|\arg(z-a)-\pi|\le\frac{\pi}{4}$*, one has*

$$\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^\to+\delta\}}(z)\le C\delta^{\frac12}|\Im\mathfrak{m}\,(z-a)|\,|z-a|^{-\frac32}. \tag{3.6}$$

(ii) For all neighboring $z,z'\in\mathbb{X}_\delta$*, one has*

$$\delta^{-1}\left|\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^\to+\delta\}}(z')-\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^\to+\delta\}}(z)\right|\le C\delta^{\frac12}|z-a|^{-\frac32}. \tag{3.7}$$

(iii) Being normalized by the value $\vartheta(\delta)$ *at the proper lattice approximation of the point* $a+1$*, the functions* $(\vartheta(\delta))^{-1}\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^\to+\delta\}}$ *converge to* $\Re\mathfrak{e}\,[1/\sqrt{z-a}\,]$ *as* $\delta\to0$*, uniformly on compact subsets of* $\mathbb{C}\setminus L_a$*. Moreover, this convergence holds true in* C^1*-sense meaning that the discrete derivatives (3.7) converge to the corresponding partial derivatives as well.*

Proof. (i) In order to prove (3.6), note that the probability for the random walk started at z to leave the ball of radius $\frac12|z-a|$ around z before hitting $\partial\mathbb{X}_\delta$ is $O(|\Im\mathfrak{m}\,(z-a)|\cdot|z-a|^{-1})$, and once this has happened the probability to hit $a^\to+\delta$ is uniformly bounded by $O(\delta^{\frac12}|z-a|^{-\frac12})$ due to the Beurling estimate (3.4).

(ii) The estimate (3.7) for the discrete derivatives follows from (3.4), (3.6) and the (discrete) Harnack estimate (e.g., see [ChSm11, Proposition 2.7]), applied to the ball of radius $\frac12|z-a|$ (or $\frac12|\Im\mathfrak{m}\,(z-a)|$, if z is close to L_a) around z.

(iii) This is essentially a special case of [ChSm11, Theorem 3.13] which claims the C^1-convergence of discrete Poisson kernels normalized at some inner point to their continuous counterparts. The fact that our domain is unbounded plays no role here as, for any $r>0$, the positive discrete harmonic functions $f_a^\delta:=(\vartheta(\delta))^{-1}\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^\to+\delta\}}$ are uniformly bounded in the annulus $\{z:|z-a|\ge r\}$. Indeed, if f_a^δ is big at some point v in this annulus, then, by the maximum principle, f_a^δ is also big along some nearest-neighbor path running from v to $a^\to+\delta$. Since the discrete harmonic measure of such a path as seen from $a+1$ is bounded from below (by a constant depending on r but not on δ), this leads to a contradiction with $f_a^\delta(a+1)=1$. □

3.2.2. *Construction of the full-plane spinor* $F_{[\mathbb{C}_\delta,a]}$. Now we are ready to construct $F_{[\mathbb{C}_\delta,a]}$, as announced in Lemma 2.13.

Proof of Lemma 2.13. Let $\mathbb{X}_\delta^\pm$ denote two copies of the slit plane $\mathbb{X}_\delta\subset\mathcal{V}^1_{\mathbb{C}_\delta}$. We first define the real component $F^1_{[\mathbb{C}_\delta,a]}$ on $\mathbb{X}_\delta^\pm$ as

$$F^1_{[\mathbb{C}_\delta,a]}(z)=\pm\mathrm{hm}^{\mathbb{X}_\delta^\pm}_{\{a^\to+\delta\}}(z),\quad z\in\mathbb{X}_\delta^\pm. \tag{3.8}$$

Since $F^1_{[\mathbb{C}_\delta,a]}$ vanishes identically on L_a, by identifying the upper side of this cut in $\mathbb{X}_\delta^+$ with the lower side in $\mathbb{X}_\delta^-$ and vice versa, we obtain a function $F^1_{[\mathbb{C}_\delta,a]}$ which is defined and harmonic everywhere on $\mathcal{V}^1_{[\mathbb{C}_\delta,a]}$ except at the (two) points over $a^{\to}+\delta$. Recall that Lemma 3.4(iii) ensures the convergence

$$(\vartheta(\delta))^{-1}F^1_{[\mathbb{C}_\delta,a]}(z) \underset{\delta\to 0}{\longrightarrow} \mathfrak{Re}\,[1/\sqrt{z-a}\,] \qquad (3.9)$$

as well as the convergence of discrete derivatives of $F^1_{[\mathbb{C}_\delta,a]}(z)$ to partial derivatives of $\mathfrak{Re}\,[1/\sqrt{z-a}\,]$, uniformly on compact subsets of $\mathbb{C}\setminus L_a$.

We then define the imaginary component $F^i_{[\mathbb{C}_\delta,a]}:\mathcal{V}^i_{[\mathbb{C}_\delta,a^{\to}]}=\mathcal{V}^i_{[\mathbb{C}_\delta,a]}\setminus\{a^{\to}\}\to i\mathbb{R}$ as discrete harmonic conjugate of $F^1_{[\mathbb{C}_\delta,a]}$, that is, by integrating the identity (3.1) along paths on $\mathcal{V}^i_{[\mathbb{C}_\delta,a^{\to}]}$ starting from, say, one of the two fibers of the point $a^{\to}+2\delta$. Since $F^1_{[\mathbb{C}_\delta,a]}$ is harmonic, its discrete harmonic conjugate is well-defined at least on the universal cover of $[\mathbb{C}_\delta,a^{\to}]$. Further, let ϖ denote some simple loop in $\mathcal{V}^i_{\mathbb{C}_\delta}$, starting and ending at $a^{\to}+2\delta$, symmetric with respect to the horizontal line $\{x:\mathfrak{Im}\,(x-a)=0\}$, and surrounding the singularity (so, it lifts to $[\mathbb{C}_\delta,a^{\to}]$ as a path connecting two different fibers of $a^{\to}+2\delta$). It follows from the antisymmetry of $F^1_{[\mathbb{C}_\delta,a]}$ with respect to L_a (also, note that ϖ changes the sheet once passing across the cut), that the total increment of $F^i_{[\mathbb{C}_\delta,a]}$ along ϖ is zero. Thus, $F^i_{[\mathbb{C}_\delta,a]}$ vanishes at both fibers of the point $a^{\to}+2\delta$, and hence inherits the spinor property of its discrete harmonic conjugate $F^1_{[\mathbb{C}_\delta,a]}$. Moreover, for $b:=a^{\to}+2j\delta$, $j\geq 1$, the discrete holomorphicity equation (3.3) and the symmetry of $F^1_{[\mathbb{C}_\delta,a]}$ with respect to the half-line $R_a:=\{x+a^{\to}:x>0\}$ imply

$$F^i_{[\mathbb{C}_\delta,a]}(b+2\delta)-F^i_{[\mathbb{C}_\delta,a]}(b)=\mp i[F^1_{[\mathbb{C}_\delta,a]}(b\pm i\delta)-2F^1_{[\mathbb{C}_\delta,a]}(b+\delta)+F^1_{[\mathbb{C}_\delta,a]}(b+(2\pm i)\delta)]=0.$$

Therefore, $F^i_{[\mathbb{C}_\delta,a]}$ vanishes everywhere on R_a. Further, the estimate (3.7) guarantees that $F^i_{[\mathbb{C}_\delta,a]}$ is uniformly bounded: just take a path of discrete integration in the definition of $F^i_{[\mathbb{C}_\delta,a]}$ running from R_a to z along the circular arc centered at 0.

It is worth to note that $F^i_{[\mathbb{C}_\delta,a]}$ also admits a discrete harmonic measure representation similar to (3.8). Namely, let $\mathbb{Y}^\pm_\delta$ denote the two sheets of $\mathcal{V}^i_{[\mathbb{C}_\delta,a^{\to}]}\setminus R_a$, where signs in the notation are chosen so that $F^1_{[\mathbb{C}_\delta,a]}>0$ in the upper half-plane on $\mathbb{Y}^+_\delta$ and in the lower half-plane on $\mathbb{Y}^-_\delta$. Then,

$$F^i_{[\mathbb{C}_\delta,a]}(z)=\mp i\cdot \mathrm{hm}^{\mathbb{Y}^\pm_\delta}_{\{a^{\to}\}}(z),\quad z\in\mathbb{Y}^\pm_\delta. \qquad (3.10)$$

Indeed, on each of the sheets $\mathbb{Y}^\pm_\delta$ one can further extend $F^i_{[\mathbb{C}_\delta,a]}$ (as a harmonic conjugate of $F^1_{[\mathbb{C}_\delta,a]}$) to the point $a^{\to}$: the only obstruction to do so on $[\mathbb{C}_\delta,a]$ was that the increment of $F^i_{[\mathbb{C}_\delta,a]}$ along the smallest loop surrounding $a^{\to}+\delta$ would be non-zero as $F^1_{[\mathbb{C}_\delta,a]}$ is not harmonic at $a^{\to}+\delta$, but now this loop intersects the cut R_a. Since the bounded function $F^i_{[\mathbb{C}_\delta,a]}$ is harmonic on $\mathbb{Y}^\pm_\delta$ and vanishes on R_a, it is proportional to $\mathrm{hm}^{\mathbb{Y}^\pm_\delta}_{\{a^{\to}\}}$. Finally, it follows from symmetry arguments that

$$\begin{aligned}\pm\mathrm{hm}^{\mathbb{Y}^\pm_\delta}_{\{a^{\to}\}}(a^{\to}+(1+i)\delta)\;-\;(\pm 1) &= \pm\mathrm{hm}^{\mathbb{X}^\pm_\delta}_{\{a^{\to}+\delta\}}(a^{\to}+i\delta)-(\pm 1)\\ &= F^1_{[\mathbb{C}_\delta,a]}(a^{\to}+i\delta)-F^1_{[\mathbb{C}_\delta,a]}(a^{\to}+\delta)\\ &= i\cdot[F^i_{[\mathbb{C}_\delta,a]}(a^{\to}+(1+i)\,\delta)-F^i_{[\mathbb{C}_\delta,a]}(a^{\to})]\end{aligned}$$

which fixes the multiplicative constant $\mp i$ in (3.10).

Similarly to (3.9), we have

$$(\vartheta(\delta))^{-1}F^{i}_{[\mathbb{C}_\delta,a]}(z) \underset{\delta\to 0}{\longrightarrow} \Im\mathfrak{m}\,[1/\sqrt{z-a}\,] \tag{3.11}$$

together with the convergence of discrete derivatives, uniformly on compact subsets of $\mathbb{C}\setminus R_a$. Now, following Remark 3.1, we extend $F_{[\mathbb{C}_\delta,a]}$ to $\mathcal{V}^{\mathrm{m}}_{[\mathbb{C},a]}$ and $\mathcal{V}^{\lambda,\overline{\lambda}}_{[\mathbb{C},a]}$ as an s-holomorphic function. The convergence (2.15) readily follows from (3.9) and (3.11) (for points lying near L_a, one can approximate $F^{1}_{[\mathbb{C}_\delta,a]}$ by summing discrete derivatives of $F^{i}_{[\mathbb{C}_\delta,a]}$ and vice versa near R_a). □

The following lemma explains the future role of $F_{[\mathbb{C}_\delta,a]}$: it has the same "discrete singularity" (whatever it means) at $a^{\rightarrow}$ as $F_{[\Omega_\delta,a_1,\dots,a_n]}$ has at $a_1^{\rightarrow}$. This allows one to handle this singularity by taking the difference $F_{[\Omega_\delta,a_1,\dots,a_n]}-F_{[\mathbb{C}_\delta,a_1]}$

Lemma 3.5. *The spinor* $F^{\dagger}_{[\Omega_\delta,a_1,\dots,a_n]} := F_{[\Omega_\delta,a_1,\dots,a_n]}-F_{[\mathbb{C}_\delta,a_1]}$, *extended to be zero at* $a_1^{\rightarrow}$, *is s-holomorphic in any simply connected neighborhood of* a_1 *which do not contain other branching points* $a_2,\dots,a_n$.

Proof. By Lemma 3.2, it suffices to check that $\mathsf{P}_{i\mathbb{R}}\left[F_{[\mathbb{C}_\delta,a_1]}\left(a_1+\tfrac{1\pm i}{2}\delta\right)\right]=\mp i$ on the sheet $\mathbb{X}^{+}_\delta$. The considerations given in the previous proof show that, being considered on $\mathbb{Y}^{\pm}_\delta$ instead of $\mathbb{X}^{+}_\delta$, the spinor $F_{[\mathbb{C}_\delta,a_1]}$ can be extended at $a_1^{\rightarrow}$ in an s-holomorphic way and

$$\mathsf{P}_{i\mathbb{R}}\left[F_{[\mathbb{C}_\delta,a_1]}\left(a_1+\tfrac{1\pm i}{2}\delta\right)\right]=F^{i}_{[\mathbb{C}_\delta,a_1]}(a_1^{\rightarrow})=\mp i \;\text{ on }\; \mathbb{Y}^{\pm}_\delta.$$

However, by definition, $\mathbb{X}^{+}_\delta$ coincides with $\mathbb{Y}^{+}_\delta$ in the upper half-plane, and with $\mathbb{Y}^{-}_\delta$ in the lower half-plane. □

3.2.3. *Construction of the harmonic spinor* $G_{[\mathbb{C}_\delta,a]}$. Now we "integrate" the real component of $F_{[\mathbb{C}_\delta,a]}$ in order to construct a discrete counterpart of the harmonic spinor $\Re\mathfrak{e}\sqrt{z-a}$, as announced in Lemma 2.16. Along the way, we also prove the double-sided estimate (2.17) of the normalizing factor $\vartheta(\delta)$. Note that the upper bound $\vartheta(\delta)\le \mathrm{C}_+\sqrt{\delta}$ directly follows from the discrete Beurling estimate (3.4), so we need to prove the lower bound only.

Proof of Lemma 2.16 and the estimate (2.16). We use the notation for the sheets and the cuts of $[\mathbb{C}_\delta,a]$ introduced above. For $z\in\mathbb{X}^{+}_\delta\subset\mathcal{V}^{1}_{[\mathbb{C}_\delta,a]}$, define

$$G_{[\mathbb{C}_\delta,a]}(z) := \delta\cdot\sum_{j=0}^{\infty}F^{1}_{[\mathbb{C}_\delta,a]}(z-2j\delta). \tag{3.12}$$

Due to the estimate (3.6) for $F^{1}_{[\mathbb{C}_\delta,a]}(\,\cdot\,)=\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\rightarrow}+\delta\}}(\,\cdot\,)$, this sum always converges. Note that $F^{1}_{[\mathbb{C}_\delta,a]}=0$ on the cut L_a, so $G_{[\mathbb{C}_\delta,a]}$ vanishes on L_a too.

We are going to prove that $G_{[\mathbb{C}_\delta,a]}$ is harmonic *everywhere* inside $\mathbb{X}^{+}_\delta$, including the point $a^{\rightarrow}+\delta$ right near the cut L_a. For z outside R_a, this harmonicity readily follows from the harmonicity of $F^{1}_{[\mathbb{C}_\delta,a]}$, so let $z\in R_a$. Denote by $\mathcal{S}_N(z)$,

$$\operatorname{Int}\mathcal{S}_N(z) := \mathcal{V}^{1}_{\mathbb{C}_\delta}\cap\{z : |\Re\mathfrak{e}\,(w-z)|,\ |\Im\mathfrak{m}\,(w-z)|\le 2N\delta\}$$

a sufficiently large square centered at z and write the discrete Green formula:

$$\begin{aligned}\sum_{j=\lfloor z/2\delta\rfloor}^{N} \Delta F^1_{[\mathbb{C}_\delta,a]}(z-2j\delta) &= \sum_{w\in\mathcal{S}_N(z)} \Delta F^1_{[\mathbb{C}_\delta,a]}(w)\\ &= \sum_{(ww')\in\partial\mathcal{S}_N(z)} (F^1_{[\mathbb{C}_\delta,a]}(w')-F^1_{[\mathbb{C}_\delta,a]}(w)).\end{aligned}$$

Let N be large enough so that $|w-a|\geq N\delta$ for all boundary edges of the square $\mathcal{S}_N(z)$. Then, it immediately follows from the estimate (3.7) that the last sum is $O(N\cdot\delta^{\frac{3}{2}}(N\delta)^{-\frac{3}{2}})=O(N^{-\frac{1}{2}})$. Passing to a limit as $N\to\infty$, we conclude that

$$\Delta G_{[\mathbb{C}_\delta,a]}(z)=\delta\cdot\sum_{j=\lfloor z/2\delta\rfloor}^{\infty}\Delta F^1_{[\mathbb{C}_\delta,a]}(z-2j\delta)=0.$$

Thus, $G_{[\mathbb{C}_\delta,a]}$ is indeed discrete harmonic in $\mathbb{X}^+_\delta$. Moreover, since it vanishes on L_a, one can harmonically extend $G_{[\mathbb{C}_\delta,a]}$ to the double cover $\mathcal{V}^1_{[\mathbb{C}_\delta,a]}$ by symmetry.

Let $\nu(\delta)$ denote the value of $G_{[\mathbb{C}_\delta,a]}$ in a lattice approximation of the point $a+1$. Arguing as in the proof of Lemma 3.4(iii), we see that, uniformly on compact subsets of $\mathbb{C}\setminus L_a$,

$$(\nu(\delta))^{-1}G_{[\mathbb{C}_\delta,a]}(z)\underset{\delta\to 0}{\longrightarrow}\mathfrak{Re}\sqrt{z-a} \tag{3.13}$$

(since $G_{[\mathbb{C}_\delta,a]}$ vanishes everywhere on L_a and remains bounded near 0 by the maximum principle, in this case the limiting positive harmonic function should be proportional to $\mathfrak{Re}\sqrt{z-a}$, and the multiplicative normalization is fixed at $a+1$). Moreover, the similar convergence holds true for discrete derivatives, yielding

$$(\nu(\delta))^{-1}\cdot\tfrac{1}{2}F^1_{[\mathbb{C}_\delta,a]}(z)\underset{\delta\to 0}{\longrightarrow}\partial_x\mathfrak{Re}\sqrt{z-a}=\tfrac{1}{2}\,\mathfrak{Re}\left[1/\sqrt{z-a}\,\right].$$

In particular, $\nu(\delta)\sim\vartheta(\delta)$ as $\delta\to 0$ which allows us to give a simple proof of the lower bound in (2.17): as discrete harmonic functions $(\nu(\delta))^{-1}G_{[\mathbb{C}_\delta,a]}$ are uniformly bounded near the unit circle around a and vanish identically on L_a, discrete Beurling estimate(3.5) implies

$$(\nu(\delta))^{-1}\cdot\delta=(\nu(\delta))^{-1}G_{[\mathbb{C}_\delta,a]}(a^{\rightarrow}+\delta)\leq C\sqrt{\delta}.$$

Finally, the convergence (3.13) near L_a follows from the convergence of $F^1_{[\mathbb{C}_\delta,a]}(z)$, since the tails in (3.12) are uniformly small due to the estimate (3.6). □

3.3. **The boundary value problem for spinors.** In this section we reformulate the Riemann-type boundary value problem for holomorphic spinors (both discrete and continuous) using primitives of their squares. Note that this approach is not completely straightforward, since the square of a discrete holomorphic function, in general, does not have discrete primitive. However, it was noted in [Smi06] that one can naturally define the real part of this primitive, using the *s-holomorphicity* of observables. Moreover, a technique developed in [ChSm12] (see, in particular, sections 3.4 and 3.5 therein) allows one to treat this real part essentially if it were a harmonic function. Below, we summarize the tools we will use. We warn the reader that all our definitions are equivalent to those of [ChSm12] *after the multiplication of the spinor by* $\sqrt{i}$, which means that imaginary part, sub-/super-harmonicity and positivity of functions and their (inner) normal derivatives used in [ChSm12] should be replaced by real part, super-/sub-harmonicity and negativity, respectively.

3.3.1. *Discrete integration of the squared spinor observables.* Let Δ°_δ be the standard (unnormalized) discrete Laplacian acting on functions $H^\circ_\delta : \mathcal{V}^\circ_{\Omega_\delta} \to \mathbb{R}$ (which are defined on faces of Ω_δ):

$$\Delta^\circ_\delta H^\circ_\delta(z) \ := \ \sum_{w\sim z} \left(H^\circ_\delta(w) - H^\circ_\delta(z)\right), \quad z \in \mathrm{Int}\mathcal{V}^\circ_{\Omega_\delta},$$

where the sum is over the four neighbors $w \in \mathcal{V}^\circ_{\Omega_\delta}$ of z. Similarly, for functions $H^\bullet_\delta : \mathcal{V}^\bullet_{\Omega_\delta} \to \mathbb{R}$ defined on vertices of Ω_δ, let $\Delta^\bullet_\delta$ denote the slightly modified discrete Laplacian:

$$\Delta^\bullet_\delta H^\bullet_\delta(z) \ := \ \sum_{w\sim z} c_{zw} \cdot \left(H^\bullet_\delta(w) - H^\bullet_\delta(z)\right), \quad z \in \mathrm{Int}\mathcal{V}^\bullet_{\Omega_\delta},$$

where the conductance c_{zw} is equal to 1 for inner edges (i.e., for $w \in \mathrm{Int}\mathcal{V}^\bullet_{\Omega_\delta}$) and $c_{zw} = 2(\sqrt{2}-1)$ for the boundary edges (see Section 3.6 in [ChSm12] or [DHN11] for the reason of this "boundary modification" of $\Delta^\bullet_\delta$).

Proposition 3.6. *For an s-holomorphic spinor observable $F_\delta = F_{[\Omega_\delta,a_1,\dots,a_n]} : [\Omega_\delta, a_1^{\rightarrow}, \dots, a_n] \to \mathbb{C}$ satisfying boundary conditions (2.3), one can define a real-valued function $H_\delta = H_{[\Omega_\delta,a_1,\dots,a_n]} : \mathcal{V}^{\bullet\circ}_{\Omega_\delta} \to \mathbb{R}$ which is a discrete analogue of the primitive $\mathfrak{Re}\int(F_\delta(z))^2dz$, so that the following properties are fulfilled:*

- *for any adjacent $w \in \mathcal{V}^\circ_{\Omega_\delta}$ and $v \in \mathcal{V}^\bullet_{\Omega_\delta}$, one has*
$$H^\circ_\delta(w) - H^\bullet_\delta(v) = 2\delta \left|F_\delta\left(\tfrac{1}{2}(w+v)\right)\right|^2, \qquad (3.14)$$
where $\frac{1}{2}(w+v) \in \mathcal{V}^{\mathrm{c}}_{\Omega_\delta}$ is the corner between v and w (in accordance with Lemma 3.2, we set $|F_\delta(a_1^{\rightarrow})| := 1$ in the case $w = a_1$ and $v = a_1 + \delta$);
- *H_δ satisfies Dirichlet boundary conditions: $H^\circ_\delta(v) = 0$ for any $w \in \partial\mathcal{V}^\circ_{\Omega_\delta}$, and $H^\bullet_\delta(v) = 0$ for any $v \in \partial\mathcal{V}^\bullet_{\Omega_\delta}$;*
- *$H^\bullet_\delta$ has a "negative inner normal derivative", i.e. $H_\delta(v) \le 0$ for any vertex $v \in \mathcal{V}^\bullet_{\Omega_\delta}$ adjacent to a boundary vertex;*
- *H°_δ is Δ°_δ-subharmonic on $\mathcal{V}^\circ_{\Omega_\delta} \setminus \{a_1,\dots,a_n\}$, while $H^\bullet_\delta$ is $\Delta^\bullet_\delta$-superharmonic on $\mathcal{V}^\bullet_{\Omega_\delta} \setminus \{a_1+\delta\}$.*

Proof. All the claims follow directly from the results of Section 3.3 in [ChSm12]. Since all listed properties are local, the spinor nature of F_δ plays no role here (note that the right-hand side of (3.14) does not depend on the sheet). □

Remark 3.7. By construction, $H^\circ_\delta(w) \ge H^\bullet_\delta(v)$ for adjacent w and v. Combined with sub-/super-harmonicity, this implies a maximum principle for H_δ: if $\Omega'_\delta \subset \Omega_\delta$ does not contain $a_1+\delta$ (respectively, any of $a_1,\dots,a_n$), then

$$\min_{\Omega'_\delta} H_\delta = \min_{\partial\Omega'_\delta} H^\bullet_\delta \quad \text{(respectively, } \max_{\Omega'_\delta} H_\delta = \max_{\partial\Omega'_\delta} H^\circ_\delta\text{)}.$$

Moreover, if $\mathrm{hm}_A(z)$ denotes the discrete harmonic measure of a set A in Ω'_δ viewed from z, then

$$H_\delta(z) \ge (1 - \mathrm{hm}^\bullet_A(z)) \min_{\partial\Omega'_\delta} H^\bullet_\delta + \mathrm{hm}^\bullet_A(z) \min_A H^\bullet_\delta;$$
$$H_\delta(z) \le (1 - \mathrm{hm}^\circ_A(z)) \max_{\partial\Omega'_\delta} H^\circ_\delta + \mathrm{hm}^\circ_A(z) \max_A H^\circ_\delta.$$

Remark 3.8. The subharmonicity of H_δ° fails at $a_1, \dots, a_n$ because $F_{[\Omega_\delta, a_1 \dots a_n]}$ branches at those points, while the superharmonicity of $H_\delta^\bullet$ fails at $a_1 + \delta$ because of the discrete singularity of $F_{[\Omega_\delta, a_1 \dots a_n]}$ which is not defined at $a_1^\rightarrow$. Due to Lemma 3.5, one can remove this singularity subtracting the full-plane observable $F_{[\mathbb{C}_\delta, a_1]}$. Then, the function

$$H_\delta^\dagger = H^\dagger_{[\Omega_\delta, a_1 \dots, a_n]} := \Re\mathfrak{e} \textstyle\int (F_\delta^\dagger(z))^2 dz, \quad F_\delta^\dagger := F_{[\Omega_\delta, a_1 \dots, a_n]} - F_{[\mathbb{C}_\delta, a_1]}$$

(defined in the same way as H_δ accordingly to Proposition 3.6) is subharmonic on $\mathcal{V}^\circ_{\Omega_\delta} \setminus \{a_1\}$ and superharmonic on $\mathcal{V}^\bullet_{\Omega_\delta}$ everywhere in a neighborhood of a_1. Moreover, since $F_\delta^\dagger(a_1^\rightarrow) = 0$ on both sheets, the values of $F_\delta^\dagger$ at the nearby corners $a_1 \pm \frac{\delta}{2}, a_1 \pm \frac{i\delta}{2}$ and midedges $a_1 \pm \frac{1 \pm i}{2}\delta$ satisfy the s-holomorphicity conditions as if it were nonbranching at a_1. Thus, $H_\delta^\dagger$ is subharmonic at the point $a_1 \in \mathcal{V}^\circ_{\Omega_\delta}$ too.

3.3.2. *Integration of squared spinors in the continuous setup.* We now give a characterization of the continuous spinors solving boundary value problem (2.7)–(2.9) in terms of the primitives of their squares. In the next section, this characterization will be used in the proofs of main convergence results.

Proposition 3.9. *Let Ω be a simply connected domain, and suppose a holomorphic spinor $f = f_{[\Omega, a_1, \dots, a_n]}$ solves the boundary value problem (2.7)–(2.9) (or is defined according to Remark 2.9(ii), if Ω is not smooth). Define two harmonic functions*

$$h := \Re\mathfrak{e} \textstyle\int (f(z))^2 dz \quad \textit{and} \quad h^\dagger := \Re\mathfrak{e} \textstyle\int (f(z) - f_{[\mathbb{C}, a_1]}(z))^2 dz,$$

where $f_{[\mathbb{C}, a_1]}(z) := 1/\sqrt{z - a_1}$. Then, the following holds true:

1. *h is a single-valued function in $\Omega \setminus \{a_1, \dots, a_n\}$, continuous up to $\partial\Omega$, which satisfies Dirichlet boundary conditions $h \equiv const$ on $\partial\Omega$ (since h is defined up to an additive constant, below we assume that $h \equiv 0$ on $\partial\Omega$);*
2. *there is no point z_0 on $\partial\Omega$ such that $h(z) \geq 0$ in a neighborhood of z_0;*
3. *h is bounded from below in a neighborhood of each $a_2, \dots, a_n$;*
4. *$h^\dagger$ is single-valued and bounded in a neighborhood of a_1.*

Moreover, if h and $h^\dagger$ satisfy (1)–(4), then f solves the problem (2.7)–(2.9).

Proof. Note that, being integrated, the covariance property (2.11) claims the conformal invariance of both h and $h^\dagger$. As the properties (1)–(4) are preserved under conformal mappings too, we will further assume that $\partial\Omega$ is smooth. Note that the property (2.7) is equivalent to (1) and (2): it states that $f^2(z)\nu_{\mathrm{out}}(z)$ is positive on the boundary, which yields $\partial_{\tau(z)} h \equiv 0$ (where $\tau(z)$ denotes a tangent vector) and $\partial_{\nu_{\mathrm{out}}(z)} h \geq 0$ everywhere on $\partial\Omega$. A straightforward integration of the asymptotics of f near a_k given by the conditions (2.8) and (2.9) yields

$$\begin{array}{rcll} h^\dagger(z) & = & O(1), & z \to a_1, \\ h(z) & = & -c_k \log|z - a_k| + O(1), & z \to a_k, \ 2 \leq k \leq n. \end{array} \tag{3.15}$$

for some $c_k \geq 0$, giving (3) and (4). Vice versa, (3),(4) imply (2.9), (2.8) by differentiating and taking the square root. □

3.4. **Convergence of discrete observables away from singularities.** In this section we prove the convergence of (properly normalized) discrete spinor observables to their continuous counterparts on compact subsets of $\Omega \setminus \{a_1, \dots, a_n\}$.

Proof of Theorem 2.15. Let the discrete integrals

$$H_\delta := \mathfrak{Re} \int \left((\vartheta(\delta))^{-1} F_\delta(z) \right)^2 dz, \quad F_\delta = F_{[\Omega_\delta, a_1, \ldots, a_n]},$$

be defined on $\mathcal{V}^{\bullet\circ}_{\Omega_\delta}$ accordingly to Proposition 3.6. Given $\epsilon > 0$, denote

$$\Omega_\delta(\epsilon) := \Omega_\delta \cap \{z : \operatorname{dist}(z; \{a_1, \ldots, a_n\}) > \epsilon\}.$$

Assume for a moment that,

$$\begin{array}{l}\text{for any } \epsilon > 0, \text{ the functions } H_\delta \text{ remain uniformly} \\ \text{bounded on } \Omega_\delta(\epsilon) \text{ by some constant } C(\epsilon) \text{ as } \delta \to 0.\end{array} \tag{3.16}$$

Then by [ChSm12, Theorem 3.12], the functions $(\vartheta(\delta))^{-1} F_\delta$ are equicontinuous on $\Omega_\delta(\epsilon)$ for any $\epsilon > 0$. Therefore, by passing to a subsequence and applying the diagonal process, we can assume that $(\vartheta(\delta))^{-1} F_\delta$ tends to a limit $\tilde{f}$ and $H_\delta \to \tilde{h} := \mathfrak{Re} \int \tilde{f}^2$ uniformly on compact subsets of $\Omega \backslash \{a_1, \ldots, a_n\}$. Our goal is to check that $\tilde{f}$ satisfies the properties (1) – (4) given in Proposition 3.9. Then, the uniqueness of a solution to the boundary value problem (2.7)–(2.9) proven in Remark 2.9(i) implies $\tilde{f} = f_{[\Omega, a_1, \ldots, a_n]}$.

Clearly, $\tilde{f}$ is a holomorphic spinor on $[\Omega, a_1, \ldots, a_n]$. By superhamonicity of $H_\delta^\bullet$, for $2 \le k \le n$, we have

$$\min_{|z - a_k| \le \epsilon} H_\delta(z) \ \ge \min_{\epsilon < |z - a_k| \le 2\epsilon} H_\delta^\bullet(z),$$

thus $\tilde{h}$ is bounded from below in the neighborhoods of $a_2, \ldots, a_n$, so the property (3) holds true. By the maximum principle for H_δ (see Remark 3.7), taking into account that $H_\delta \equiv 0$ on $\partial\Omega_\delta$, we have that

$$\begin{array}{rcl} H_\delta^\bullet(z) & \ge & -C(\epsilon)(1 - \operatorname{hm}_{\partial\Omega_\delta}^{\Omega_\delta(\varepsilon)}(z)) \\ H_\delta^\circ(z) & \le & C(\epsilon)(1 - \operatorname{hm}_{\partial\Omega_\delta}^{\Omega_\delta(\varepsilon)}(z)) \end{array} \tag{3.17}$$

Since $\operatorname{hm}_{\partial\Omega_\delta}^{\Omega_\delta(\varepsilon)}(z) \to 1$ uniformly in δ as z approaches the boundary of Ω_δ, this implies $\tilde{h} \equiv 0$ on $\partial\Omega$, giving (1). Moreover, thanks to Remark 6.3 in [ChSm12], we also have (2): there is no point on $\partial\Omega$ such that $\tilde{h} \ge 0$ in its neighborhood.

Consider now the function discussed (up to normalization) in Remark 3.8:

$$H_\delta^\dagger := \mathfrak{Re} \int ((\vartheta(\delta))^{-1} (F_\delta(z) - F_{[\mathbb{C}_\delta, a_1]}(z)))^2 dz,$$

which is well-defined in the disc $\{z : |z - a_1| < r\}$ provided that r is small enough. Since $(\vartheta(\delta))^{-1} F_\delta$ and $(\vartheta(\delta))^{-1} F_{[\mathbb{C}_\delta, a_1]}$ converge to $\tilde{f}$ and $f_{[\mathbb{C}, a_1]}$, respectively (see Lemma 2.13), uniformly on compact subsets of $A_r := \{z : 0 < |z - a_1| < r\}$, we conclude that $H_\delta^\dagger$ converges to $\tilde{h}^\dagger := \mathfrak{Re} \int (\tilde{f}(z) - f_{[\mathbb{C}, a_1]}(z))^2 dz$ everywhere in A_r. By Remark 3.8 (sub-/super-harmonicity of $H_\delta^\dagger$ on $\mathcal{V}^\circ / \mathcal{V}^\bullet$ near the point a_1), the functions $H_\delta^\dagger$ are uniformly bounded in A_r, so $\tilde{h}^\dagger$ is bounded in A_r too, which concludes the proof of the property (4). Therefore, $\tilde{f} = f_{[\Omega, a_1, \ldots, a_n]}$.

It remains to justify (3.16). On the contrary, suppose that

$$M_\delta(\epsilon) := \max_{\Omega_\delta(\epsilon)} |H_\delta| \underset{\delta \to 0}{\longrightarrow} \infty$$

for some $\epsilon > 0$ and along some subsequence of δ's. Then the re-normalized functions $(M_\delta(\epsilon))^{-1} H_\delta$ are uniformly bounded in $\Omega_\delta(\varepsilon)$, and thus $(M_\delta(\epsilon))^{-1/2} \cdot (\vartheta(\delta))^{-1} F_\delta$

and $(M_\delta(\epsilon))^{-2}H_\delta$ have subsequential limits $\tilde{f}$ and $\tilde{h} = \mathfrak{Re}\int \tilde{f}^2$ which are holomorphic and harmonic in $\Omega_\delta(\epsilon)$, respectively. An important observation, proven in Lemma 3.10 below, is that $\tilde{h}$ cannot be identically zero. In particular, for any $0<\epsilon'<\epsilon$, we have $M_\delta(\epsilon')\le CM_\delta(\epsilon)$ with some $C=C(\epsilon',\epsilon)$ independent of δ.

Applying the diagonal procedure, we may assume that $(M_\delta(\epsilon))^{-1/2}\cdot(\vartheta(\delta))^{-1}F_\delta$ tends to a limit $\tilde{f}$ (and $(M_\delta(\epsilon))^{-1}H_\delta$ tends to $\tilde{h} = \mathfrak{Re}\int\tilde{f}^2$) uniformly on *each* of $\Omega_\delta(\epsilon')$. Arguing as above, we see that $\tilde{h}$ is harmonic in $\Omega\backslash\{a_1,\dots,a_n\}$, satisfies Dirichlet boundary conditions, has positive outer normal derivative, and is bounded from below near $a_1,\dots,a_n$. Moreover, repeating the last step of the proof given above we see that the function

$$\begin{aligned}\tilde{h}^\dagger &= \lim_{\delta\to 0}(M_\delta(\epsilon))^{-1}\mathfrak{Re}\int((\vartheta(\delta))^{-1}(F_\delta(z)-F_{[\mathbb{C}_\delta,a_1]}(z)))^2dz\\ &= \lim_{\delta\to 0}(M_\delta(\epsilon))^{-1}\mathfrak{Re}\int((\vartheta(\delta))^{-1}F_\delta(z))^2dz = \tilde{h}\end{aligned}$$

is also bounded in a neighborhood of a_1 (one can neglect $F_{[\mathbb{C}_\delta,a_1]}(z)$ in the last expression since $(\vartheta(\delta))^{-1}F_{[\mathbb{C}_\delta,a_1]}(z)$ tends to $f_{[\mathbb{C},a_1]}$ and $M_\delta(\epsilon)\to\infty$). Thus, $\tilde{h}$ is bounded from below near *all* $a_1,\dots,a_n$ and has positive outer normal derivative which contradicts to the maximum principle, if it does not vanish identically. □

Lemma 3.10. *In the notation of the proof above, none of the subsequential limits of $(M_\delta(\epsilon))^{-1}H_\delta$ is identically zero in $\Omega(\epsilon)$.*

Proof. Suppose by contradiction that $(M_\delta(\epsilon))^{-1}H_\delta\to 0$ uniformly on compact subsets of $\Omega_\delta(\epsilon)$. Let $z_\delta^{\max}$ be the point of $\Omega_\delta(\epsilon)$ where the maximum of $|H_\delta|$ is attained. Since H_δ vanishes on $\partial\Omega_\delta$, the sub-/super-harmonicity of H_δ implies that $z_\delta^{\max}$ belongs to one of the "discrete circles"

$$\varpi_k(\epsilon) := \{z : \epsilon\le|z-a_k|\le\epsilon+5\delta\}$$

of radius ϵ around a_k. Passing to a subsequence, one can assume that $z_\delta^{\max}\to z^{\max}$. Recall that $H^\circ\ge H^\bullet$ at adjacent points (see (3.14)). Hence, either $z_\delta^{\max}\in\mathcal{V}^\circ_{\Omega_\delta}$ and $M_\delta(\epsilon)=H^\circ_\delta(z_\delta^{\max})$, or $z_\delta^{\max}\in\mathcal{V}^\bullet_{\Omega_\delta}$ and $M_\delta(\epsilon)=-H^\bullet_\delta(z_\delta^{\max})$.

Suppose that $z_\delta^{\max}\in\varpi_k(\epsilon)$ for some $2\le k\le n$. Denote

$$m_\delta(2\epsilon) := \min_{z:|z-a_k|\le 2\epsilon} H^\bullet_\delta(z).$$

As $H^\bullet_\delta$ is superharmonic inside $\varpi_k(2\epsilon)$ and $(M_\delta(\epsilon))^{-1}H^\bullet_\delta$ tends to zero uniformly on $\varpi_k(2\epsilon)$ by our assumption, we have

$$(M_\delta(\epsilon))^{-1}\cdot m_\delta(2\epsilon)\underset{\delta\to 0}{\longrightarrow} 0. \tag{3.18}$$

Therefore, $z_\delta^{\max}\in\mathcal{V}^\circ_{\Omega_\delta}$. By subharmonicity of H°_δ, we can find a discrete nearest-neighbor path $\gamma^\circ := \{z_\delta^{\max}=z_1\sim z_2\sim\dots\}\subset\mathcal{V}^\circ_{\Omega_\delta}$ with $M_\delta(\epsilon)\le\dots\le H^\circ_\delta(z_j)\le H^\circ_\delta(z_{j+1})\le\dots$, which may only end up at a_k, where the subharmonicity fails. By [ChSm12, Remark 3.10], the functions $H^\bullet_\delta-m_\delta(2\epsilon)$ and $H^\circ_\delta-m_\delta(2\epsilon)$ are uniformly comparable at adjacent points inside $\varpi_k(2\epsilon)$. Taking into account (3.18), this implies $H^\bullet_\delta(z)\ge cM_\delta(\epsilon)$ for some absolute constant $c>0$ and all $z\in\gamma^\bullet$, where $\gamma^\bullet$ is the set of vertices adjacent to γ°. Further, the maximum principle gives

$$H^\bullet_\delta(z)\ge cM_\delta(\epsilon)\mathrm{hm}^\bullet_\gamma(z)+(1-\mathrm{hm}^\bullet_\gamma(z))m_\delta(2\epsilon)\quad\text{for } z:|z-a_k|\le 2\epsilon,$$

where $\mathrm{hm}_\gamma^\bullet$ denotes the discrete harmonic measure of γ in $\{z : |z - a_k| \le 2\epsilon\}$. It follows from the discrete Beurling estimate that $\mathrm{hm}_\gamma^\bullet(z) \ge \frac{1}{2}$, if z is chosen close enough (but at fixed distance independent of δ) to $z^{\max}$ where the path γ starts. For such z, one has $(M_\delta(\epsilon))^{-1}H_\delta^\bullet(z) \ge \frac{1}{2}c + o(1)$ as $\delta \to 0$, and hence the limit of $(M_\delta(\epsilon))^{-1}H_\delta(z)$ cannot be identically zero in $\Omega(\epsilon)$.

It remains to treat the case $z_\delta^{\max} \in \varpi_1(\epsilon)$. Consider the function

$$(M_\delta(\epsilon))^{-1}H_\delta^\dagger = (M_\delta(\epsilon))^{-1}\mathfrak{Re}\int((\vartheta(\delta))^{-1}(F_\delta(z) - F_{[\mathbb{C}_\delta,a_1]}(z)))^2dz.$$

Note that it tends to zero on $\varpi_1(2\epsilon)$, since one can neglect the term $F_{[\mathbb{C}_\delta,a_1]}(z)$ (recall that $(\vartheta(\delta))^{-1}F_{[\mathbb{C}_\delta,a_1]}(z) \to f_{[\mathbb{C},a_1]}$ and $M_\delta(\epsilon) \to \infty$). Therefore, by Remark 3.8 and the maximum principle, it also tends to zero in a neighborhood of $\varpi_1(\epsilon)$, so we consequently derive that each of the functions

$$\frac{F_\delta - F_{[\mathbb{C}_\delta,a_1]}}{(M_\delta(\epsilon))^{1/2}\vartheta(\delta)}, \quad \frac{F_\delta}{(M_\delta(\epsilon))^{1/2}\vartheta(\delta)} \quad \text{and} \quad \frac{H_\delta}{M_\delta(\epsilon)}$$

tends to zero uniformly on $\varpi_1(\epsilon)$. In particular, $1 = (M_\delta(\epsilon))^{-1}|H_\delta(z_\delta^{\max})| \to 0$, which is a contradiction. □

3.5. **Analysis near the singularities.** We now pass to the most delicate part of our analysis: matching the second-order terms in the values $F_\delta(a_1 + \frac{3\delta}{2})$ with the second coefficient in the expansion of the continuous spinor near a_1. For shortness, below we use the notation $a := a_1$, $F_\delta = F_{[\Omega_\delta,a,a_2,\dots,a_n]}$ and $\mathcal{A} = \mathcal{A}_{[\Omega,a,a_2,\dots,a_n]}$.

Proof of Theorem 2.17. Let $\mathcal{R}$ denote the reflection with respect to the horizontal line $\{x : \mathfrak{Im}\,(x - a) = 0\}$ and Λ_δ be a small neighborhood of a in $\Omega_\delta \cap \mathcal{R}\,(\Omega_\delta)$. Recall the notation $L_a = \{x + a + \frac{3\delta}{2} : x < 0\}$, and denote by $\Lambda_\delta^+ \subset \mathcal{V}_{\Lambda_\delta}^1$ one of two sheets of $[\Lambda_\delta, a] \setminus L_a$ such that $F_\delta(a + \frac{3\delta}{2}) > 0$ on Λ_δ^+. We define a *real-valued* function $S_\delta : \Lambda_\delta^+ \to \mathbb{R}$ by

$$S_\delta := (\vartheta\,(\delta))^{-1}(\tfrac{1}{2}(F_\delta + F_\delta^{(\mathcal{R})}) - F_{[\mathbb{C}_\delta,a]} - 2\mathfrak{Re}\,\mathcal{A}\cdot G_{[\mathbb{C}_\delta,a]}),$$

where $F_\delta^{(\mathcal{R})} = F_{[\mathcal{R}(\Omega_\delta),a,\mathcal{R}(a_2),\dots,\mathcal{R}(a_n)]}$ and the functions $F_{[\mathbb{C}_\delta,a]}$, $G_{[\mathbb{C}_\delta,a]}$ were constructed in Section 3.2. By symmetry, one has $F_\delta^{(\mathcal{R})}(a + \frac{3\delta}{2}) = F_\delta(a + \frac{3\delta}{2})$. Thus,

$$S_\delta(a + \tfrac{3\delta}{2}) = (\vartheta\,(\delta))^{-1}\left(F_\delta(a + \tfrac{3\delta}{2}) - 1 - 2\mathfrak{Re}\,\mathcal{A}\cdot\delta\right) \tag{3.19}$$

and our goal is to estimate this value. Note that S_δ vanishes on the cut L_a: both $F_{[\mathbb{C}_\delta,a]}$ and $G_{[\mathbb{C}_\delta,a]}$ vanish by construction, and $F_\delta^{(\mathcal{R})} = -F_\delta$ on L_a due to the spinor property (F_δ changes the sign between opposite sides of L_a, since they belong to different sheets). It is clear that S_δ is discrete harmonic everywhere in Λ_δ^+ except at the point $a + \frac{3\delta}{2}$ since all terms are discrete harmonic there. Moreover, due to Lemma 3.5, it is discrete harmonic at $a + \frac{3\delta}{2}$ also. Therefore, for any (small, but fixed) $r > 0$, discrete Beurling estimate (3.4) implies

$$|S_\delta(a + \tfrac{3\delta}{2})| \le C\delta^{\frac{1}{2}}\cdot r^{-\frac{1}{2}}\max_{\varpi(r)}|S_\delta|, \tag{3.20}$$

where $\varpi(r) := \{z : r \le |z - a| \le r + 5\delta\}$ denotes the "discrete circle" of radius r around a. Further, it follows from Theorem 2.15 and Lemmas 2.13, 2.16 that

$$S_\delta \underset{\delta\to 0}{\longrightarrow} s := \mathfrak{Re}\,[\tfrac{1}{2}(f + f^{(\mathcal{R})}) - f_{[\mathbb{C},a]}] - 2\mathfrak{Re}\,\mathcal{A}\cdot g_{[\mathbb{C},a]},$$

uniformly on $\varpi(r)$, where $f = f_{[\Omega,a_1,\dots,a_n]}$, $f^{(\mathcal{R})} = f_{[\mathcal{R}(\Omega),a,\mathcal{R}(a_2),\dots,\mathcal{R}(a_n)]}$, and $f_{[\mathbb{C},a]}(z) = \mathfrak{Re}\,[1/\sqrt{z-a}\,]$ and $g_{[\mathbb{C},a]}(z) = \mathfrak{Re}\,\sqrt{z-a}$. Recall that, by definition of the coefficient $\mathcal{A}$, one has

$$f - f_{[\mathbb{C},a]} = 2\mathcal{A}\sqrt{z-a} + O(|z-a|^{3/2}), \quad z \to a.$$

It is easy to check that $f^{(\mathcal{R})}(z) \equiv \overline{f(\mathcal{R}(z))}$ (since this spinor solves the corresponding boundary value problem), hence

$$f^{(\mathcal{R})} - f_{[\mathbb{C},a]} = 2\overline{\mathcal{A}}\sqrt{z-a} + O(|z-a|^{3/2}), \quad z \to a.$$

Thus, we arrive at $s(z) = O(|z-a|^{3/2})$ as $z \to a$, which means

$$r^{-\frac{1}{2}} \max_{\varpi(r)} |S_\delta| \underset{\delta\to 0}{\longrightarrow} r^{-\frac{1}{2}} \cdot O(r^{3/2}) = O(r). \tag{3.21}$$

Combining (3.19)–(3.21), one concludes that, for any given $r > 0$,

$$|F_\delta(a + \tfrac{3\delta}{2}) - 1 - 2\mathfrak{Re}\,\mathcal{A}\cdot\delta| \le C\vartheta(\delta)\delta^{\frac{1}{2}} r,$$

if δ is small enough. Since $\vartheta(\delta) = O(\delta^{\frac{1}{2}})$ by (2.17), and r can be chosen arbitrary small, this yields (2.19). All estimates are uniform with respect to $a_1, \dots, a_n$ at definite distance from the boundary and each other. □

To prove Theorem 2.19, we need to analyze the discrete spinor $F_{[\Omega_\delta,a,b]}$ near the point b. In contrast to Theorem 2.17, where second-order information near a was extracted, here we only need to match the first-order coefficients. Note that the situation is slightly different from the first-order analysis near a, as $F_{[\Omega_\delta,a,b]}$ does not have an explicit discrete singularity, remaining s-holomorphic near the branching point b, while its limit blows up at b. Still, the strategy for the identification resembles the one used above and appeals to the symmetrization with respect to the horizontal line passing through b. Let $L'_b := \{x + b + \frac{\delta}{2} : x < 0\}$.

Proof of Theorem 2.19. Let $\mathcal{R}$ denote the reflection with respect to the line $\{x : \mathfrak{Im}(x-b) = 0\}$ and $\Lambda_\delta \subset \Omega_\delta \cap \mathcal{R}(\Omega_\delta)$ be a small neighborhood of b. Fix the sheet $\Lambda_\delta^+ \subset \mathcal{V}^i_{\Lambda_\delta}$ of $[\Omega_\delta, a, b] \setminus L'_b$ so that $F_\delta(b + \frac{\delta}{2}) = i\mathcal{B}_\delta$ with $\mathcal{B}_\delta > 0$ (recall that the values of discrete spinors on $\mathcal{V}^i_{\mathbb{C}_\delta}$ are purely imaginary), and do the same for the continuous spinor: fix a sheet so that $\mathfrak{Im}\, f_{[\Omega,a,b]}(b + x) > 0$ as $x \downarrow 0$ (see (2.13)).

Let $W_\delta := \mathrm{hm}_{\{b+\frac{\delta}{2}\}}(\,\cdot\,)$ denote the harmonic measure of the point $b + \frac{\delta}{2}$ in the slit discrete plane $\mathbb{C}^i_\delta \setminus L'_b$ and

$$T_\delta := (\vartheta(\delta))^{-1}(\tfrac{1}{2}(F_\delta + F^{(\mathcal{R})}_\delta) - i\mathcal{B}_\delta \cdot W_\delta) \ : \Lambda^+_\delta \to i\mathbb{R},$$

where $F_\delta = F_{[\Omega_\delta,a,b]}$ and $F^{(\mathcal{R})}_\delta = F_{[\mathcal{R}(\Omega),\mathcal{R}(a),b]}$. By symmetry, $F^{(\mathcal{R})}_\delta(b + \frac{\delta}{2}) = i\mathcal{B}_\delta$ (if one fixes the sheet for $F^{(\mathcal{R})}_\delta$ by the same condition $\mathfrak{Im}\, F^{(\mathcal{R})}_\delta(b + \frac{\delta}{2}) > 0$). We have to prove that $\mathcal{B}_\delta \to \mathcal{B} = \mathcal{B}_{[\Omega,a,b]}$ as $\delta \to 0$. Note that, by passing to a subsequence, one may assume that $\mathcal{B}_\delta \to \widetilde{\mathcal{B}}$ for some $\widetilde{\mathcal{B}} \in [0, +\infty]$.

Suppose that $\widetilde{\mathcal{B}}$ is finite. Then, for any fixed (small) $r > 0$, Theorem 2.15 and Lemma 3.4(iii) imply the uniform convergence

$$T_\delta(z) \underset{\delta\to 0}{\longrightarrow} t(z) := i \cdot \mathfrak{Im}\,[\tfrac{1}{2}(f(z) + f^{(\mathcal{R})}(z))] - i\widetilde{\mathcal{B}} \cdot \mathfrak{Re}\,[1/\sqrt{z-b}\,] \tag{3.22}$$

on compact subsets of $\{z : 0 < |z-b| \le r\}$, where $f = f_{[\Omega,a,b]}$ and $f^{(\mathcal{R})}(z) = f_{[\mathcal{R}(\Omega),\mathcal{R}(a),b]}(z) = -\overline{f(\mathcal{R}(z))}$ due to our conventions about sheets. Note that

$$t(z) = i(\mathcal{B}-\widetilde{\mathcal{B}})\cdot \mathfrak{Re}\,[1/\sqrt{z-b}\,] + O(|z-b|^{1/2}), \quad z\to b. \tag{3.23}$$

Clearly, the function T_δ is harmonic everywhere in Λ_δ^+ except at the point $b+\frac{\delta}{2}$, and $T_\delta = 0$ on L_b': indeed, $F_\delta^{(\mathcal{R})} = -F_\delta$ on L_b' due to the spinor property of F_δ and symmetry reasons. Moreover, $T_\delta(b+\frac{\delta}{2}) = 0$ as $W_\delta(b+\frac{\delta}{2}) = 1$. Therefore, $\widetilde{\mathcal{B}} \ne \mathcal{B}$ would contradict the maximum principle for the discrete harmonic function T_δ: in this case (3.22) and (3.23) imply that, for sufficiently small δ, the values of T_δ near b are bigger than those near the circle $\{z : |z-b| = r\}$.

The similar argument works, if $\widetilde{\mathcal{B}} = +\infty$: in this case one would have

$$\mathcal{B}_\delta^{-1}T_\delta(z) \underset{\delta\to 0}{\longrightarrow} -i\cdot\mathfrak{Re}\,[1/\sqrt{z-b}\,]$$

which contradicts to the maximum principle for T_δ, just as before. $\square$

A. Appendix. Spinors in the half-plane and decorrelations identities

In this Appendix we explicitly construct the continuous spinor which satisfies Definition 2.8 in the half-plane. Recall that, along with (2.7)–(2.9), one should also impose the condition $f_{[\mathbb{H},a_1,\dots,a_n]}(z) = O(|z|^{-1})$ at infinity, so that after a conformal mapping to a bounded domain with smooth boundary the spinor remains bounded. We argue that the construction of $f_{[\mathbb{H},a_1,\dots,a_n]}$ boils down to a system of n linear equations, which is always non-degenerate. Consequently, we derive the formulae for $\mathcal{A}_{[\mathbb{H},a_1,\dots,a_n]}$ and, by integration, the continuous correlation functions. Although for small n it is possible to check by ugly brute force computations that the functions we get coincide with those predicted by the Conformal Field Theory arguments (recall that the computations of 1-point and 2-point functions were done in Section 2.7), we prove this coincidence only for some particular configurations when all $a_1,\dots,a_n$ are on the imaginary axis. This turns out to be sufficient to prove Lemma 2.22.

A.1. **The explicit formulae for spinors.** Let us introduce some notation. For $a\in\mathbb{H}$, denote

$$p_a(z) := (z-a)(z-\overline{a}).$$

Given $a_1,\dots,a_n\in\mathbb{H}$, we look for $f_{[\mathbb{H},a_1,\dots,a_n]}$ in the following form:

$$f_Q(z) := e^{\frac{\pi i}{4}}\cdot\frac{Q(z)}{\sqrt{p_{a_1}(z)\dots p_{a_n}(z)}},$$

where $Q(z) = \sum_{s=0}^{n-1} q_s z^s$ is a polynomial of degree $n-1$ with *real* coefficients. Clearly, f_Q is a holomorphic spinor on $[\mathbb{H},a_1,\dots,a_n]$ which satisfies the boundary condition (2.7), hence we have to determine the coefficients $q_0,q_1,\dots,q_{n-1}$ from (2.8) and (2.9). Since $e^{\frac{\pi i}{4}}/\sqrt{p_{a_k}(z)} = 1/\sqrt{(2\mathfrak{Im}\,a_k)(z-a_k)} + O(|z-a_k|^{\frac12})$ as $z\to a_k$, these conditions imply

$$\begin{aligned} \sum_{s=1}^{n} M_{1,s}(a_1,\dots,a_n)q_{s-1} &= \sqrt{2\mathfrak{Im}\,a_1}, \\ \sum_{s=1}^{n} M_{k,s}(a_1,\dots,a_n)q_{s-1} &= 0, \quad 2\le k\le n, \end{aligned} \tag{A.1}$$

where

$$M_{k,s}=M_{k,s}(a_1,\dots,a_n)\ :=\ \mathfrak{Re}\left[a_k^{s-1}\cdot\left[\prod_{l\neq k}p_{a_l}(a_k)\right]^{-\frac{1}{2}}\right],\quad 1\le k,s\le n.$$

Once this linear system is non-degenerate, the Cramer's rule gives, for $1\le s\le n$,

$$q_{s-1}=\sqrt{2\mathfrak{Im}\,a_1}\cdot(-1)^{s-1}\det\left[M_{k,m}\right]_{k\neq 1,m\neq s}\cdot D^{-1},\quad D=\det\left[M_{k,m}\right]_{k,m=1}^{n}. \tag{A.2}$$

Proposition A.1. *For any n pairwise distinct points $a_1,\dots,a_n\in\mathbb{H}$, the linear system (A.1) has a unique solution which is given by (A.2). The spinor*

$$f_{[\mathbb{C},a_1,\dots,a_n]}(z):=e^{\frac{\pi i}{4}}\cdot\frac{Q_{a_1,\dots,a_n}(z)}{\sqrt{p_{a_1}(z)\dots p_{a_n}(z)}},\qquad Q_{a_1,\dots,a_n}(z):=\sum_{s=0}^{n-1}q_sz^s, \tag{A.3}$$

solves the boundary value problem (2.7)–(2.9) and satisfies the additional regularity condition $f_{[\mathbb{C},a_1,\dots,a_n]}(z)=O(|z|^{-1})$ at infinity. Moreover, all the spinors $f_{[\mathbb{C},a_1,\dots,a_n]}(z)$ are uniformly bounded, if z is ϵ-away from $a_1,\dots,a_n$, and

$$\mathcal{A}_{\mathbb{H}}(a_1,\dots,a_n)=-\frac{1}{4(a_1-\overline{a}_1)}-\frac{1}{4}\sum_{k=2}^{n}\left[\frac{1}{a_1-a_k}+\frac{1}{a_1-\overline{a}_k}\right]+\left.\frac{\partial_zQ_{a_1,\dots,a_n}(z)}{2Q_{a_1,\dots,a_n}(z)}\right|_{z=a_1}. \tag{A.4}$$

Proof. On the contrary, let us suppose that the system (A.1) is degenerate. Then, a nontrivial solution to its homogeneous counterpart gives rise to a spinor f_Q that satisfies (2.7) and (2.9), as well as $\mathfrak{Re}\,[\lim_{z\to a_1}\sqrt{z-a_1}f_Q(z)]=0$, and thus vanishes identically, since

$$0\le i^{-1}\int_{\mathbb{R}}(f_Q(z))^2dz=2\pi\sum_{k=1}^{n}\lim_{z\to a_k}(z-a_k)(f_Q(z))^2\le 0, \tag{A.5}$$

which is a contradiction. Hence, (A.1) is non-degenerate and has a solution (A.2). Let $f_{[\mathbb{C},a_1,\dots,a_n]}$ be given by (A.3), thus satisfying the boundary conditions (2.7), and denote

$$\beta_k=\beta_k(a_1,\dots,a_n):=\lim\nolimits_{z\to a_k}\sqrt{z-a_k}\,f_{[\mathbb{C},a_1,\dots,a_n]}(z),\quad 1\le k\le n. \tag{A.6}$$

Note that the linear system (A.1) means $\mathfrak{Re}\,\beta_1=1$ and $\mathfrak{Re}\,\beta_k=0$ for $2\le k\le n$, in particular the condition (2.9) is satisfied. The contour integration argument (A.5) ensures that

$$\beta_1^2-(|\beta_2|^2+\dots+|\beta_n|^2)\ge 0.$$

Thus, β_1 should be purely real (and so $\beta_1=1$), i.e. the condition (2.8) is satisfied too. Moreover, one has

$$h_{[\mathbb{C},a_1,\dots,a_n]}(z):=\mathfrak{Re}\int(f_{[\mathbb{C},a_1,\dots,a_n]}(z))^2dz=G_{\mathbb{H}}(z;a_1)-\sum_{k=2}^{n}|\beta_k|^2G_{\mathbb{H}}(z;a_k),$$

where $G_{\mathbb{H}}(z;a)$ denotes the Green's function in the half-plane. Thus, all spinors

$$(f_{[\mathbb{C},a_1,\dots,a_n]}(z))^2=2\left[\partial_zG_{\mathbb{H}}(z;a_1)-\sum_{k=2}^{n}|\beta_k|^2\partial_zG_{\mathbb{H}}(z;a_k)\right]$$

are uniformly bounded, since the gradients of the Green's functions are uniformly bounded if z is ϵ-away from the singularities, and $|\beta_k|^2\le\beta_1^2=1$. Finally, the formula (A.4) readily follows from (A.3) and (2.8). ☐

A.2. **Conformal Field Theory predictions and the case $a_1,\ldots,a_n \in i\mathbb{R}_+$.** It was predicted by the methods of Conformal Field Theory [BuGu93] that in the continuum limit, the spin correlation functions take the following form:

$$\langle \sigma_{a_1}\ldots\sigma_{a_n}\rangle^+_{\mathbb{H},\mathrm{CFT}} = \prod_{k=1}^{n} \frac{1}{(2\Im\mathfrak{m}\, a_k)^{\frac18}} \left[2^{-\frac{n}{2}} \sum_{\mu=\pm1} \prod_{s<m} \left| \frac{a_s-a_m}{a_s-\overline{a}_m}\right|^{\frac{\mu_s\mu_m}{2}} \right]^{\frac12}, \qquad \text{(A.7)}$$

here and below we use a shorthand $\mu=\pm1$ instead of $\mu_1=\pm1,\ldots,\mu_n=\pm1$, so the sum $\sum_{\mu=\pm1}$ contains 2^n terms. In particular, this sum equals 2, if $n=1$.

Remark A.2. Recall that we have considered two simplest cases $n=1$ and $n=2$ in Section 2.7. Namely, we found the explicit solutions (2.21) and (2.22) to the boundary value problem (2.7)–(2.9), computed their asymptotics near a_1 and obtained the corresponding coefficients $\mathcal{A}_{\mathbb{H}}(a_1)$ and $\mathcal{A}_{\mathbb{H}}(a_1,a_2)$. Then, a direct check shows that those coefficients coincide with the logarithmic derivatives of the quantities (A.7), thus establishing this Conformal Field Theory prediction. Unfortunately, this procedure becomes much harder as n grows. Having in hand explicit formulae given in the previous Section, one *can* compute $\mathcal{A}_{\mathbb{H}}(a_1,\ldots,a_n)$ and integrate the answer, but it is not easy to *check* whether it coincides with (A.7) or not. Note that the case $n=3$ is still doable by hand. In this case, for a proper real constant C, it is easy to check that the spinor

$$f_{[\mathbb{H},a_1,a_2,a_3]}(z) := Ce^{\frac{\pi i}{4}} \cdot \frac{c_{13}p_{a_2}(z)+c_{12}p_{a_3}(z)-c_{23}p_{a_1}(z)}{\sqrt{p_{a_1}(z)p_{a_2}(z)p_{a_3}(z)}},$$

where $c_{sm}=|a_s-a_m||a_s-\overline{a}_m|$, satisfies the conditions (2.7)–(2.9), which allows us to verify the CFT prediction (A.7) for the 3-point function by brute force computations. In the next lemma we prove (A.7) for all n in the special case when all points are purely imaginary, which is sufficient for our purposes.

Lemma A.3. *If $a_k=iw_k\in i\mathbb{R}_+$ for all $k=1,\ldots,n$, then*

$$\partial_{w_1}\log\langle \sigma_{a_1}\ldots\sigma_{a_n}\rangle^+_{\mathbb{H},\mathrm{CFT}} = i\mathcal{A}_{\mathbb{H}}(a_1,\ldots,a_n).$$

Proof. One can rewrite (A.7) in the following form:

$$\langle \sigma_{a_1}\ldots\sigma_{a_n}\rangle^+_{\mathbb{H},\mathrm{CFT}} = 2^{-\frac{3n}{8}} \prod_{k=1}^{n} (\Im\mathfrak{m}\, a_k)^{-\frac18} \prod_{s<m} |a_s-a_m|^{-\frac14}\, |a_s-\overline{a}_m|^{-\frac14}$$
$$\times \left[\sum_{\mu=\pm1} \prod_{s<m} |a_s-a_m|^{\frac{1+\mu_s\mu_m}{2}} |a_s-\overline{a}_m|^{\frac{1-\mu_s\mu_m}{2}} \right]^{\frac12}$$

Observe that the logarithmic derivative of the first factor exactly matches the first two terms of (A.4). Hence it suffices to show that

$$\partial_{w_1}\log\left[\sum_{\mu=\pm1}\prod_{s<m} |w_s-w_m|^{\frac{1+\mu_s\mu_m}{2}} |w_s+w_m|^{\frac{1-\mu_s\mu_m}{2}}\right]$$
$$= i\partial_z \log Q_{a_1,\ldots,a_n}(z)\big|_{z=a_1} = \partial_w \log Q_{a_1,\ldots,a_n}(iw)\big|_{w=w_1}. \qquad \text{(A.8)}$$

Let, say, $w_1 < \dots < w_n$, the other cases are treated by similar computations. Then,

$$\begin{aligned}\sum_{\mu=\pm1}\prod_{s<m}|w_s-w_m|^{\frac{1+\mu_s\mu_m}{2}}|w_s+w_m|^{\frac{1-\mu_s\mu_m}{2}} &= \sum_{\mu=\pm1}\prod_{s<m}(w_m-\mu_s\mu_m w_s)\\ = \sum_{\mu=\pm1}\prod_{k=1}^n\mu_k^{k-1}\cdot\prod_{s<m}(\mu_m w_m-\mu_s w_s) &= \sum_{\mu=\pm1}\prod_{k=1}^n\mu_k^{k-1}\cdot\det[\,(\mu_k w_k)^{m-1}\,]_{m,k=1}^n\\ = \sum_{\mu=\pm1}\det[\,\mu_k^{m+k-2}w_k^{m-1}\,]_{m,k=1}^n &= \det[\,(1+(-1)^{m+k-2})w_k^{m-1}\,]_{m,k=1}^n.\end{aligned} \tag{A.9}$$

On the other hand, $p_{a_l}(a_k)=\operatorname{sign}(l-k)\cdot|w_l-w_k||w_l+w_k|$, thus

$$M_{k,m}=\mathfrak{Re}\,\left[(iw_k)^{m-1}\cdot i^{k-1}R_k\right],\quad\text{where}\quad R_k=\left[\prod_{l\neq k}|w_l-w_k||w_l+w_k|\right]^{-\frac12}$$

are real and do not depend on m. Note that $\mathfrak{Re}\,[i^{m+k-2}]=\frac12(1+(-1)^{m+k-2})i^{m+k-2}$. Therefore, (A.2) reads

$$\begin{aligned}q_{s-1} &= \sqrt{2w_1}\,D^{-1}\cdot(-1)^{s-1}\det[\,M_{k,m}\,]_{k\neq1,m\neq s}\\ &= C\cdot(-1)^{s-1}\det\left[\,i^{m+k-2}(1+(-1)^{m+k-2})w_k^{m-1}\,\right]_{k\neq1,m\neq s}\\ &= C\cdot(-1)^{s-1}i^{n(n-1)-(s-1)}\det\left[\,(1+(-1)^{m+k-2})w_k^{m-1}\,\right]_{k\neq1,m\neq s}\\ &= (-1)^{n(n-1)/2}C\cdot i^{s-1}\det\left[\,(1+(-1)^{m+k-2})w_k^{m-1}\,\right]_{k\neq1,m\neq s},\end{aligned}$$

where $C=\sqrt{2w_1}D^{-1}2^{-n+1}R_2\dots R_n$, and

$$Q_{a_1,\dots,a_n}(iw)=\pm C\cdot\sum_{s=1}^n(-w)^{s-1}\det\left[\,(1+(-1)^{m+k-2})w_k^{m-1}\,\right]_{k\neq1,m\neq s}. \tag{A.10}$$

Note that $\det\left[\,(1+(-1)^{m+k})x_{k,m}\,\right]_{k\neq1,m\neq s}=0$ for all even s and any $x_{k,m}$. Thus, the sum in (A.10) is actually the same determinant as (A.9) with w_1 replaced by w, while the prefactor $\pm C$ may depend on $w_1,\dots,w_n$ but not on w, and thus does not affect the logarithmic derivative with respect to w. Hence, (A.8) follows. □

A.3. **Decorrelation identities and proof of Lemma 2.22.** We start by proving that the half-plane spinors $f_{\mathbb{H},a_1,\dots,a_n}$ behave in a continuous fashion, when one of the points $a_2,\dots,a_n$ approaches the real line and "disappears" there.

Lemma A.4. *For $n\geq2$ and $\mathfrak{Im}\,a_n\to0$, one has $f_{[\mathbb{H},a_1,\dots,a_n]}(z)\to f_{[\mathbb{H},a_1,\dots,a_{n-1}]}(z)$ and $\mathcal{A}_{[\Omega,a_1,\dots,a_n]}\to\mathcal{A}_{[\Omega,a_1,\dots,a_{n-1}]}$, uniformly with respect to the positions of points $a_1,\dots,a_{n-1},z$, provided they are at least ϵ-away from the boundary and each other.*

Proof. This can be easily proven by compactness arguments, but we prefer to give an explicit construction which also allows one to estimate the convergence rate, if necessarily. Denote $f(z):=f_{[\mathbb{H},a_1,\dots,a_n]}(z)$, $f^-(z):=f_{[\mathbb{H},a_1,\dots,a_{n-1}]}(z)$ and let

$$g:=f^-(z)\cdot r_{a_n}(z),\quad\text{where}\quad r_{a_n}(z):=\frac{z-\mathfrak{Re}\,a_n}{\sqrt{(z-a_n)(z-\overline{a}_n)}}\,.$$

By definition, g satisfies the spinor property on $[\mathbb{C},a_1,\dots,a_n]$, the boundary conditions (2.7) and $g(z)=O(|z|^{-1})$ at infinity, but the conditions (2.8), (2.9) fail since the values $r_{a_n}(a_k)$ are nonreal. Let $\beta_k^- = \beta_k(a_1,\dots,a_{n-1})$ be defined by (A.6), and

$$\begin{aligned} \beta_1 :=&\ \lim_{z\to a_1}\sqrt{z-a_1}\,g(z) = \beta_1^-\cdot r_{a_n}(a_1) = r_{a_n}(a_1),\\ \beta_k :=&\ \lim_{z\to a_k}\sqrt{z-a_k}\,g(z) = \beta_k^-\cdot r_{a_n}(a_k), \quad 2\le k\le n-1,\\ \beta_n :=&\ \lim_{z\to a_n}\sqrt{z-a_n}\,g(z) = f^-(a_n)\cdot(\tfrac12\Im\mathfrak{m}\,a_n)^{\frac12}, \end{aligned}$$

Note that g can be represented as

$$g = \sum_{k=1}^n \Re\mathfrak{e}\,\beta_k\cdot f_{[\mathbb{H},a_k,a_1,\dots,\hat{a}_k,\dots,a_n]}.$$

Indeed, the difference of two sides solves the homogeneous version of the boundary value problem (2.7)–(2.9) and hence is identically zero. Therefore,

$$g - f = (\Re\mathfrak{e}\,\beta_1 - 1)f + \sum_{k=2}^n \Re\mathfrak{e}\,\beta_k\cdot f_{[\mathbb{H},a_k,a_1,\dots,\hat{a}_k,\dots,a_n]}.$$

It is easy to see that $\beta_1 = r_{a_n}(a_1)\to 1$, $\Re\mathfrak{e}\,\beta_k = -\Im\mathfrak{m}\,\beta_k^-\,\Im\mathfrak{m}\,r_{a_n}(a_k)\to 0$ and $\beta_n\to 0$ as $\Im\mathfrak{m}\,a_n\to 0$, and all the spinors in the right-hand side are uniformly bounded due to Proposition A.1. Thus, we conclude that

$$f(z) - f^-(z) = [f(z)-g(z)] - [1-r_{a_n}(z)]f^-(z)\to 0$$

since both terms vanish as $\Im\mathfrak{m}\,a_n\to 0$. The convergence of $\mathcal{A}_{[\Omega,a_1,\dots,a_n]}$ follows by the Cauchy integral formula. □

Now we are in the position to prove Lemma 2.22.

Proof of Lemma 2.22. We first define the correlation functions $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_{\mathbb{H}}^+$ in the half-plane. For $n=1,2$ we use (1.4), and we define them to be $\exp[\int\mathcal{L}_{\Omega,n}]$ in the general case, where the differential form $\mathcal{L}_{\Omega,n}$ is given by (2.24). Note that this form is symmetric with respect to $a_1,\dots,a_n$ and exact on the manifold $\widetilde{\Omega}^n$ of all n-tuples of distinct points in Ω due to Proposition 2.20. Therefore, the function $\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_{\mathbb{H}}^+$ is symmetric with respect to $a_1,\dots,a_n$ and well-defined on $\widetilde{\Omega}^n$ up to a multiplicative constant which we fix so that

$$\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_{\mathbb{H}}^+ = \langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_{\mathbb{H},\mathrm{CFT}}^+ \quad\text{for}\quad a_1,\dots,a_n\in i\mathbb{R}_+. \tag{A.11}$$

As it was established in Lemma A.3, this normalization (everywhere on the imaginary line) does not contradict our definition of the correlation function as an exponent of the primitive of $\mathcal{L}_{\Omega,n}$. Moreover, (A.11) holds true for any $a_1,\dots,a_n$ lying on the same vertical line: indeed, the multiplicative normalization cannot depend on the x-coordinate of this line, since Lemma A.3 guarantees $\Re\mathfrak{e}\,\mathcal{A}_{\mathbb{H}}(a_1\dots a_n)=0$ for such configurations.

Let $a_1,\dots,a_n\in\mathbb{H}$, $\Im\mathfrak{m}\,a_1\to 0$, and $b_k := \Re\mathfrak{e}\,a_1 + ik$ for $k=2,\dots,n$. Write

$$\frac{\langle\sigma_{a_1}\dots\sigma_{a_n}\rangle_{\mathbb{H}}^+}{\langle\sigma_{a_1}\rangle_{\mathbb{H}}^+\langle\sigma_{a_2}\dots\sigma_{a_n}\rangle_{\mathbb{H}}^+} = \frac{\langle\sigma_{a_1}\sigma_{b_2}\dots\sigma_{b_n}\rangle_{\mathbb{H}}^+}{\langle\sigma_{a_1}\rangle_{\mathbb{H}}^+\langle\sigma_{b_2}\dots\sigma_{b_n}\rangle_{\mathbb{H}}^+}\cdot\frac{\langle\sigma_{a_1}\sigma_{a_2}\dots\sigma_{a_n}\rangle_{\mathbb{H}}^+}{\langle\sigma_{a_1}\sigma_{b_2}\dots\sigma_{b_n}\rangle_{\mathbb{H}}^+}\cdot\frac{\langle\sigma_{b_2}\dots\sigma_{b_n}\rangle_{\mathbb{H}}^+}{\langle\sigma_{a_2}\dots\sigma_{a_n}\rangle_{\mathbb{H}}^+}$$

Since $a_1,b_2,\dots,b_n$ are on the same vertical line, we know from the explicit formulae (A.7) that the first term tends to 1 as $\Im\mathfrak{m}\,a_1\to 0$. Also, we have

$$\log\left[\frac{\langle\sigma_{a_1}\sigma_{a_2}\dots\sigma_{a_n}\rangle_{\mathbb{H}}^+}{\langle\sigma_{a_1}\sigma_{b_2}\dots\sigma_{b_n}\rangle_{\mathbb{H}}^+}\cdot\frac{\langle\sigma_{b_2}\dots\sigma_{b_n}\rangle_{\mathbb{H}}^+}{\langle\sigma_{a_2}\dots\sigma_{a_n}\rangle_{\mathbb{H}}^+}\right] = \int_{(a_1,b_2,\dots,b_n)}^{(a_1,a_2,\dots,a_n)}\mathcal{L}_{\Omega,n} - \int_{(b_2,\dots,b_n)}^{(a_2,\dots,a_n)}\mathcal{L}_{\Omega,n-1}.$$

According to definition (2.24), the differential forms $\mathcal{L}_{\Omega,n}$ and $\mathcal{L}_{\Omega,n-1}$ consist of the terms $\mathfrak{Re}\,[\mathcal{A}_\Omega(z_k, z_2, ..., \hat{z}_k, ..., z_n, a_1)dz_k]$ and $\mathfrak{Re}\,[\mathcal{A}_\Omega(z_k; z_2, ..., \hat{z}_k, ..., z_n)dz_k]$, respectively, which are uniformly close to each other for all $k = 2, \ldots, n$ due to Lemma A.4, as $(z_2, \ldots, z_n)$ runs from $(a_2, \ldots, a_n)$ to $(b_2, \ldots, b_n)$ and $\mathfrak{Im}\, a_1 \to 0$. Note that, since the convergence in Lemma A.4 is uniform, it does not matter whether a_1 approaches $\partial\mathbb{H}$ along a fixed vertical line or not.

Thus, we have defined the half-plane functions $\langle\sigma_{a_1} \ldots \sigma_{a_n}\rangle^+_{\mathbb{H}}$ so that decorrelation identities (2.28) are fulfilled. We then define the correlation functions $\langle\sigma_{a_1} \ldots \sigma_{a_n}\rangle^+_\Omega$ in a given bounded domain Ω using a conformal map $\varphi : \Omega \to \mathbb{H}$ and the covariance rule (1.3). The computation given in Remark 2.21 shows that, being defined in this way, $\log\langle\sigma_{a_1} \ldots \sigma_{a_n}\rangle^+_\Omega$ is indeed a primitive of the form $\mathcal{L}_{\Omega,n}$. Since the covariance rule (1.3) implies the conformal invariance of the ratios $\langle\sigma_{a_1} \ldots \sigma_{a_n}\rangle^+_\Omega / \langle\sigma_{a_1}\rangle^+_\Omega \langle\sigma_{a_2} \ldots \sigma_{a_n}\rangle^+_\Omega$, one obtains decorrelation identities (2.28) as $a_1 \to \partial\Omega$ (which means $\varphi(a_1) \to \partial\mathbb{H}$). The second part of (2.28) was proven in Remark 2.23.

It is easy to see that conditions (2.28) fix the unknown multiplicative normalization of the correlation functions uniquely: the full-plane normalization fixes the two-point function and decorellation identities inductively fix all others. Therefore, the functions $\langle\sigma_{a_1} \ldots \sigma_{a_n}\rangle^+_\Omega$ constructed above do not depend on a particular choice of the conformal mapping φ. □

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