ON BOUSFIELD PROBLEM FOR THE CLASS OF METABELIAN GROUPS

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Abstract. The homological properties of localizations and completions of metabelian groups are studied. It is shown that, for $R = \mathbb{Q}$ or $R = \mathbb{Z}/n$ and a finitely presented metabelian group G, the natural map from G to its R-completion induces an epimorphism of homology groups $H_2(-, R)$. This answers a problem of A.K. Bousfield for the class of metabelian groups.

1. Introduction

The subject of investigation of this paper is the relation between inverse limits of groups and the second homology $H_2(-, K)$ for certain coefficients K. One of the results of the paper is the following. Let G be a finitely presented metabelian group, $\{\gamma_i(G)\}_{i\geq 0}$ its lower central series, then, for any $n > 0$, there is a natural isomorphism

$$H_2(\varprojlim\ G/\gamma_i(G), \mathbb{Z}/n) \simeq \varprojlim\ H_2(G/\gamma_i(G), \mathbb{Z}/n).$$

That is, in this particular case, the inverse limit commutes with the second homology functor.

The problem of relation between inverse limit and second homology of groups appears in different areas of algebra and topology. Recall two related open problems, one from [6], the second from [11]. A.K. Bousfield posed the following question in [6], Problem 4.10:

Problem. *(Bousfield)* Is $E^R X \to \hat{X}_R$ iso when X is a finitely presented group when $R = \mathbb{Q}$ or $R = \mathbb{Z}/n$?

In the above problem, E^R is the HR-localization functor defined in [6] and $\hat{X}_R$ is the R-completion of the group X. It follows immediately from the construction of E^R, that, for a finitely presented group X, the map $E^R X \to \hat{X}_R$ is isomorphism if and only if the completion map $X \to \hat{X}_R$ induces an epimorphism $H_2(X; R) \to H_2(\hat{X}_R, R)$. In this paper we prove the following (see Corollary 9.2)

Theorem. For a finitely presented metabelian group X, the natural map $E^R X \to \hat{X}_R$ is an isomorphism for $R = \mathbb{Q}$ or $R = \mathbb{Z}/n$.

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Observe that, the above result can not be generalized to the case $R = \mathbb{Z}$ as the following simple example shows. For the Klein bottle group $G = \langle a, b \mid a^{-1}bab = 1\rangle$, the second homology $H_2(\hat{G}, \mathbb{Z})$ is isomorphic to the exterior square of the 2-adic integers and therefore is uncountable (see [6]).

The second problem of a similar type is the question of comparison between discrete and continuous homology of pro-p-groups. For any pro-p-group P, one can look at homology of P in two different ways: as homology of a discrete group and as homology of a topological group. There are natural comparison maps between these homology groups (see [11] for detailed discussion):

$$\phi_n : H_n^{discrete}(P, \mathbb{Z}/p) \to H_n^{cont}(P, \mathbb{Z}/p)$$

Analogously, there are maps between cohomology groups $\phi^n : H^n_{cont}(P, \mathbb{Z}/p) \to H^n_{discrete}(P, \mathbb{Z}/p)$. In [11], G.A. Fernandez-Alcober, I.V. Kazachkov, V.N. Remeslennikov, and P. Symonds asked the following question.

Problem. Does there exist a finitely presented pro-p group P for which $\phi^2 : H^2_{\text{cont}}(P, \mathbb{Z}/p) \to H^2_{\text{discrete}}(P, \mathbb{Z}/p)$ is not an isomorphism?

See [20] for background on the continuous cohomology of pro-p groups. It is shown in [11] that, for a finitely presented pro-p-group P, the following two conditions are equivalent:

(1) the map $\phi^2 : H^2_{\text{cont}}(P, \mathbb{Z}/p) \to H^2_{\text{discrete}}(P, \mathbb{Z}/p)$ is an isomorphism;

(2) the map $\phi_2 : H_2^{\text{discrete}}(P, \mathbb{Z}/p) \to H_2^{\text{cont}}(P, \mathbb{Z}/p)$ is an isomorphism.

Our contribution to this problem is the following: for a finitely presented metabelian group G, there is a natural isomorphism (see Corollary 8.2)

$$H_2^{\text{discrete}}(\hat{G}_p, \mathbb{Z}/p) \xrightarrow{\cong} H_2^{\text{cont}}(\hat{G}_p, \mathbb{Z}/p)$$

Observe that, it is not possible to extend the above results to the class of all finitely generated metabelian groups. For example, one can consider the p-Lamplighter group

$$L_p \coloneqq \mathbb{Z}/p \wr \mathbb{Z} = \langle a, b \mid a^p = [a, a^{b^i}] = 1,\ i \in \mathbb{Z}\rangle$$

It is easy to check that the second homology of the pro-p-completion $H_2(\hat{L}_p, \mathbb{Z}/p)$ is an uncountable $\mathbb{Z}/p$-vector space. Despite the fact that all finitely generated metabelian groups are subgroups of finitely presented metabelian groups (see [1] and [19]), the finite presentability is a crucial point in the results like Corollary 8.2 and Corollary 9.2.

Let G be a finitely generated metabelian group with a metabelian decomposition $M \rightarrowtail G \twoheadrightarrow A$. The group G is finitely presented if and only if the $\mathbb{Z}[A]$-module M is tame in the sense of Bieri-Strebel [4]. Tame modules are characterized via geometric properties (see section 7) and certain group-theoretic properties of finitely generated metabelian groups can be formulated in the language of commutative algebra. In this paper, the properties of tame modules are used for investigation of homological behavior of R-completions.

The paper is organized as follows. In section 2 we recall the general properties of inverse limits of groups as well as the properties of their derived functors. In section 3 we recall the definitions and properties of I-adic completions of modules and describe the structure of a *twisted exterior square* of a completed and localized module. The structure of the twisted exterior square plays a central role in the study of the second homology of the completed and localized metabelian groups. Section 4 is about R-completions of finitely generated metabelian groups, where R is either a subring of rationals or a finite ring $\mathbb{Z}/n$. A natural way to compare homology of a metabelian group with homology of its R-completion it to consider the standard homology spectral sequences for corresponding metabelian decompositions. Sections 5,6 contain technical properties of functors which appear in these spectral sequences. Section 7 is a technical section, where the finite presentability appears. The properties of exterior squares of tame modules in the sense of Bieri-Strebel play a key role in the whole picture (see Proposition 7.2). The main results of the paper are essentially based on the fact that one can control the $E^2_{0,2}$-term of the standard homology spectral sequence for a metabelian decomposition of R-completions.

The main results of the paper are theorems 8.1 and 9.1. These theorems are proved in sections 8 and 9. Theorem 8.1 is formulated as follows. Let G be a finitely presented metabelian group, $R = \mathbb{Z}[J^{-1}]$ or $R = \mathbb{Z}/n$ and K be an Artinian quotient ring of R. Then the R-lower central quotient maps induce the isomorphisms

$$H_2(\hat{G}_R, K) \cong \varprojlim H_2(G/\gamma_i^R(G), K).$$

As a simple corollary of this theorem, we get the natural isomorphism between discrete and continuous homology groups of pro-p-completions of finitely presented groups, which gives an answer to a particular case of the problem from [11] mentioned above. Theorem 9.1 is the following. Assuming that G, R and K are as in theorem 8.1, for $i \gg 0$, there is a short exact sequence

$$0 \longrightarrow \Phi_i^R H_2(G, K) \longrightarrow H_2(G, K) \longrightarrow H_2(\hat{G}_R, K) \longrightarrow 0,$$

where the epimorphism is induced by the homomorphism $G \to \hat{G}_R$. Here $\Phi_i^R H_2(G,K)$ is the ith term of an R-analog of the Dwyer filtration of $H_2(G,K)$. That is, the kernel of the map of $H_2(-,K)$ induced by R-completion, is described.

In the final section 10 we consider the second homology with finite coefficients of the Telescope functor (see [2] for properties and applications of the Telescope). We prove that, for a finitely presented metabelian group G, the inclusion of the Telescope of G into pronilpotent completion $\bar{G} \hookrightarrow \hat{G}$ induces a natural isomorphism $H_2(\bar{G}, \mathbb{Z}/n) \cong H_2(\hat{G}, \mathbb{Z}/n)$.

Recall that, for a free group F of rank 2, the second homology $H_2(\hat{F}, \mathbb{Z})$ is uncountable [6]. The proof is based on the construction of a free simplicial resolution of the Klein bottle group and uses the homology spectral sequence for a bisimplicial group. Observe that, the results of this paper show that the same type of proof can not be used for homology with finite coefficients. This motivates the following:

Problem. *Is it true that, for every $n \geq 1$ and a free group F, $H_2(\hat{F}, \mathbb{Z}/n) = 0$?*

2. Inverse limits of abelian groups and modules.

Denote by ω the category, whose objects are natural numbers and

$$\omega(n,m) = \begin{cases} \{(n,m)\}, & \text{if } n \le m \\ \varnothing, & \text{if } n > m. \end{cases}$$

Then an inverse sequence of abelian groups

$$A_0 \longleftarrow A_1 \longleftarrow A_2 \longleftarrow \dots$$

can be considered as a functor $A : \omega^{\rm op} \to \mathsf{Ab}$. Consider the functor category $\mathsf{Ab}^{\omega^{\rm op}}$. Since, the category $\omega^{\rm op}$ is a free category generated by a graph, the global dimension of the abelian category $\mathsf{Ab}^{\omega^{\rm op}}$ is equal to 1. The functor

$$\varprojlim : \mathsf{Ab}^{\omega^{\rm op}} \to \mathsf{Ab}$$

is left exact and its derived functors we denote by $\varprojlim^i = \mathbf{R}^i \varprojlim$. Since the global dimension is equal to 1, we get $\varprojlim^i = 0$ for $i \ge 2$. Moreover, if the Mittag-Leffler condition holds for the inverse sequence $\{A_i\}$, we have $\varprojlim^1 A_i = 0$. For example, the Mittag-Leffler condition holds if the homomorphisms $A_{i+1} \to A_i$ are epimorphisms.

Let $C_i^\bullet$ be an inverse sequence of (not necessarily bounded) complexes of abelian groups:

$$C_0^\bullet \longleftarrow C_1^\bullet \longleftarrow C_2^\bullet \longleftarrow \dots.$$

It can be considered as a complex $C^\bullet$ in the abelian category $\mathsf{Ab}^{\omega^{\rm op}}$. Let $I^{\bullet\bullet}$ be a right Cartan-Eilenberg resolution of $C^\bullet$ in $\mathsf{Ab}^{\omega^{\rm op}}$. Since the global dimension is 1, we can chose $I^{\bullet\bullet}$ so that $I^{\bullet i} = 0$ for $i \ge 2$. It follows that the totalisations are equal $\mathsf{Tot}^{\oplus}(I^{\bullet\bullet}) = \mathsf{Tot}^{\prod}(I^{\bullet\bullet})$ and we denote it by $\mathsf{Tot}(I^{\bullet\bullet})$. Then we have

$$\mathbf{R}^i \varprojlim(C^\bullet) = H^i(\varprojlim \mathsf{Tot}(I^{\bullet\bullet})) = H^i(\mathsf{Tot}(\varprojlim I^{\bullet\bullet})),$$

where $\mathbf{R}^i \varprojlim$ is the right hyper derived functor of $\varprojlim$ (see [21, 5.7.9]).

Lemma 2.1. *There are two spectral sequences E and $'E$ such that*

$$E_2^{pq} = H^p(\varprojlim_i{}^q\ C_i^\bullet) \ \Rightarrow\ \mathbf{R}^{p+q} \varprojlim(C^\bullet),$$

$$'E_2^{pq} = \varprojlim_i{}^p\ H^q(C_i^\bullet) \ \Rightarrow\ \mathbf{R}^{p+q} \varprojlim(C^\bullet),$$

and $E_1^{pq} = \varprojlim^q C_i^p$.

Proof. The double complex $\varprojlim I^{\bullet\bullet}$ has only two nonzero rows, and hence the canonical filtrations of $\varprojlim \mathsf{Tot}(I^{\bullet\bullet})$ are bounded. It follows that the both sequences of a double complex E and $'E$ converge to $\mathbf{R}^{p+q} \varprojlim(C^\bullet)$. Further, as in [21, 5.7.9], we get $E_1^{pq} = \varprojlim^q C_i^p$, $E_2^{pq} = H^p(\varprojlim^q C_i^\bullet)$ and $'E_2^{pq} = \varprojlim^p H^q(C_i^\bullet)$. □

Remark 2.2. The difference between the general statement in [21, 5.7.9] and Lemma 2.1 is that in our case all the spectral sequences converge in the strict sense. This lemma can be proved for any left exact functor from an abelian category of finite global dimension.

Corollary 2.3. *If morphisms $C^\bullet_{i+1} \to C^\bullet_i$ satisfy the Mittag-Leffler condition, then there is a short exact sequence*

$$0 \longrightarrow \varprojlim_i{}^1\; H^{q-1}(C^\bullet_i) \longrightarrow H^q(\varprojlim_i\; C^\bullet_i) \longrightarrow \varprojlim_i\; H^q(C^\bullet_i) \longrightarrow 0.$$

Proposition 2.4. *Let $\{A_i\}$ be an inverse sequence of abelian groups and B a finitely generated abelian group. Then the natural morphisms are isomorphisms*

$$\varprojlim \mathsf{Tor}(A_i, B) \cong \mathsf{Tor}(\varprojlim A_i, B), \qquad (\varprojlim{}^1 A_i) \otimes B \cong \varprojlim{}^1 (A_i \otimes B)$$

and for the natural morphisms

$$\alpha : (\varprojlim A_i) \otimes B \to \varprojlim\, (A_i \otimes B), \qquad \beta : \varprojlim{}^1\, \mathsf{Tor}(A_i, B) \to \mathsf{Tor}(\varprojlim{}^1 A_i, B)$$

there are natural isomorphisms

$$\mathrm{Ker}(\alpha) \cong \mathrm{Ker}(\beta), \qquad \mathrm{Coker}(\alpha) \cong \mathrm{Coker}(\beta).$$

Proof. Let $0 \to P_1 \to P_0 \to B \to 0$ be a free presentation of B, where P_0, P_1 are finitely generated free abelian groups. Consider the acyclic complex

$$C^\bullet = (\dots \to 0 \to \mathsf{Tor}(A_i, B) \to A_i \otimes P_1 \to A_i \otimes P_0 \to A_i \otimes B \to 0 \to \dots).$$

Then by Lemma 2.1 we get a spectral sequence E that converges to zero and $E_1^{pq} = \varprojlim{}^q C_i^p$. The first page E_1 looks as follows

$$\begin{array}{ccccccc}
\varprojlim{}^1 \mathsf{Tor}(A_i, B) & \longrightarrow & \varprojlim{}^1 (A_i \otimes P_1) & \xrightarrow{f^1} & \varprojlim{}^1 (A_i \otimes P_0) & \longrightarrow & \varprojlim{}^1 (A_i \otimes B) \\
\\
\varprojlim \mathsf{Tor}(A_i, B) & \longrightarrow & \varprojlim\, (A_i \otimes P_1) & \xrightarrow{f^0} & \varprojlim\, (A_i \otimes P_0) & \longrightarrow & \varprojlim\, (A_i \otimes B).
\end{array}$$

Since P_j is a finitely generated abelian group and $\varprojlim{}^q$ is an additive functor, we get $\varprojlim{}^q (A_i \otimes P_j) \cong (\varprojlim{}^q A_i) \otimes P_j$. It follows that $\mathrm{Ker}(f^q) = \mathsf{Tor}(\varprojlim{}^q A_i, B)$ and $\mathrm{Coker}(f^q) = (\varprojlim{}^q A_i) \otimes B$. We can replace the middle four terms in the spectral sequence with the kernels and cokernels of f^q so that the new spectral sequence still converges to zero:

$$\begin{array}{ccccccc}
\varprojlim{}^1 \mathsf{Tor}(A_i, B) & \xrightarrow{\beta} & \mathsf{Tor}(\varprojlim{}^1 A_i, B) & \xrightarrow{0} & (\varprojlim{}^1 A_i) \otimes B & \longrightarrow & \varprojlim{}^1 (A_i \otimes B) \\
\\
\varprojlim \mathsf{Tor}(A_i, B) & \longrightarrow & \mathsf{Tor}(\varprojlim A_i, B) & \xrightarrow{0} & (\varprojlim A_i) \otimes B & \xrightarrow{\alpha} & \varprojlim\, (A_i \otimes B).
\end{array}$$

Analysing the second page of this spectral sequence we obtain the required isomorphisms. $\square$

Corollary 2.5. *Let $\{A_i\}$ be an inverse sequence of abelian groups that satisfies the Mittag-Leffler condition. Then there is the following short exact sequence*

$$0 \longrightarrow \varprojlim{}^1 \mathsf{Tor}(A_i, B) \longrightarrow (\varprojlim\; A_i) \otimes B \xrightarrow{\alpha} \varprojlim\, (A_i \otimes B) \longrightarrow 0.$$

The following proposition is a generalisation of the previous corollary.

Proposition 2.6. *Let Λ be an associative right Notherian ring, M a finitely generated right Λ-module and $\{N_i\}$ an inverse sequence of left Λ-modules that satisfies the Mittag-Leffler condition. Then, for $m \geq 0$, there is the following short exact sequence*

$$0 \longrightarrow \varprojlim{}^1 \mathsf{Tor}^{\Lambda}_{m+1}(M, N_i) \longrightarrow \mathsf{Tor}^{\Lambda}_m(M, \varprojlim N_i) \longrightarrow \varprojlim \mathsf{Tor}^{\Lambda}_m(M, N_i) \longrightarrow 0.$$

Proof. Since, Λ is right Notherian and M is finitely generated, there exists a projective resolution $P_\bullet$ of M that consists of finitely generated projective modules. Let $C_i^\bullet$ be the complex $P_\bullet \otimes_\Lambda N_i$. The Mittag-Leffler condition for N_i implies the Mittag-Leffler condition for $P_\bullet \otimes_\Lambda N_i$. Then by Corollary 2.3 we get the short exact sequence

$$0 \longrightarrow \varprojlim{}^1 \mathsf{Tor}^{\Lambda}_{m+1}(M, N_i) \longrightarrow H_m(\varprojlim (P_\bullet \otimes_\Lambda N_i)) \longrightarrow \varprojlim \mathsf{Tor}^{\Lambda}_m(M, N_i) \longrightarrow 0.$$

Since $P_\bullet$ consists of finitely generated projective modules, we get the isomorphism

$$\varprojlim (P_\bullet \otimes_\Lambda N_i) \cong P_\bullet \otimes_\Lambda (\varprojlim N_i).$$

Thus the middle term is isomorphic to $\mathsf{Tor}^{\Lambda}_m(M, \varprojlim N_i)$ □

3. Completion and localization of rings and modules.

First we remind main concepts concerned to I-adic topology [22, VIII], [5, III]. Throughout the section all rings are assumed to be Noetherian and commutative. Let I be an ideal of a Noetherian commutative ring Λ. A (right) Λ-module we endow by the I-adic topology i.e. the topology such that the submodules $\{MI^n\}$ form a fundamental system of neighbourhoods of zero. In particular, the ring Λ is endowed by the I-adic topology. The closure of a submodule $N \leq M$ is given by $\mathsf{cl}(N) = \bigcap (N + MI^n)$. The submodule N is open if and only if $N \supseteq MI^n$ for some n. We put $MI^\infty := \bigcap MI^i = \mathsf{cl}(0)$. The module M is said to be nilpotent if $MI^i = 0$ for $i \gg 0$. The module M is said to be residually nilpotent if $MI^\infty = 0$. The module is residually nilpotent if and only if it is Hausdorff in the I-adic topology. The ideal I has the Artin-Rees property i.e. for a finitely generated module M and its submodule N the I-adic topology on the N coincides with the induced topology. In particular,

$$MI^\infty \cdot I = MI^\infty. \tag{3.1}$$

We put

$$M_{\mathsf{rn}} := M/MI^\infty.$$

Then the projection $M \to M_{\mathsf{rn}}$ is the universal homomorphism from M to a residually nilpotent module.

The R-completion of M is the inverse limit $\hat{M} = \hat{M}_I = \varprojlim M/MI^i$ with the natural structure of $\hat{\Lambda} = \varprojlim \Lambda/I^i$-module and the natural Λ-homomorphism $\varphi_M : M \to \hat{M}$. The ring $\hat{\Lambda}$ is still Noetherian, the morphism $\varphi : \Lambda \to \hat{\Lambda}$ is flat and for a finitely generated Λ-module M there is an isomorphism $\hat{M} \cong M \otimes_\Lambda \hat{\Lambda}$. The ideal $\hat{I} = \mathrm{Ker}(\hat{\Lambda} \to \Lambda/I)$ is equal to $\hat{\Lambda} \cdot \varphi(I)$, and there are isomorphisms $\Lambda/I^n \cong \hat{\Lambda}/\hat{I}^n$.

The notion of the I-adic completion is related to the notion of localization by the multiplicative set $1 + I$. We put $\Lambda^\ell = \Lambda^\ell_I = \Lambda[(1+I)^{-1}]$ and $M^\ell = M^\ell_I = M[(1+I)^{-1}]$. It is well-known that the morphism $\Lambda \to \Lambda^\ell$ is flat and $M^\ell = M \otimes_\Lambda \Lambda^\ell$. Moreover, if we denote $I^\ell = I[(1+I)^{-1}]$, then $\Lambda/I^n \cong \Lambda^\ell/(I^\ell)^n$. Since every element of $1+\hat{I}$ is invertible $\big((1+x)^{-1} = \sum_{i=0}^\infty (-1)^i x^i\big)$, the morphism $\varphi : \Lambda \to \hat{\Lambda}$ lifts to the morphism $\Lambda^\ell \to \hat{\Lambda}$.

The ring Λ is said to be *Zariski ring* with respect to the ideal I if one of the following equivalent properties holds (see [22, VII §4]):

- every submodule of every finitely generated Λ-module is closed;
- every finitely generated Λ-module M is residually nilpotent;
- every ideal of Λ is closed;
- every element of $1+I$ is invertible.

For any ring Λ and an ideal $I \lhd \Lambda$ there are two constructions that give examples of Zariski rings: the I-adic completion $\hat{\Lambda}$ and the localization Λ^ℓ by the set $1+I$. We are interested in both of these situations, so we work in the following general case.

Consider a ring homomorphism $\varphi : \Lambda \to \tilde{\Lambda}$ that satisfies the following conditions:

$$\begin{aligned} &1)\ \tilde{\Lambda} \text{ is a Zariski ring with respect to the ideal } \tilde{I} = \varphi(I)\cdot\tilde{\Lambda};\\ &2)\ \varphi \text{ is flat i.e. } \tilde{\Lambda} \text{ is a flat } \Lambda\text{-module};\\ &3)\ \varphi \text{ induces the isomorphism } \Lambda/I^n \cong \tilde{\Lambda}/\tilde{I}^n \text{ for } n \geq 0. \end{aligned} \tag{3.2}$$

We assume that $\tilde{\Lambda}$ is endowed by the $\tilde{I}$-adic topology. It is easy to see that φ is continuous, $\varphi(\Lambda)$ is dense in $\tilde{\Lambda}$ and $\tilde{I}^n = \varphi(I^n)\tilde{\Lambda}$. For a Λ-module M we set $\tilde{M} = M^\sim = M \otimes_\Lambda \tilde{\Lambda}$ and $\varphi_M = 1 \otimes \varphi : M \to \tilde{M}$. Then the functor $(\tilde{-}) : \mathsf{Mod}(\Lambda) \to \mathsf{Mod}(\tilde{\Lambda})$ is exact. The sequence of isomorphisms

$$\tilde{M}/\tilde{M}\tilde{I}^n \cong \tilde{M} \otimes_{\tilde{\Lambda}} \tilde{\Lambda}/\tilde{I}^n \cong M \otimes_\Lambda \tilde{\Lambda} \otimes_{\tilde{\Lambda}} \tilde{\Lambda}/\tilde{I}^n \cong M \otimes_\Lambda \tilde{\Lambda}/\tilde{I}^n \cong M \otimes_\Lambda \Lambda/I^n \cong M/MI^n$$

together with $\tilde{M}I^n = \tilde{M}\cdot\varphi(I^n) = \tilde{M}\cdot\tilde{\Lambda}\cdot\varphi(I^n) = \tilde{M}\tilde{I}^n$ give isomorphisms

$$M/MI^n \cong \tilde{M}/\tilde{M}\tilde{I}^n \cong \tilde{M}/\tilde{M}I^n \tag{3.3}$$

for any Λ-module M and $n \geq 0$. It follows that $\tilde{M} = \varphi_M(M) + \tilde{M}\tilde{I}^n$. Since, every finitely generated $\tilde{\Lambda}$-module is residually nilpotent, we get an isomorphism

$$(M_{\mathsf{rn}})^\sim \cong M^\sim.$$

It follows that the morphism $\varphi_M : M \to \tilde{M}$ is the composition of morphisms $M \to M_{\mathsf{rn}} \to \tilde{M}$.

Lemma 3.1. *Let $f : \Gamma \to \Lambda$ be a ring homomorphism, M, N be Λ-modules, X be a Γ-module and $i, j \geq 0$. Then there are the following isomorphisms.*

1. $\mathsf{Tor}_i^\Gamma(\tilde{M}, X) \cong \mathsf{Tor}_i^\Gamma(M, X)^\sim$.
2. $\mathsf{Tor}_i^\Lambda(\tilde{M}, N) \cong \mathsf{Tor}_i^\Lambda(M, \tilde{N}) \cong \mathsf{Tor}_i^\Lambda(M, N)^\sim$.
3. *The morphisms $M \to M_{\mathsf{rn}} \to \tilde{M}$ induce isomorphisms*
$$\mathsf{Tor}_i^\Lambda(\Lambda/I, \mathsf{Tor}_j^\Gamma(M, X)) \cong \mathsf{Tor}_i^\Lambda(\Lambda/I, \mathsf{Tor}_j^\Gamma(M_{\mathsf{rn}}, X)) \cong \mathsf{Tor}_i^\Lambda(\Lambda/I, \mathsf{Tor}_j^\Gamma(\tilde{M}, X))$$

Proof. (1) Let $P_\bullet$ be a finitely generated Γ-projective resolution of X. Then we have

$$\mathsf{Tor}_*^\Gamma(\tilde{M}, X) = H_*(\tilde{M} \otimes_\Gamma P_\bullet) \cong H_*(\tilde{\Lambda} \otimes_\Lambda M \otimes_\Gamma P_\bullet) \cong \tilde{\Lambda} \otimes_\Lambda H_*(M \otimes_\Gamma P_\bullet) \cong \mathsf{Tor}_*^\Gamma(M, X)^\sim.$$

(2) It follows from the previous formula and the isomorphism $\mathsf{Tor}_*(M, N) \cong \mathsf{Tor}_*(N, M)$.

(3) Using the previous isomorphisms and $(\Lambda/I)\tilde{} \cong \Lambda/I$, we get

$$\mathsf{Tor}_i^{\Lambda}(\Lambda/I, \mathsf{Tor}_j^{\Gamma}(\tilde{M}, X)) \cong \mathsf{Tor}_i^{\Lambda}(\Lambda/I, \mathsf{Tor}_j^{\Gamma}(M, X)\tilde{}) \cong \mathsf{Tor}_i^{\Lambda}((\Lambda/I)\tilde{}, \mathsf{Tor}_j^{\Gamma}(M, X)) \cong \mathsf{Tor}_i^{\Lambda}(\Lambda/I, \mathsf{Tor}_j^{\Gamma}(M, X))$$

and similarly

$$\mathsf{Tor}_i^{\Lambda}(\Lambda/I, \mathsf{Tor}_j^{\Gamma}((M_{\mathsf{rn}})\tilde{}, X)) \cong \mathsf{Tor}_i^{\Lambda}(\Lambda/I, \mathsf{Tor}_j^{\Gamma}(M_{\mathsf{rn}}, X)).$$

Using that $(M_{\mathsf{rn}})\tilde{} \cong \tilde{M}$, we obtain the required isomorphisms. □

Since $\tilde{\Lambda}$ is a Zariski ring, the annihilator $\mathsf{Ann}_{\tilde{\Lambda}} \tilde{M}$ is a closed ideal of $\tilde{\Lambda}$ and hence

$$\mathsf{cl}(\mathsf{Ann}_{\Lambda} M) \subseteq \mathsf{Ann}_{\Lambda} \tilde{M}. \tag{3.4}$$

Lemma 3.2. *If $\mathsf{cl}(\mathsf{Ann}_{\Lambda} M) \supseteq I^n$ then $MI^n = MI^{n+1}$ and there is the short exact sequence*

$$0 \longrightarrow MI^n \longrightarrow M \xrightarrow{\varphi_M} \tilde{M} \longrightarrow 0.$$

In particular, $\tilde{M} \cong M/MI^n$.

Proof. The inclusion $\mathsf{cl}(\mathsf{Ann}_{\Lambda} M) = \bigcap(\mathsf{Ann}_{\Lambda} M + I^i) \supseteq I^n$ implies $\mathsf{Ann}_{\Lambda} M + I^n = \mathsf{Ann}_{\Lambda} M + I^{n+1}$ and hence

$$MI^n = M(\mathsf{Ann}_{\Lambda} M + I^n) = M(\mathsf{Ann}_{\Lambda} M + I^{n+1}) = MI^{n+1}.$$

Using (3.4), we get the inclusion $\tilde{I}^n \subseteq \mathsf{Ann}_{\tilde{\Lambda}} \tilde{M}$. Therefore, we obtain $\tilde{M} = \tilde{M}/\tilde{M}\tilde{I}^n \cong M/MI^n$. It is easy to see that the composition of $1 \otimes \varphi$ with this isomorphism is the canonical projection. □

Proposition 3.3. *Let $\varphi : \Lambda \to \tilde{\Lambda}$ be a ring homomorphism satisfying* (3.2)*, and M and N be Λ-modules such that $\mathsf{cl}(\mathsf{Ann}_{\Lambda} M + \mathsf{Ann}_{\Lambda} N) \supseteq I^n$. Then $(M \otimes_{\Lambda} N)I^n = (M \otimes_{\Lambda} N)I^{n+1}$ and the obvious morphisms induce isomorphisms*

$$\tilde{M} \otimes_{\Lambda} \tilde{N} \cong \tilde{M} \otimes_{\tilde{\Lambda}} \tilde{N} \cong (M \otimes_{\Lambda} N)\tilde{} \cong (M \otimes_{\Lambda} N)/(M \otimes_{\Lambda} N)I^n \cong (M/MI^n) \otimes_{\Lambda} (N/NI^n).$$

Proof. Endow $\tilde{M} \otimes_{\Lambda} \tilde{N}$ by the structure of a $\tilde{\Lambda}$-module as follows: $(m \otimes n)a = m \otimes (na)$ for $a \in \tilde{\Lambda}$, $m \in \tilde{M}$, $n \in \tilde{N}$. Then the induced action of Λ on $\tilde{M} \otimes_{\Lambda} \tilde{N}$ coincides with the standard action. Hence, $\mathsf{Ann}_{\Lambda}(\tilde{M} \otimes_{\Lambda} \tilde{N}) = \varphi^{-1}(\mathsf{Ann}_{\tilde{\Lambda}}(\tilde{M} \otimes_{\Lambda} \tilde{N}))$. Since all ideals in $\tilde{\Lambda}$ are closed and $\varphi : \Lambda \to \tilde{\Lambda}$ is continuous, $\mathsf{Ann}_{\Lambda}(\tilde{M} \otimes_{\Lambda} \tilde{N})$ is a closed ideal. Thus, $\mathsf{Ann}_{\Lambda}(\tilde{M} \otimes_{\Lambda} \tilde{N}) \supseteq I^n$ and $\mathsf{Ann}_{\tilde{\Lambda}}(\tilde{M} \otimes_{\Lambda} \tilde{N}) \supseteq \tilde{I}^n$.

Now we prove that for any $b \in \tilde{I}^n$ and $m \in \tilde{M}, n \in \tilde{N}$ the elements $m \otimes nb$ and $mb \otimes n$ are equal to zero in $\tilde{M} \otimes_{\Lambda} \tilde{N}$. The first equality is obvious because $b \in \mathsf{Ann}_{\tilde{\Lambda}}(\tilde{M} \otimes_{\Lambda} \tilde{N})$ and $m \otimes nb = (m \otimes n)b = 0$. The ring Λ is Noetherian and hence the ideal I^n is finitely generated $I^n = (\lambda_1, \dots, \lambda_s)$. Then $\tilde{I}^n = (\varphi(\lambda_1), \dots, \varphi(\lambda_s))$. Consider $b = \sum_{i=1}^s a_i \varphi(\lambda_i) \in \tilde{I}^n$, where $a_i \in \tilde{\Lambda}$. Since $\varphi(\lambda_i)$ annihilates $\tilde{M} \otimes_{\Lambda} \tilde{N}$, we get

$$mb \otimes n = \sum_{i=1}^{s} ma_i \varphi(\lambda_i) \otimes n = \sum_{i=1}^{s} (ma_i \otimes n)\varphi(\lambda_i) = 0.$$

Therefore, we have that the image of $(\tilde{M} \otimes_{\Lambda} \tilde{N}\tilde{I}^n) \oplus (\tilde{M}\tilde{I}^n \otimes_{\Lambda} \tilde{N})$ in $\tilde{M} \otimes_{\Lambda} \tilde{N}$ vanishes. It follows that $\tilde{M} \otimes_{\Lambda} \tilde{N} \cong (\tilde{M}/\tilde{M}\tilde{I}^n) \otimes_{\Lambda} (\tilde{N}/\tilde{N}\tilde{I}^n)$. Using (3.3), we get the isomorphism $\tilde{M} \otimes_{\Lambda} \tilde{N} \cong (M/MI^n) \otimes_{\Lambda} (N/NI^n)$.

The isomorphisms

$$\tilde{M} \otimes_{\tilde{\Lambda}} \tilde{N} \cong (M \otimes_\Lambda N)\tilde{} \cong (M \otimes_\Lambda N)/(M \otimes_\Lambda N)I^n$$

and the equality $(M \otimes_\Lambda N)I^n = (M \otimes_\Lambda N)I^{n+1}$ follow from lemma 3.2 and the inclusion

$$\mathsf{cl}(\mathsf{Ann}_\Lambda(M \otimes_\Lambda N)) \supseteq \mathsf{cl}(\mathsf{Ann}_\Lambda M + \mathsf{Ann}_\Lambda N) \supseteq I^n.$$

Then we only need to prove the isomorphism $\tilde{M} \otimes_\Lambda \tilde{N} \cong \tilde{M} \otimes_{\tilde{\Lambda}} \tilde{N}$. It is sufficient to prove that for any $a \in \tilde{\Lambda}$ and $m \in \tilde{M}, n \in \tilde{N}$ the equality $ma \otimes n = m \otimes na$ holds in $\tilde{M} \otimes_\Lambda \tilde{N}$. Since $\tilde{\Lambda} = \varphi(\Lambda) + \tilde{I}^n$, a can be presented as follows $a = \varphi(\lambda) + b$, where $\lambda \in \Lambda$ and $b \in \tilde{I}^n$. Then the equality $ma \otimes n = m \otimes na$ follows from the equalities $mb \otimes n = 0$, $m \otimes nb = 0$ and $m\lambda \otimes n = m \otimes n\lambda$. □

Let $\sigma : \Lambda \to \Lambda$ be an automorphism such that $\sigma(I) = I$ and $\sigma^2 = \mathsf{id}$. In particular, σ is continuous in the I-adic topology. For a Λ-module M we denote by M_σ the Λ-module with the same underlying abelian group and the following action $m * \lambda = m\sigma(\lambda)$. Define the *twisted exterior square* $\wedge^2_\sigma M$ as the quotient Λ-module

$$\wedge^2_\sigma M = \frac{M \otimes_\Lambda M_\sigma}{\langle \{m \otimes m \mid m \in M\} \rangle_\Lambda},$$

where $\langle X \rangle_\Lambda$ means the Λ-submodule generated by X.

Corollary 3.4. *Let M be a Λ-module such that $\mathsf{cl}(\mathsf{Ann}_\Lambda M + \sigma(\mathsf{Ann}_\Lambda M)) \supseteq I^n$. Then $(\wedge^2_\sigma M)I^n = (\wedge^2_\sigma M)I^{n+1}$ and the obvious morphisms induce isomorphisms*

$$\wedge^2_\sigma \tilde{M} \cong (\wedge^2_\sigma M)\tilde{} \cong (\wedge^2_\sigma M)/(\wedge^2_\sigma M)I^n \cong \wedge^2_\sigma (M/MI^n).$$

Proof. For an Λ-module N we set

$$D(N) := \langle \{n \otimes n \mid n \in N\} \rangle_\Lambda \le N \otimes_\Lambda N_\sigma.$$

It is sufficient to prove the isomorphisms

$$\tilde{M} \otimes_\Lambda \tilde{M}_\sigma \cong (M \otimes_\Lambda M_\sigma)\tilde{} \cong (M/MI^n) \otimes_\Lambda (M/MI^n)_\sigma$$

induce isomorphisms $D(\tilde{M}) \cong D(M)\tilde{} \cong D(M/MI^n)$. First we prove that the isomorphism

$$\tilde{M} \otimes_\Lambda \tilde{M}_\sigma \cong (M/MI^n) \otimes_\Lambda (M/MI^n)_\sigma$$

induces the isomorphism $D(\tilde{M}) \cong D(M/MI^n)$. It is easy to see that the functor D takes epimorphisms to epimorphisms. Hence the epimorphism $\tilde{M} \twoheadrightarrow M/MI^n$ induces the epimorphism $D(\tilde{M}) \to D(M/MI^n)$. From the other hand $D(\tilde{M}) \to D(M/MI^n)$ is a monomorphism because

$$\tilde{M} \otimes_\Lambda \tilde{M}_\sigma \to (M/MI^n) \otimes_\Lambda (M/MI^n)_\sigma$$

is an isomorphism.

Then we only need to prove that the isomorphism $(M \otimes_\Lambda M_\sigma)\tilde{} \cong \tilde{M} \otimes_\Lambda \tilde{M}_\sigma$ induces the isomorphism $D(M)\tilde{} \cong D(\tilde{M})$. The image of $D(M)\tilde{}$ in $\tilde{M} \otimes_\Lambda \tilde{M}_\sigma$ is generated by $\varphi_M(m) \otimes \varphi_M(m)$ for $m \in M$ and hence $D(\tilde{M})$ includes the image. We need to prove that for any $x \in \tilde{M}$ the element $x \otimes x$ lies in the image of $D(M)\tilde{}$. Since $\tilde{M} = \varphi_M(M) + \tilde{M}\tilde{I}^n$, we get $x = \varphi_M(m) + y$, for some $m \in M, y \in \tilde{M}\tilde{I}^n$. Isomorphisms

$$\tilde{M} \otimes_\Lambda \tilde{M}_\sigma \cong (M/MI^n) \otimes_\Lambda (M/MI^n)_\sigma \cong (\tilde{M}/\tilde{M}\tilde{I}^n) \otimes_\Lambda (\tilde{M}/\tilde{M}\tilde{I}^n)_\sigma$$

imply that the elements $y \otimes \varphi_M(m)$, $\varphi_M(m) \otimes y$, $y \otimes y$ vanish in $\tilde{M} \otimes_\Lambda \tilde{M}_\sigma$ and hence $x \otimes x = \varphi_M(m) \otimes \varphi_M(m)$ lies in the image of $D(M)\tilde{}$. □

Corollary 3.5. *Let Λ be a commutative Noetherian ring, I be an ideal of Λ and M be a finitely generated Λ-module such that $\mathsf{cl}(\mathsf{Ann}_\Lambda M + \sigma(\mathsf{Ann}_\Lambda M)) \supseteq I^n$. Then there are isomorphisms*

$$\wedge^2_\sigma \hat{M} \cong \wedge^2_\sigma M^\ell \cong (\wedge^2_\sigma M)/(\wedge^2_\sigma M)I^n \cong \wedge^2_\sigma(M/MI^n),$$

where $\hat{M}$ is the I-adic completion and M^ℓ is the localization $M^\ell = M[(1+I)^{-1}]$.

Proof. It follows from corollary 3.4, the isomorphisms $\hat{M} \cong M \otimes_\Lambda \hat{\Lambda}$, $M^\ell \cong M \otimes_\Lambda \Lambda^\ell$ and the fact that the morphisms $\Lambda \to \hat{\Lambda}$ and $\Lambda \to \Lambda^\ell$ satisfy (3.2). □

4. R-completion of a metabelian group.

Let G be a metabelian group and

$$0 \longrightarrow M \longrightarrow G \xrightarrow{\pi} A \longrightarrow 1 \tag{4.1}$$

is a short exact sequence of groups, where A is an abelian group and M is a right A-module with the action defined by conjugation. We assume that $M = \ker(\pi) \subseteq G$. We use the multiplicative notation for A and G but for M we use both the multiplicative and the additive notations. We use $*$ for the action of $\mathbb{Z}[G]$ and $\mathbb{Z}[A]$ on M in order to separate it from the multiplication in the group. Therefore, for $m, m_1, m_2 \in M$, $g, g_1, \dots, g_l \in G$ and $k_1, \dots, k_l \in \mathbb{Z}$ we have $(m^{k_1})^{g_1} \dots (m^{k_l})^{g_l} = m * (\sum_{i=1}^l k_i g_i) = m * \pi(\sum_{i=1}^l k_i g_i)$ and $m_1^g m_2^g = (m_1 + m_2) * g = (m_1 + m_2) * \pi(g)$.

There is a notion of R-completion of a group for subrings of $\mathbb{Q}$ and for $R = \mathbb{Z}/n$ (see [7]). All subrings of $\mathbb{Q}$ have the form $R = \mathbb{Z}[J^{-1}]$, where J is a set of prime numbers. We are going to describe the R-completion $\hat{G}_R$ of the metabelian group G in terms of A and M in these two cases separately. For the case $R = \mathbb{Z}[J^{-1}]$ we need some information about Malcev R-completion.

4.1. Malcev R-completion.

In this subsection by R we denote the ring $\mathbb{Z}[J^{-1}] \subseteq \mathbb{Q}$. Recall the notion of Malcev R-completion [13], [14], [18]. A group G is said to be *J-local* or *uniquely J-divisible* if the map $g \mapsto g^p$ is a bijection for $p \in J$. The embedding of the category of all J-local nilpotent groups to the category of all nilpotent groups $\mathsf{Nil}_J \hookrightarrow \mathsf{Nil}$ has the left adjoint functor called Malcev R-completion

$$- \otimes R : \mathsf{Nil} \to \mathsf{Nil}_J.$$

Thus, if H is a nilpotent group and H' is J-local nilpotent group, there is a natural isomorphism $\mathsf{Hom}(H, H') \cong \mathsf{Hom}(H \otimes R, H')$, and the unit of the adjunction $\eta_H : H \to H \otimes R$ is the universal homomorphism from H to a J-local nilpotent group. If H is abelian, then $H \otimes R$ is the ordinary tensor product. The functor $- \otimes R$ preserves short exact sequences.

Let K denote an Artinian quotient ring of R. In other words,

$$K = \begin{cases} \mathbb{Q}, & \text{if } R = \mathbb{Q} \\ \mathbb{Z}/n, & \text{if } R = \mathbb{Z}[J^{-1}] \neq \mathbb{Q} \end{cases},$$

where n is a natural number such that the prime divisors do not lie in J.

Lemma 4.1. *Let $H' \rightarrowtail H \twoheadrightarrow H''$ be a short exact sequence of finitely generated nilpotent groups. If N is a nilpotent $K[H]$-module finitely generated over K, then the homology group $H_i(H', N)$ is a nilpotent $K[H'']$-module finitely generated over K.*

Proof. Since H', H, H'' are finitely generated nilpotent groups, then the group rings $K[H'], K[H], K[H'']$ are Noetherian [9]. It follows that there exists a free resolution $P_\bullet$ of the trivial $K[H']$-module K that consists of finitely generated free $K[H']$-modules. Since N is a nilpotent $K[H]$-module finitely generated over K, the module $P_i \otimes_{KH'} N$ has this property too. The homology group $H_i(H', N)$ is a subquotient of $P_i \otimes_{K[H']} N$, and hence it has this property too. □

Proposition 4.2. *Let H be a nilpotent finitely generated group and N be a nilpotent $K[H \otimes R]$-module that is finitely generated over K, then the homomorphism $H \to H \otimes R$ induces the isomorphism*

$$H_*(H, N) \cong H_*(H \otimes R, N).$$

Proof. Let $R = \mathbb{Z}[J^{-1}] \neq \mathbb{Q}$, and hence $K = \mathbb{Z}/n$. First we prove the proposition for an abelian group H and $N = \mathbb{Z}/p$, where p is a prime divisor of n. Homology $H_*(X, \mathbb{Z}/p)$ of an abelian group X is isomorphic to $\bigwedge^*(X/p) \otimes \Gamma_*({}_pX)$ (see [8, V, 6.6]). Then we only need to note that $H/p \cong (H \otimes R)/p$ and ${}_pH \cong {}_p(H \otimes R)$.

Let now H is abelian and N is a finite nilpotent $K[H]$-module. All nilpotent $K[H \otimes R]$-modules finitely generated over $\mathbb{Z}/n$ can be obtained by a sequence of extensions from the trivial modules $\mathbb{Z}/p$, where p is a prime divisor of n. Then we need to prove that the class of $K[H \otimes R]$-modules with the property $H_*(H, N) \cong H_*(H \otimes R, N)$ is closed under extensions. It follows easily from the homology long exact sequence and the five lemma.

Prove the general case. We need to prove that the class of groups with this property is closed under extensions. Let $H' \rightarrowtail H \twoheadrightarrow H''$ is a short exact sequence of finitely generated nilpotent groups such that the proposition holds for H' and H''. We prove it for H. Let N be a nilpotent KH-module finitely generated over K. Consider the morphism of the short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H' & \longrightarrow & H & \longrightarrow & H'' & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H' \otimes R & \longrightarrow & H \otimes R & \longrightarrow & H'' \otimes R & \longrightarrow & 1. \end{array}$$

It induces the the morphism of the corresponding Lyndon-Hochschild-Serre spectral sequences $E \to E^R$. It is sufficient to prove that the morphism $E \to E^R$ is an isomorphism. By induction hypothesis we know $H_q(H', N) \cong H_q(H' \otimes R, N)$. By Lemma 4.1 the KH''-module $H_q(H', N)$ is finite and nilpotent. Then again by induction hypothesis we have $H_p(H'', H_q(H', N)) \cong H_p(H'' \otimes R, H_q(H' \otimes R, N))$. It follows that the morphism $E \to E^R$ is an isomorphism.

The case of $R = K = \mathbb{Q}$ can be proved similarly, using the formula $H_*(X, \mathbb{Q}) \cong \bigwedge^*(X \otimes \mathbb{Q})$ for an abelian group X. □

4.2. **$\mathbb{Z}[J^{-1}]$-completion of a metabelian group.** In this section we assume $R = \mathbb{Z}[J^{-1}]$. For $\alpha \in \mathbb{R}$ we denote $\binom{\alpha}{n} := \alpha(\alpha - 1) \dots (\alpha - n + 1)/n!$.

Lemma 4.3. *If $\alpha \in \mathbb{Z}[J^{-1}]$ then $\binom{\alpha}{n} \in \mathbb{Z}[J^{-1}]$.*

Proof. Let $q \notin J$ be a prime number, v be the q-adic value of $n!$ and $l \in \mathbb{Z}$ such that $l\alpha^{-1} \in \mathbb{Z}$, $l \cdot \alpha^{-1} \equiv 1 \pmod{q^v}$ and $l \geq n$. Consider the epimorphism $\mathbb{Z}[J^{-1}] \twoheadrightarrow \mathbb{Z}/q^v$. The image of α coincides with the image of l, and hence the image of $\alpha(\alpha-1)\dots(\alpha-n+1)$ coincides with the image of $l(l-1)\dots(l-n+1)$. Since $l(l-1)\dots(l-n+1)$ is divisible by $n!$, we obtain that the image of $\alpha(\alpha-1)\dots(\alpha-n+1)$ vanishes, and hence the q-adic value of $\alpha(\alpha-1)\dots(\alpha-n+1)$ is greater than or equal to v. It follows that the q-adic value of $\binom{\alpha}{n}$ is non-negative for any $q \notin J$. Hence $\binom{\alpha}{n} \in \mathbb{Z}[J^{-1}]$. □

Let Λ be a complete $\mathbb{Z}[J^{-1}]$-algebra with respect to an ideal $\mathfrak{a}$ i.e. $\varphi : \Lambda \to \hat{\Lambda}_{\mathfrak{a}}$ is an isomorphism. For $x \in 1 + \mathfrak{a}$ we denote

$$x^{[\alpha]} = \sum_{n=0}^{\infty} \binom{\alpha}{n} (x-1)^n. \tag{4.2}$$

Lemma 4.4. *For any $\alpha, \beta \in \mathbb{Z}[J^{-1}]$, $n \in \mathbb{N}$ and $x \in 1 + \mathfrak{a}$ the following equalities hold*

$$(x^{[\alpha]})^{[\beta]} = x^{[\alpha\beta]}, \qquad x^{[\alpha]}x^{[\beta]} = x^{[\alpha+\beta]}, \qquad x^n = x^{[n]}.$$

In particular, $(x^p)^{[1/p]} = x = (x^{[1/p]})^p$, and $x \cdot x^{[-1]} = 1 = x^{[-1]} \cdot x$. Therefore, $1+\mathfrak{a}$ is a J-local group. Moreover, $\gamma_n(1+\mathfrak{a}) \subseteq 1+\mathfrak{a}^n$, where $\{\gamma_n(1+\mathfrak{a})\}$ is the lower central series of the group $1+\mathfrak{a}$.

Proof. The equality $x^n = x^{[n]}$ follows from the binomial theorem. From the standard course of mathematical analysis we know that for $t > 0$ and $\alpha \in \mathbb{R}$ the equality $t^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n}(t-1)^n$ holds. Denote by $\mathfrak{b}$ the ideal of $R[t]$ generated by the element $(t-1)$. Then for $\alpha, \beta \in R$ the equalities $(t^\alpha)^\beta = t^{\alpha\beta}$ and $t^{\alpha+\beta} = t^\alpha t^\beta$ imply that in the algebra $\widehat{R[t]}_{\mathfrak{b}}$ of 'power series of $(t-1)$' the corresponding identities hold $(t^{[\alpha]})^{[\beta]} = t^{[\alpha\beta]}, t^{[\alpha+\beta]} = t^{[\alpha]}t^{[\beta]}$. Consider the homomorphism $\xi : R[t] \to \Lambda$, that takes t to x. Endow $R[t]$ with the $\mathfrak{b}$-adic topology. Then ξ is continuous, and hence it induces a continuous homomorphism $\hat{\xi} : \widehat{R[t]}_{\mathfrak{b}} \to \Lambda$. Since $\hat{\xi}$ is continuous, we get $\hat{\xi}(f^{[\alpha]}) = \hat{\xi}(f)^{[\alpha]}$ for all $f \in 1 + \hat{\mathfrak{b}}$. Therefore, the equalities $(t^{[\alpha]})^{[\beta]} = t^{[\alpha\beta]}, t^{[\alpha+\beta]} = t^{[\alpha]}t^{[\beta]}$ imply $(x^{[\alpha]})^{[\beta]} = x^{[\alpha\beta]}, x^{[\alpha+\beta]} = x^{[\alpha]}x^{[\beta]}$.

Prove the inclusion $\gamma_n(1+\mathfrak{a}) \subseteq 1+\mathfrak{a}^n$ by induction on n. For $n = 1$ it is obvious. Assume that $\gamma_{n-1}(1+\mathfrak{a}) \subseteq 1+\mathfrak{a}^{n-1}$. For two elements $a, b \in 1+\mathfrak{a}$ we write $a \equiv b$ if their images in the quotient group $(1+\mathfrak{a})/(1+\mathfrak{a}^n)$ are equal. Then for $y \in \mathfrak{a}$ and $z \in \mathfrak{a}^n$ we have $1+y+z \equiv 1+y$ because $(1+y)^{-1}(1+y+z) = 1+(1+y)^{-1}z$ and $(1+y)^{-1}z \in \mathfrak{a}^n$. Chose $1+x \in \gamma_{n-1}(G)$ and $1+y \in 1+\mathfrak{a}$. Then $x \in \mathfrak{a}^{n-1}, y \in \mathfrak{a}$ and we have

$$[1+x, 1+y] = (1+x)^{-1}(1+y)^{-1}(1+x)(1+y) =$$

$$= \left(\sum_{i=0}^{\infty}(-1)^i x^i\right)\left(\sum_{i=0}^{\infty}(-1)^i y^i\right)(1+x)(1+y) \equiv (1-x)(1-y)(1+x)(1+y) \equiv$$

$$\equiv 1 - x - y + x + y = 1.$$

Therefore $[1+x, 1+y] \in 1+\mathfrak{a}^n$, and hence $\gamma_n(1+\mathfrak{a}) \subseteq 1+\mathfrak{a}^n$.

□

Corollary 4.5. *Let A be an abelian group and $\widehat{R[A]}$ be the I_R-adic completion of the group algebra, where $I_R = I_R(A)$ is the augmentation ideal. Then the homomorphism $\tilde{\theta} : A \otimes R \to 1 + \hat{I}_R$ defined by the formula $\tilde{\theta}(a \otimes \alpha) = a^{[\alpha]}$ is the unique homomorphism such that the following diagram is commutative*

$$\begin{tikzcd} & A \arrow[dl, "\eta"'] \arrow[dr, "\iota"] & \\ A \otimes R \arrow[rr, "\tilde{\theta}"] & & 1 + \hat{I}_R. \end{tikzcd} \tag{4.3}$$

Let G be a metabelian group with a metabelian decomposition $M \rightarrowtail G \twoheadrightarrow A$. G is nilpotent if and only if M is a nilpotent $\mathbb{Z}[A]$-module (i.e. $M * I^m = 0$ for $m \gg 0$). In this case $(M \otimes R)^{\wedge}_{I_R} = M \otimes R$, and hence $M \otimes R$ has the natural structure of $\widehat{R[A]}$-module. The composition of homomorphisms $A \otimes R \xrightarrow{\tilde{\theta}} 1 + \hat{I}_R \hookrightarrow \widehat{R[A]}$ we denote by

$$\theta_R : A \otimes R \longrightarrow \widehat{R[A]}. \tag{4.4}$$

Observe that $\theta_R(x^{-1}) = \hat{\sigma}(\theta_R(x))$, where $\sigma : R[A] \to R[A]$ is the antipode.

Lemma 4.6. *Let G be a nilpotent metabelian group with a metabelian decomposition $M \rightarrowtail G \twoheadrightarrow A$. Then the group $G \otimes R$ is a metabelian nilpotent group with the metabelian decomposition $M \otimes R \rightarrowtail G \otimes R \twoheadrightarrow A \otimes R$, and the action of $A \otimes R$ on $M \otimes R$ is induced by the structure of $\widehat{R[A]}$-module via the homomorphism $\theta_R : A \otimes R \to \widehat{R[A]}$.*

Proof. Since $- \otimes R$ takes short exact sequences to short exact sequences, and on abelian groups it is the usual tensor product, we get the metabelian decomposition $M \otimes R \rightarrowtail G \otimes R \twoheadrightarrow A \otimes R$. Using the morphism of sequences

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\eta_M"'] & G \arrow[r, "\pi"] \arrow[d, "\eta_G"'] & A \arrow[r] \arrow[d, "\eta_A"'] & 1 \\ 0 \arrow[r] & M \otimes R \arrow[r] & G \otimes R \arrow[r, "\pi \otimes 1"] & A \otimes R \arrow[r] & 1. \end{tikzcd}$$

we get the identity $(m \otimes 1) * (a \otimes 1) = (m * a) \otimes 1$. An endomorphism of the abelian group M lifts uniquely to an endomorphism of the abelian group $M \otimes R$. Hence, $(m \otimes r) * (a \otimes 1) = (m * a) \otimes r$ for all $m \in M, a \in A, r \in R$. Therefore, the action of $A \otimes R$ on $M \otimes R$ extends the induced action of A.

Consider the module $M \otimes R$ as a filtered R-module, where the filtration is given by $\mathcal{F}_i = [M \otimes R, \gamma_i(G \otimes R)]$ where $\{\gamma_i(G \otimes R)\}$ is the lower central series of $G \otimes R$. Since $G \otimes R$ is nilpotent, the filtration is finite. Observe that $\mathcal{F}_i$ is a normal subgroup of $G \otimes R$ and equivalently a $\widehat{R[A]}$-submodule of $M \otimes R$. Consider the R-algebra $\Lambda = \mathsf{End}_{R\text{-}\mathsf{Filt}}(M \otimes R)$ of R-endomorphisms preserving filtration, and the ideal $\mathfrak{a} = \{f \in \Lambda \mid f(\mathcal{F}_i) \subseteq \mathcal{F}_{i+1}\}$. Since $\mathfrak{a}^m = 0$ for $m \gg 0$, the R-algebra Λ is complete with respect to the $\mathfrak{a}$-adic topology. By Lemma 4.4, $1 + \mathfrak{a}$ is a J-local nilpotent group. Since M is a $R[A]$-module and $\{\mathcal{F}_i\}$ are submodules, we obtain the algebra homomorphism $R[A] \to \Lambda$. The ideal I_R is generated by elements $(a - 1)$

for $a \in A$. Let $g \in G$ such that $\pi(g) = a$. Thus, $x * (a-1) = -x + x * a = x^{-1} \cdot x^g = [x, g]$ for $x \in M \otimes R$, and hence, the image of I_R lies in $\mathfrak{a}$. Then the obtain the continuous homomorphism $\widehat{R[A]} \to \Lambda$ that induces the group homomorphism $\sigma : 1 + \hat{I}_R \to 1 + \mathfrak{a}$. Similarly, the action of $A \otimes R$ on $M \otimes R$ induces a homomorphism $\tau : A \otimes R \to 1 + \mathfrak{a}$. Then we obtain the following diagram.

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, "\eta"'] & 1 + \hat{I}_R \arrow[d, "\sigma"] \\ A \otimes R \arrow[r, "\tau"] \arrow[ur, "\tilde{\theta}"] & 1 + \mathfrak{a} \end{tikzcd}$$

By Corollary 4.5 we have $\tilde{\theta}\eta = \iota$. Since the action of $A \otimes R$ extends the action of A, we have $\tau\eta = \sigma\iota$. Thus we have $\tau\eta = \sigma\tilde{\theta}\eta$, and using the universal property of η, we get $\tau = \sigma\tilde{\theta}$. Therefore, the action of $A \otimes R$ on $M \otimes R$ is induced by the structure of $\widehat{R[A]}$-module via the homomorphism θ_R. □

The R-*completion* of a group G is defined as follows

$$\hat{G}_R := \varprojlim \, (G/\gamma_i(G)) \otimes R.$$

Proposition 4.7. *Let R be the ring $\mathbb{Z}[J^{-1}]$, G be a metabelian group as in* (4.1). *Denote by $(M \otimes R)^\wedge_{I_R}$ the I_R-adic completion of the $R[A]$-module $M \otimes R$. Then there is a short exact sequence*

$$0 \longrightarrow (M \otimes R)^\wedge_{I_R} \longrightarrow \hat{G}_R \longrightarrow A \otimes R \longrightarrow 1,$$

where the action by conjugation of $A \otimes R$ on $(M \otimes R)^\wedge_{I_R}$ coincides with the action that induced by the structure of $\widehat{R[A]}$-module via the homomorphism $\theta_R : A \otimes R \to \widehat{R[A]}$.

Proof. For $g \in G$ and $m \in M$ we have $m * (g-1) = -m + m * g = m^{-1}m^g = [m, g]$. Thus, we have $M * I^i = [M, G, \dots, G]$ and hence

$$M * I^i \subseteq \gamma_i(G) \subseteq M * I^{i-1}. \tag{4.5}$$

Therefore, we obtain an isomorphism $\hat{G}_R = \varprojlim \, (G/\gamma_i(G)) \otimes R \cong \varprojlim \, (G/(M * I^i)) \otimes R$. By Lemma 4.6 the short exact sequences $M/(M * I^i) \rightarrowtail G/(M * I^i) \twoheadrightarrow A$ give the short exact sequences $(M/(M*I^i)) \otimes R \rightarrowtail (G/(M*I^i)) \otimes R \twoheadrightarrow A \otimes R$, and the action on $(M/(M*I^i)) \otimes R$ is induced by the structure of $\widehat{R[A]}$-module via the homomorphism $\theta : A \otimes R \to \widehat{R[A]}$. The inverse sequence $(M/(M*I^i)) \otimes R$ satisfies the Mittag–Leffler condition, and hence we have the following short exact sequence:

$$0 \longrightarrow \varprojlim \, (M/(M * I^i)) \otimes R \longrightarrow \hat{G}_R \longrightarrow A \otimes R \longrightarrow 1.$$

Since R is a flat $\mathbb{Z}$-module, we have $(M/(M*I^i)) \otimes R \cong (M \otimes R)/((M \otimes R) * I_R^i)$. Therefore, we get

$$\varprojlim \, (M/(M * I^i)) \otimes R \cong (M \otimes R)^\wedge_{I_R}.$$

The action of $A \otimes R$ on $(M \otimes R)^\wedge_R$ is induced by θ, because the action of $A \otimes R$ on the quotients

$$(M/(M * I^i)) \otimes R \cong (M \otimes R)/((M \otimes R) * I_R^i)$$

is induced by θ. □

4.3. **$\mathbb{Z}/n$-completion of a metabelian group.** In this subsection we denote by R the ring $\mathbb{Z}/n$. The *n-lower central series* of a group G is defined as follows: $\gamma_1^{[n]}(G) = G$ and $\gamma_{i+1}^{[n]}(G) = \ker\big(\gamma_i^{[n]}(G) \to (\gamma_i^{[n]}(G)/[G,\gamma_i^{[n]}(G)]) \otimes \mathbb{Z}/n\big)$. Then R-completion of a group G is

$$\hat{G}_R = \varprojlim G/\gamma_i^{[n]}(G).$$

If p is a prime number and G is a finitely generated group, the $\mathbb{Z}/p$-completion $\hat{G}_{\mathbb{Z}/p}$ coincides with the p-profinite completion $\hat{G}_p$ [7, IV,2.3]

$$\hat{G}_{\mathbb{Z}/p} \cong \hat{G}_p. \tag{4.6}$$

Further we will assume that A is a finitely generated abelian group. For a normal subgroup H of G we denote by $\mathsf{p}_n(H)$ the normal subgroup generated by the set of powers $\{h^n \mid h \in H\}$ and we set $\mathcal{P}_n(H) = \mathsf{p}_n(H)\cdot[H,G]$. It is easy to verify that

$$\gamma_{i+1}^{[n]}(G) = \mathcal{P}_n(\gamma_i^{[n]}(G)).$$

By I_n we denote the kernel of the composition of the augmentation map and the canonical projection

$$I_n = \ker(\mathbb{Z}[A] \longrightarrow \mathbb{Z}/n).$$

Equivalently we can describe it as follows $I_n = I + (n)$, where (n) is the ideal of $\mathbb{Z}[A]$ generated by n. It is easy to see that $\sum_{i=1}^n a_i \in I_n$, for any sequence $a_1,\dots,a_n \in A$.

Since A is finitely generated we can fix a natural number T such that $a^{n^{T+1}} = 1$ implies $a^{n^T} = 1$ for $a \in A$. In other words,

$$(a \in A^{n^T} \quad \text{and} \quad a^n = 1) \qquad \Rightarrow \qquad a = 1. \tag{4.7}$$

Lemma 4.8. *Let G be a metabelian group as in* (4.1)*, N be a submodule of M and H be a normal subgroup of G. Then the following holds.*

(1) $N * I_n = \mathcal{P}_n(N)$;
(2) $\mathsf{p}_n(H\cdot N) \subseteq \mathsf{p}_n(H)\cdot(N * I_n)$;
(3) *if* $H\cap M \subseteq N$ *then* $[\mathsf{p}_n(H),G] \subseteq N * I_n$;
(4) *if* $H\cap M\subseteq N$ *and* $H \subseteq \pi^{-1}(A^{n^T})$ *then* $\mathsf{p}_n(\mathsf{p}_n(H))\cap M \subseteq N * I_n$;
(5) *if* $H\cap M\subseteq N$ *and* $H \subseteq \pi^{-1}(A^{n^T})$ *then* $\mathcal{P}_n(\mathcal{P}_n(H))\cap M \subseteq N * I_n$;

Proof. (1) It follows from the equalities $N * I = [N,G]$ and $N * (n) = \mathsf{p}_n(N)$.

(2) It follows from the equality $(hm)^n = h^n\cdot(\prod_{i=0}^{n-1} m^{h^i}) = h^n\cdot(m * (\sum_{i=0}^{n-1} h^i))$ and the fact that $\pi(\sum_{i=0}^{n-1} h^i) \in I_n$.

(3) Using the equalities $[x_1x_2,y] = [x_1,y]^{x_2}\cdot[x_2,y]$ and $[x^z,y] = [x,y^{z^{-1}}]^{z^{-1}}$ we get that the normal subgroup $[\mathsf{p}_n(H),G]$ is generated by the elements of the form $[h^n,g]$ as a normal subgroup, where $h\in H$ and $g \in G$. Moreover, we have

$$[h^n,g] = \prod_{i=0}^{n-1}[h,g]^{h^i} = [h,g] * (\sum_{i=0}^{n-1} h^i).$$

Using the inclusions $[G,G]\subseteq M$ and $H\cap M\subseteq N$ we obtain $[h,g]\in N$ and hence $[h^n,g] \in N * I_n$.

(4) Consider an element $x_1^n x_2^n \dots x_l^n \in \mathsf{p}_n(\mathsf{p}_n(H)) \cap M$ where $x_i \in \mathsf{p}_n(H)$. Thus $\pi(x_1^n x_2^n \dots x_l^n) = \pi(x_1 \dots x_l)^n = 1$. Since $H \subseteq \pi^{-1}(A^{n^t})$, using (4.7) we obtain $\pi(x_1 \dots x_l) = 1$ and hence $x_1 \dots x_l \in M$. Moreover, using the inclusion $H \cap M \subseteq N$, we obtain $x_1 \dots x_l \in N$. From the other hand, we have

$$x_1^n \dots x_l^n (x_1 \dots x_l)^{-n} \in [\mathsf{p}_n(H), \mathsf{p}_n(H)] \subseteq [\mathsf{p}_n(H), G] \subseteq N * I_n$$

and $(x_1 \dots x_l)^n \in N * (n) \subseteq N * I_n$. Therefore, $x_1^n \dots x_l^n \in N * I_n$.

(5) Since $H \cap M \subseteq N$ and $[G,G] \subseteq M$, we have $[H,G] \subseteq N$ and hence $[\mathcal{P}_n(H), G] \subseteq N$. Thus, we obtain

$$\mathcal{P}_n(\mathcal{P}_n(H)) \cap M = \big(\mathsf{p}_n(\mathcal{P}_n(H)) \cdot [\mathcal{P}_n(H), G]\big) \cap M = \big(\mathsf{p}_n(\mathcal{P}_n(H)) \cap M\big) \cdot [\mathcal{P}_n(H), G].$$

Therefore, we need to prove the inclusions $\mathsf{p}_n(\mathcal{P}_n(H)) \cap M \subseteq N * I_n$ and $[\mathcal{P}_n(H), G] \subseteq N * I_n$. Using the inclusion $[H,G] \subseteq N$, the inclusion (2), the inclusion $N * I_n \subseteq M$, and (4) we obtain

$$\mathsf{p}_n(\mathcal{P}_n(H)) \cap M = \mathsf{p}_n(\mathsf{p}_n(H) \cdot [H,G]) \cap M \subseteq \mathsf{p}_n(\mathsf{p}_n(H) \cdot N) \cap M \subseteq$$

$$\subseteq \big(\mathsf{p}_n(\mathsf{p}_n(H)) \cdot (N * I_n)\big) \cap M = \big(\mathsf{p}_n(\mathsf{p}_n(H)) \cap M\big) \cdot (N * I_n) \subseteq N * I_n.$$

□

Lemma 4.9. *The following inclusions hold for $i \geq 0$:*

$$M * I_n^i \subseteq \gamma_{i+1}^{[n]}(G) \cap M, \qquad \gamma_{2i+T+1}^{[n]}(G) \cap M \subseteq M * I_n^i.$$

Proof. The proof is by induction on i. The base is obvious for both cases. Suppose that these inclusions hold for $i = j$ and prove them for $i = j + i$. In order to prove the first inclusion we only need to prove $M * I_n^{j+1} \subseteq \gamma_{j+2}^{[n]}(G)$. Using lemma 4.8, we get the first required inclusion

$$M * I_n^{j+1} = \mathcal{P}_n(M * I_n^j) \subseteq \mathcal{P}_n(\gamma_j^{[n]}(G)) = \gamma_{j+1}^{[n]}(G).$$

Since $\gamma_{2j+T+1}^{[n]}(G) \subseteq \gamma_{T+1}^{[n]}(G) \subseteq \pi^{-1}(A^{n^T})$ and $\gamma_{2j+T+1}^{[n]}(G) \cap M \subseteq M * I_n^j$, we can use (5) of lemma 4.8 and obtain the second required inclusion

$$\gamma_{2j+2+T+1}^{[n]}(G) \cap M = \mathcal{P}_n(\mathcal{P}_n(\gamma_{2j+T+1}^{[n]}(G))) \cap M \subseteq (M * I_n^j) * I_n = M * I_n^{j+1}.$$

□

Corollary 4.10. $\hat{M}_{I_n} = \varprojlim M/(\gamma_i^{[n]}(G) \cap M)$.

Corollary 4.11. *For any $i, n \in \mathbb{N}$ we have the following.*

1. *Let $\mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials. Then*

$$t^{n^{2i+1}} \equiv 1 \mod (n, (t-1))^i.$$

2. *Let $a \in A$. Then $a^{n^{2i+1}} - 1 \in I_n^i$.*
3. *Let $\mathfrak{R}$ be an associative ring whose characteristic divides n, and r be an invertible element of $\mathfrak{R}$. Then the element $r^{n^{2i+1}} - 1$ is divisible by $(r-1)^i$.*

Proof. It is easy to see that (2) and (3) follow from (1). Prove (1). Let A be the infinite cyclic group $\langle t\rangle$. Then $\mathbb{Z}[A]=\mathbb{Z}[t,t^{-1}]$, and $T=0$. Consider the semidirect product $H:=\langle t\rangle\ltimes\mathbb{Z}[t,t^{-1}]$ and the short exact sequence

$$\mathbb{Z}[A]/(\gamma_{2i+1}^{[n]}(H)\cap\mathbb{Z}[A])\rightarrowtail H/\gamma_{2i+T+1}^{[n]}(H)\twoheadrightarrow A/A^{n^{2i+T+1}}.$$

Then the action of A on $\mathbb{Z}[t,t^{-1}]$ induces an action of $A/A^{n^{2i+T+1}}$ on $\mathbb{Z}[A]/(\gamma_i^{[n]}(H)\cap\mathbb{Z}[A])$. We have $I_n=(n,t-1)$. By Lemma 4.9 we get the action of A on $\mathbb{Z}[t,t^{-1}]$ induces an action of $A/A^{n^{2i+1}}$ on $\mathbb{Z}[t,t^{-1}]/I_n^i$. Hence $t^{n^{2i+1}}$ acts trivially on $\mathbb{Z}[t,t^{-1}]/(n,t-1)^i$ by multiplication. □

Corollary 4.11 implies that the multiplicative homomorphism $A\to\mathbb{Z}[A]$ induces a homomorphism $\theta_i':A/A^{n^{2i+T+1}}\to\mathbb{Z}[A]/I_n^i$. Then applying the inverse limit we get the continuous homomorphism

$$\theta_R:\hat{A}_n\to\widehat{\mathbb{Z}[A]}_{I_n}. \tag{4.8}$$

Observe that $\theta_R(x^{-1})=\hat{\sigma}(\theta_R(x))$, where $\sigma:\mathbb{Z}[A]\to\mathbb{Z}[A]$ is the antipode.

Proposition 4.12. *Let R be the ring $\mathbb{Z}/n$, G be a metabelian group as in* (4.1)*, where A is a finitely generated abelian group. Denote by $\hat{M}_{I_n}$ the I_n-adic completion of the $\mathbb{Z}[A]$-module M. Then there is a short exact sequence*

$$0\longrightarrow\hat{M}_{I_n}\longrightarrow\hat{G}_R\longrightarrow\hat{A}_n\longrightarrow 1,$$

whose morphisms are induced by (4.1)*, and the action by conjugation of $\hat{A}_n$ on $\hat{M}_{I_n}$ coincides with the action that induced by the structure of $\widehat{\mathbb{Z}[A]}_{I_n}$-module via the homomorphism $\theta_R:\hat{A}_n\to\widehat{\mathbb{Z}[A]}_{I_n}$.*

Proof. Denote $M_i=M/(\gamma_i^{[n]}(G)\cap M)$ and consider the short exact sequence $M_i\rightarrowtail G/\gamma_i^{[n]}(G)\twoheadrightarrow A/A^{n^i}$. The inverse sequence M_i satisfies Mittag-Leffler condition and hence we get the short exact sequence $\varprojlim M_i\rightarrowtail\hat{G}_R\twoheadrightarrow\hat{A}_n$. Finally, by lemma 4.9 we obtain the required isomorphism $\varprojlim M_i\cong\varprojlim M/(M*I_n^i)=\hat{M}_{I_n}$. Note that $M_i*I_n^i=0$. Then the module M_i has the natural structure of a $\widehat{\mathbb{Z}[A]}_{I_n}$-module. In order to prove that the action of $\hat{A}_n$ on M is induced by θ', we only need to prove it for M_i. But it is obvious, because $\hat{A}_n/\hat{A}_n^{n^i}\cong A/A^{n^i}$. □

Now we give a slight different formulation of Proposition 4.13 that will be convenient further. Let us set

$$J=\{p\mid p\text{ does not divide }n\},\qquad S=\mathbb{Z}[J^{-1}].$$

Then we have the isomorphism $S/n\cong\mathbb{Z}/n$. Consider the ideal

$$\mathcal{I}_S=\operatorname{Ker}(S[A]\twoheadrightarrow\mathbb{Z}/n).$$

Then we have the isomorphism $S[A]\cong(\mathbb{Z}[A])[J^{-1}]\cong S\otimes\mathbb{Z}[A]$ and the ideal $\mathcal{I}_S$ corresponds to $I_n[J^{-1}]$. Since elements of J are invertible modulo n^m, we get the isomorphism $S[A]/\mathcal{I}_S^m\cong\mathbb{Z}[A]/I_n^m\cong S\otimes\mathbb{Z}[A]/I_n^m$. It follows that $\widehat{S[A]}\cong\widehat{\mathbb{Z}[A]}_{I_n}$, where $\widehat{S[A]}=\widehat{S[A]}_{\mathcal{I}_S}$.

Similarly, we have $\hat{M}_{I_n} \cong (M \otimes S)^{\wedge}_{\mathcal{I}_S}$. Using the isomorphism $\widehat{S[A]} \cong \widehat{\mathbb{Z}[A]}_{I_n}$, we can write the homomorphism θ_R as follows

$$\theta_R : \hat{A}_n \longrightarrow \widehat{S[A]}. \tag{4.9}$$

Then we get the new version of Proposition 4.13:

Proposition 4.13. *Let R be the ring $\mathbb{Z}/n$, G be a finitely generated metabelian group as in (4.1). Then there is a short exact sequence*

$$0 \longrightarrow (M \otimes S)^{\wedge}_{\mathcal{I}_S} \longrightarrow \hat{G}_R \longrightarrow \hat{A}_n \longrightarrow 1,$$

whose morphisms are induced by (4.1), and the action by conjugation of $\hat{A}_n$ on $(M \otimes S)^{\wedge}_{\mathcal{I}_S}$ coincides with the action that induced by the structure of $\widehat{S[A]}$-module via the homomorphism $\theta_R : \hat{A}_n \to \widehat{S[A]}$.

Proposition 4.14. *Let G be a finitely generated metabelian group, p and q be different prime numbers and $\hat{G}_p$ be the p-profinite completion of G. Then*

$$H_2(\hat{G}_p, \mathbb{Z}/q) = 0 \qquad \text{and} \qquad H_2(G/\gamma_m^{[p^s]}(G), \mathbb{Z}/q) = 0,$$

for any $m, s \geq 1$.

Proof. We prove the first equality. The second equality can be proved similarly. By (4.6) and Proposition 4.13, we get the metabelian decomposition $\hat{M}_{I_p} \rightarrowtail \hat{G}_p \twoheadrightarrow \hat{A}_p$. Consider the corresponding Lyndon-Hochschild-Serre spectral sequence E. It is sufficient to prove that $E^2_{i,j} = 0$ for $(i,j) \in \{(0,2),(1,1),(2,0)\}$. Since $E^2_{0,2} = H_0(\hat{A}_p, H_2(\hat{M}_{I_p}, \mathbb{Z}/q))$, using the universal coefficient theorem and the equality $H_2(\hat{M}_{I_p}) = \wedge^2 \hat{M}_{I_p}$, we obtain the following exact sequence

$$(\wedge^2 \hat{M}_{I_p} \otimes \mathbb{Z}/q)_{\hat{A}_p} \to E^2_{0,2} \to \mathsf{Tor}(\hat{M}_{I_p}, \mathbb{Z}/q)_{\hat{A}_p} \to 0.$$

The groups $M/(M * I_p^m)$ are quotients of M/Mp^m, and hence they are uniquely q-divisible. It follows that $\hat{M}_{I_p} = \varprojlim M/(M * I_p^m)$ is an uniquely q-divisible abelian group. Thus $\hat{M}_{I_p} \otimes \mathbb{Z}/q = 0$, $(\wedge^2 \hat{M}_{I_p} \otimes \mathbb{Z}/q)_{\hat{A}_p} = 0$ and $\mathsf{Tor}(\hat{M}_{I_p}, \mathbb{Z}/q) = 0$. Therefore, $E^2_{0,2} = 0$ and

$$E^2_{1,1} = H_1(\hat{A}_p, H_1(\hat{M}_{I_p}, \mathbb{Z}/q)) = H_1(\hat{A}_p, \hat{M}_{I_p} \otimes \mathbb{Z}/q) = 0.$$

Similarly, using the universal coefficient theorem, we get

$$E^2_{2,0} = H_2(\hat{A}_p, H_0(\hat{M}_{I_p}, \mathbb{Z}/q)) = H_2(\hat{A}_p, \mathbb{Z}/q) = 0.$$

□

5. Homology of an abelian group with coefficients.

In this section K denotes a commutative Notherian ring, A denotes a finitely generated abelian group and M denotes a finitely generated $K[A]$-module. Since A is a finitely generated abelian group, $K[A]$ is a commutative Notherian ring. Denote by $I = I_K(A)$ the augmentation ideal of $K[A]$. Then by (3.1) we have

$$MI^\infty \cdot I = MI^\infty \tag{5.1}$$

We put $M_{\mathsf{rn}} = M/MI^\infty$, $\hat{M} = \hat{M}_I = \varprojlim M/MI^i$ and $M^\ell = M^\ell_I = M[(1+I)^{-1}]$.

Proposition 5.1. *Let X be an abelian group. Then the homomorphisms $M \to M_{\mathsf{rn}} \to M^{\ell} \to \hat{M}$ induce isomorphisms*

$$H_*(A, M \otimes X) \cong H_*(A, M_{\mathsf{rn}} \otimes X) \cong H_*(A, M^{\ell} \otimes X) \cong H_*(A, \hat{M} \otimes X)$$

$$H_*(A, \mathsf{Tor}(M, X)) \cong H_*(A, \mathsf{Tor}(M_{\mathsf{rn}}, X)) \cong H_*(A, \mathsf{Tor}(M^{\ell}, X)) \cong H_*(A, \mathsf{Tor}(\hat{M}, X))$$

and there is the following short exact sequence:

$$0 \longrightarrow \varprojlim{}^1 H_{m+1}(A, M/MI^i) \longrightarrow H_m(A, M) \longrightarrow \varprojlim H_m(A, M/MI^i) \longrightarrow 0,$$

where the epimorphism is induced by the projections $M \twoheadrightarrow M/MI^i$.

Proof. It follows from Lemma 3.1 and Proposition 2.6. □

Corollary 5.2. *If $M = MI$, then $H_*(A, M) = 0$.*

Corollary 5.3. *If K is an Artinian commutative ring, then the projections $M \twoheadrightarrow M/MI^i$ induce the isomorphism*

$$H_*(A, M) \cong \varprojlim H_*(A, M/MI^i).$$

Proof. The homology groups $H_{m+1}(A, M/MI^i)$ are finitely generated K-modules, and hence they are Artinian K-modules. It follows that the Mittag-Leffler condition holds for the inverse sequence $H_{m+1}(A, M/MI^i)$, and hence $\varprojlim{}^1 H_{m+1}(A, M/MI^i) = 0$. □

Lemma 5.4. *Let p be a prime number, $\mathsf{char}(K) = p$, M be a nilpotent $K[A/A^{p^i}]$-module such that $MI^{p^m-1} = 0$. Then the projection $A \twoheadrightarrow A/A^{p^i}$ induces an isomorphism*

$$H_*(A, M) \cong H_*(A/A^{p^i}, M).$$

Proof. First we prove it for the first homology. Consider the short exact sequence $I(A) \rightarrowtail K[A] \twoheadrightarrow K$. The associated long exact sequence gives us the four term exact sequence

$$H_1(A, M) \rightarrowtail M \otimes_{K[A]} I(A) \to M \twoheadrightarrow M_A.$$

Similarly, we get the same sequence for A/A^{p^i}. Since, $I(A/A^{p^i}) \otimes_{K[A/A^{p^i}]} M = I(A/A^{p^i}) \otimes_{K[A]} M$ and $M_A \cong M_{A/A^{p^i}}$, we obtain the morphism of exact sequences:

$$\begin{tikzcd}
H_1(A, M) \arrow[r, tail] \arrow[d] & M \otimes_{K[A]} I(A) \arrow[r] \arrow[d] & M \arrow[r, two heads] \arrow[d, equal] & M_A \arrow[d, equal] \\
H_1(A/A^{p^m}, M) \arrow[r, tail] & M \otimes_{K[A]} I(A/A^{p^i}) \arrow[r] & M \arrow[r, two heads] & M_A.
\end{tikzcd}$$

It is sufficient to prove that the morphism $M \otimes_{K[A]} I(A) \to M \otimes_{K[A]} I(A/A^{p^i})$ is an isomorphism. Since, the functor $M \otimes_{K[A]} -$ is right exact, it is an epimorphism. The kernel of this morphism is generated by elements of the form $m \otimes (a^{p^i} - 1)$. They are equal to zero, because $MI^{p^i-1} = 0$, $a^{p^i} - 1 = (a-1)^{p^i}$ and $m \otimes (a-1)^{p^i} = m(a-1)^{p^i-1} \otimes (a-1) = 0$.

Now we generalize it for H_k using induction by k. Assume that Lemma holds for $k-1$. We have $M \otimes_{K[A]} I(A) \cong M \otimes_{K[A]} I(A/A^{p^i})$. Then using shift in homology and the assumption we get $H_k(A,M) \cong H_{k-1}(A, M \otimes_{K[A]} I(A)) \cong H_{k-1}(A, M \otimes_{K[A]} I(A/A^{p^i})) \cong H_{k-1}(A/A^{p^i}, M \otimes_{K[A]} I(A/A^{p^i})) \cong H_k(A/A^{p^i}, M)$. □

6. Notation and unification.

In this section we introduce the notation that we use in the rest of the paper. A denotes a finitely generated abelian group, M denotes a finitely generated $\mathbb{Z}[A]$-module, R denotes a fixed ring of the form $\mathbb{Z}[J^{-1}]$ or $\mathbb{Z}/n$, $R[A]$ denotes the group algebra of A over R, and I_R denotes the augmentation ideal of $R[A]$. Moreover, we denote

$$J = \begin{cases} J, & \text{if } R = \mathbb{Z}[J^{-1}] \\ \{p \mid p \nmid n\}, & \text{if } R = \mathbb{Z}/n \end{cases},$$

$$S = \mathbb{Z}[J^{-1}], \qquad \mathcal{I}_S = \begin{cases} I_R, & \text{if } R = \mathbb{Z}[J^{-1}] \\ I_S, & \text{if } R = \mathbb{Z}/n \end{cases},$$

$$\hat{M}_R = (M \otimes R)^{\wedge}_{I_R}, \qquad \hat{M}_S = (M \otimes S)^{\wedge}_{\mathcal{I}_S}.$$

$$M^{\ell}_R = (M \otimes R)[(1 + I_R)^{-1}], \qquad M^{\ell}_S = (M \otimes S)[(1 + \mathcal{I}_S)^{-1}].$$

Observe that there is the unique epimorphism $S \twoheadrightarrow R$. In general $\mathcal{I}_S$ is not the augmentation ideal of $S[A]$, but the epimorphism $S[A] \twoheadrightarrow R[A]$ takes $\mathcal{I}_S$ to I_R. It follows that there is a continuous epimorphism of completions $\widehat{S[A]} \twoheadrightarrow \widehat{R[A]}$, and the epimorphism of localizations $S[A]^{\ell} \twoheadrightarrow R[A]^{\ell}$, where

$$\begin{aligned} \widehat{S[A]} &:= \widehat{S[A]}_{\mathcal{I}_S}, \\ \widehat{R[A]} &:= \widehat{R[A]}_{I_R}, \\ S[A]^{\ell} &:= S[A][(1 + \mathcal{I}_S)^{-1}], \\ R[A]^{\ell} &:= R[A][(1 + I_R)^{-1}]. \end{aligned}$$

By $\hat{A}_R$ we denote $A \otimes R$ in the case of $R = \mathbb{Z}[J^{-1}]$, and $\hat{A}_n = \varprojlim A/A^{n^i}$ in the case of $R = \mathbb{Z}/n$.

We denote by $\sigma : S[A] \to S[A]$ the standard antipode i.e. the S-linear map with $\sigma(a) = a^{-1}$ for $a \in A$. In (4.4) and (4.9) we defined the multiplicative homomorphism

$$\theta_R : \hat{A}_R \longrightarrow \widehat{S[A]},$$

such that $\hat{\sigma}(\theta_R(x)) = \theta_R(x^{-1})$. Then we can consider $\hat{M}_S$ a $\mathbb{Z}\hat{A}_R$-module. By G we denote a finitely generated metabelian group with a metabelian decomposition

$$0 \longrightarrow M \longrightarrow G \longrightarrow A \longrightarrow 1 \tag{6.1}$$

Further, we put

$$\gamma_i^R(G) = \begin{cases} \gamma_i(G), & \text{if } R = \mathbb{Z}[J^{-1}] \\ \gamma_i^{[n]}(G), & \text{if } R = \mathbb{Z}/n \end{cases}, \qquad t_i^R(G) = \begin{cases} (G/\gamma_i(G)) \otimes R, & \text{if } R = \mathbb{Z}[J^{-1}] \\ G/\gamma_i^{[n]}(G), & \text{if } R = \mathbb{Z}/n \end{cases}$$

and

(6.2)

$$t_i^R(A) = \begin{cases} A \otimes R, & \text{if } R = \mathbb{Z}[J^{-1}] \\ A/A^{n^i}, & \text{if } R = \mathbb{Z}/n \end{cases}, \qquad \mathcal{M}^i = \begin{cases} (M \otimes R)/(\gamma_i(G) \otimes R), & \text{if } R = \mathbb{Z}[J^{-1}] \\ M/(\gamma_i^{[n]}(G) \cap M), & \text{if } R = \mathbb{Z}/n. \end{cases}$$

Then the R-completion of G is defined as follows

$$\hat{G}_R = \varprojlim \ t_i^R(G),$$

and there are short exact sequences

$$0 \longrightarrow \mathcal{M}^i \longrightarrow t_i^R(G) \longrightarrow t_i^R(A) \longrightarrow 1. \tag{6.3}$$

Then Propositions 4.7 and 4.13 can be rehash as follows.

Proposition 6.1. *There is a short exact sequence*

$$0 \longrightarrow \hat{M}_S \longrightarrow \hat{G}_R \longrightarrow \hat{A}_R \longrightarrow 1,$$

whose morphisms are induced by the sequence (6.1) *and the action by conjugation of* $\hat{A}_R$ *on* $\hat{M}_S$ *coincides with the action that induced by the structure of* $\widehat{S[A]}$*-module via the homomorphism* $\theta_R : \hat{A}_R \to \widehat{S[A]}$.

By K we denote an Artinian quotient ring of R. In other words,

(1) if $R = \mathbb{Q}$ then $K = \mathbb{Q}$;
(2) if $R = \mathbb{Z}[J^{-1}]$, then $K = \mathbb{Z}/n$, where all the primes of n do not lie in J;
(3) if $R = \mathbb{Z}/n$ then $K = \mathbb{Z}/n'$, where n' is a divisor of n.

Then we have epimorphisms

$$S \twoheadrightarrow R \twoheadrightarrow K.$$

Denote by I_K the augmentation ideal of the group algebra $K[A]$. Then the epimorphism $R[A] \twoheadrightarrow K[A]$ takes I_R to I_K, and hence we have the continuous epimorphisms of completions $\widehat{S[A]} \twoheadrightarrow \widehat{R[A]} \twoheadrightarrow \widehat{K[A]}$ and the epimorphisms of localizations $S[A]^\ell \twoheadrightarrow R[A]^\ell \twoheadrightarrow K[A]^\ell$. Further, we denote

$$M_K = M \otimes K, \qquad \hat{M}_K = (M_K)^\wedge_{I_K}, \qquad M_K^\ell = M_K[(1+I_K)^{-1}].$$

Note that, since M is finitely generated, we have the isomorphisms

$$\hat{M}_K \cong \hat{M}_S \otimes K, \qquad M_K^\ell \cong M_S^\ell \otimes K. \tag{6.4}$$

By N we denote an arbitrary finitely generated $K[A]$-module. Endow the module $\hat{N} = \hat{N}_{I_K}$ by the structure of $\mathbb{Z}[\hat{A}_R]$-module using the homomorphism $\theta_R : \hat{A}_R \to \widehat{S[A]}$ and $\widehat{S[A]} \twoheadrightarrow \widehat{K[A]}$. Then by (3.3) we get

$$N_A \cong \hat{N}_A = \hat{N}_{\hat{A}_R} \cong \hat{N} \otimes_{\widehat{K[A]}} K, \tag{6.5}$$

where $(-)_A = H_0(A,-)$ and $(-)_{\hat{A}_R} = H_0(\hat{A}_R,-)$.

For K-modules N_1, N_2 there is an isomorphism $N_1 \otimes_K N_2 \cong N_1 \otimes N_2$, where $\otimes = \otimes_{\mathbb{Z}}$. The same holds for S and R. It follows that $\wedge^2_{\sigma_K} N \cong \wedge^2_\sigma N$ for an $K[A]$-module N. From the other side, it is easy to check that $\wedge^2_\sigma M \cong (\wedge^2 M)_A$. Then we have

$$\wedge^2_{\sigma_K} N \cong \wedge^2_\sigma N \cong (\wedge^2 N)_A, \qquad \wedge^2_\sigma M \cong (\wedge^2 M)_A. \tag{6.6}$$

For abelian groups M_1, M_2 there is an isomorphism $(M_1 \otimes K) \otimes_R (M_2 \otimes K) \cong (M_1 \otimes M_2) \otimes K$. It implies the isomorphism $\wedge^2_{\sigma_K} M_K \cong (\wedge^2_\sigma M) \otimes K$. Then we get the isomorphism

$$(\wedge^2 M_K)_A \cong (\wedge^2 M)_A \otimes K. \tag{6.7}$$

7. Exterior squares and tame modules.

Remind the outcome of [3], [4] concerned with tame modules. A valuation of the group A is a homomorphism $v : A \to \mathbb{R}$ into the additive group of $\mathbb{R}$. The valuation monoid of v is the submonoid $A_v = \{a \in A \mid v(a) \geq 0\}$. The group of valuations $\mathsf{Hom}(A,\mathbb{R})$ has the natural structure of a real vector space and quotient space $\mathsf{S}(A) = \big(\mathsf{Hom}(A,\mathbb{R}) \setminus \{0\}\big)/\mathbb{R}_+$ is called the valuation sphere of A.

Let M be a finitely generated $\mathbb{Z}[A]$-module. The Bieri–Strebel invariant of M is the set $\Sigma(M) \subseteq \mathsf{S}(A)$ consisted of rays $[v]$ such that M is a finitely generated A_v-module. The equality $\Sigma(M) = \mathsf{S}(A)$ holds if and only if M is a finitely generated as an abelian group [3, theorem 2.1]. The module M is said to be tame if $\Sigma(M) \cup (-\Sigma(M)) = \mathsf{S}(A)$. The main result of the article [3] says that G is finitely presented if and only if M is a tame A-module. Moreover, it is proved in [3] that

$$\Sigma(M) = \Sigma(\mathbb{Z}[A]/\mathsf{Ann}\, M) \tag{7.1}$$

and that there is an implication

$$\mathsf{Ann}\, M_1 \subseteq \mathsf{Ann}\, M_2 \quad \Rightarrow \quad \Sigma(M_1) \subseteq \Sigma(M_2), \tag{7.2}$$

where $\mathsf{Ann} = \mathsf{Ann}_{\mathbb{Z}[A]}$. For finitely generated A-modules M_1 and M_2 the inclusions $\mathsf{Ann}(M_1 \otimes_{\mathbb{Z}[A]} M_2) \supseteq \mathsf{Ann}\, M_1$ and $\mathsf{Ann}(M_1 \otimes_{\mathbb{Z}[A]} M_2) \supseteq \mathsf{Ann}\, M_2$ and the implication (7.2) imply the inclusion $\Sigma(M_1 \otimes_{\mathbb{Z}[A]} M_2) \supseteq \Sigma(M_1) \cup \Sigma(M_2)$ and in particular

$$\Sigma(\mathbb{Z}[A]/(\mathfrak{a} + \mathfrak{b})) \supseteq \Sigma(\mathbb{Z}[A]/\mathfrak{a}) \cup \Sigma(\mathbb{Z}[A]/\mathfrak{b}) \tag{7.3}$$

for any ideals $\mathfrak{a}, \mathfrak{b} \lhd \mathbb{Z}[A]$. The $\mathbb{Z}[A]$-module M is finitely generated over A_v if and only if M_σ is finitely generated over A_{-v}. It follows that

$$\Sigma(M_\sigma) = -\Sigma(M). \tag{7.4}$$

Lemma 7.1. *If M is a tame $\mathbb{Z}[A]$-module, then*

- *$(\wedge^2 M)_A$ is a finitely generated abelian group,*
- *$H_2(M,K)_A$ is a finitely generated K-module.*

Proof. Since $\Sigma((\wedge^2 M)_A) = \Sigma(\wedge^2_\sigma M) \supseteq \Sigma(M) \cup (-\Sigma(M)) = \mathsf{S}(A)$, we get that $(\wedge^2 M)_A$ is a finitely generated abelian group. Then $(\wedge^2 M_K)_A$ is a finitely generated K-module. The exact sequence $(\wedge^2 M_K)_A \to H_2(M,K)_A \to (M_K)_A \to 0$ implies that $H_2(M,K)_A$ is an extension of finitely generated K-modules, and hence it is finitely generated itself. □

Proposition 7.2. *Let $R = \mathbb{Z}[J^{-1}]$ or $R = \mathbb{Z}/n$, M be a tame $\mathbb{Z}[A]$-module, K be an Artinian quotient ring of R. Then for $m \gg 0$ there are isomorphisms*

$$(\wedge^2 \hat{M}_K)_{\hat{A}_R} = (\wedge^2 \hat{M}_K)_A \cong (\wedge^2 M_K^\ell)_A \cong (\wedge^2 (M_K/M_K I_K^m))_A \cong (\wedge^2 \mathcal{M}_K^m)_{t_m^R(A)},$$

where $\mathcal{M}_K^m = \mathcal{M}^m \otimes K$.

Proof. First we prove that the assumptions of the Proposition 3.3 and Corollary 3.5 for the ring $K[A]$ the ideal I_K and the module M_K hold. We need to prove that

$$\mathsf{cl}(\mathsf{Ann}_{K[A]}\, M_K + \sigma_K(\mathsf{Ann}_{K[A]}\, M_K)) \supseteq I_K^{m_0}$$

for some $m_0 \in \mathbb{N}$. Denote

$$\mathfrak{a}_K := \mathsf{Ann}_{K[A]} N + \sigma_K(\mathsf{Ann}_{K[A]} N)$$
$$\mathfrak{a} := \mathsf{Ann}\, M + \sigma(\mathsf{Ann}\, M).$$

Using that M is tame, (7.3) and (7.4) we get

$$\Sigma(\mathbb{Z}[A]/\mathfrak{a}) \supseteq \Sigma(\mathbb{Z}[A]/\mathsf{Ann}\, M) \cup (-\Sigma(\mathbb{Z}[A]/\mathsf{Ann}\, M)) = \mathsf{S}(A).$$

Thus $\mathbb{Z}[A]/\mathfrak{a}$ is a finitely generated abelian group. Since the map $(\mathbb{Z}[A]/\mathfrak{a})\otimes K \twoheadrightarrow K[A]/\mathfrak{a}_K$ is an epimorphism, $K[A]/\mathfrak{a}_K$ is a finitely generated K-module. Using that K is an Artinian ring, we get that $K[A]/\mathfrak{a}_K$ is an Artinian K-module, and hence the sequence $\mathfrak{a}_K + I_K^m$ stabilizes. Therefore, $\mathsf{cl}(\mathfrak{a}_K) \supseteq I_K^{m_0}$ for some $m_0 \in \mathbb{N}$.

Hence, by Corollary 3.5, for $m \gg 0$, we have the isomorphisms

$$(\wedge^2 \hat{M}_K)_A \cong (\wedge^2 M_K^{\ell})_A \cong (\wedge^2 (M_K/M_K I_K^m))_A.$$

Using Lemma 4.9 and (4.5) we get epimorphisms $M_K/M_K I_K^{s(m)} \twoheadrightarrow \mathcal{M}_K^{t(m)} \twoheadrightarrow M_K/M_K I_K^m$ for some sequences $s(m), t(m)$ that converge to infinity. For a big enough m the epimorphism $M_K/M_K I_K^{s(m)} \twoheadrightarrow M_K/M_K I_K^m$ induces an isomorphism

$$(\wedge^2 (M_K/M_K I_K^{s(m)}))_A \cong (\wedge^2 (M_K/M_K I_K^m))_A.$$

It follows that the epimorphism $\mathcal{M}_K^{t(m)} \twoheadrightarrow M_K/M_K I_K^m$ induces an isomorphism

$$(\wedge^2 \mathcal{M}_K^m)_A \cong (\wedge^2 (M_K/M_K I_K^m))_A \cong (\wedge^2 \hat{M}_K)_A$$

for $m \gg 0$.

Note that $(\wedge^2 \mathcal{M}_K^m)_{t_m^R(A)} = (\wedge^2 \mathcal{M}_K^m)_{\hat{A}_R}$. Then we only need to prove that $(\wedge^2 \hat{M}_K)_{\hat{A}_R} = (\wedge^2 \hat{M}_K)_A$. For this it is sufficient to prove that for $x \in \hat{A}_R$ and $m_1 \wedge m_2 \in (\wedge^2 \hat{M}_K)_A$ the identity $m_1 x \wedge m_2 x = m_1 \wedge m_2$ holds. The abelian group $(\wedge^2 \hat{M}_K)_A$ is a quotient of $\hat{M}_K \otimes_{K[A]} (\hat{M}_K)_\sigma$. By Proposition 3.3 we have $\hat{M}_K \otimes_{K[A]} (\hat{M}_K)_\sigma = \hat{M}_K \otimes_{\widehat{K[A]}} (\hat{M}_K)_{\hat{\sigma}}$ and by Proposition 6.1 the action of $\hat{A}_R$ on $\hat{M}_K$ is induced by the structure of $\widehat{S[A]}$-module via the homomorphism $\theta_R : \hat{A}_R \to \widehat{S[A]}$ and $\widehat{S[A]} \twoheadrightarrow \widehat{K[A]}$. Hence,

$$m_1 x \wedge m_2 x = m_1 \theta_R(x) \wedge m_2 \theta_R(x) = m_1 \wedge m_2 \hat{\sigma}(\theta_R(x))\theta_R(x) =$$
$$= m_1 \wedge m_2 \theta_R(x^{-1})\theta_R(x) = m_1 \wedge m_2.$$

□

8. The limit formula.

Theorem 8.1. *Let G be a finitely presented metabelian group, $R = \mathbb{Z}[J^{-1}]$ or $R = \mathbb{Z}/n$ and K be an Artinian quotient ring of R. Then the homomorphisms $\hat{G}_R \to t_i^R(G)$ and $G/\gamma_i^R \to t_i^R(G)$ induce the isomorphisms*

$$H_2(\hat{G}_R, K) \cong \varprojlim H_2(t_i^R(G), K) \cong \varprojlim H_2(G/\gamma_i^R(G), K).$$

Corollary 8.2. *Let G be a finitely presented metabelian group, and p is a prime number. Then*

$$H_2(\hat{G}_p, \mathbb{Z}/p) \cong H_2^{\mathsf{cont}}(\hat{G}_p, \mathbb{Z}/p).$$

Proof. If follows from Theorem 8.1 and the formula $H_*^{\mathsf{cont}}(\hat{G}_p, \mathbb{Z}/p) \cong \varprojlim H_*(G/\gamma_i^{[p]}(G), \mathbb{Z}/p)$ [20]. □

Corollary 8.3. *Let $R = \mathbb{Z}[J^{-1}] \neq \mathbb{Q}$ and $n \in \mathbb{N}$, such that the prime divisors of n do not lie in J. Then*

$$H_2(\hat{G}_R, \mathbb{Z}/n) \cong H_2(\hat{G}_{\mathbb{Z}}, \mathbb{Z}/n).$$

Proof of Theorem 8.1. First we note that on the category of finitely generated K-modules the functor

$$\varprojlim : \mathsf{mod}(K)^{\omega^{\mathrm{op}}} \to \mathsf{mod}(K)$$

is an exact functor, because the Mittag-Leffler condition holds for all inverse sequences. In the proof we use $\varprojlim$ only in this category, and we use the exactness. Further, by Proposition 4.2 we have $H_2(t_i^R(G), K) \cong H_2(G/\gamma_i^R(G), K)$. Hence we only need to prove the isomorphism $H_2(\hat{G}_R, K) \cong \varprojlim H_2(t_i^R(G), K)$.

We reduce the theorem to a prime n in the case of $R = \mathbb{Z}/n$. Assume that the theorem holds for the case of $R = \mathbb{Z}/p$, where p is prime. Let now $R = \mathbb{Z}/n$, where $n = p_1^{s_1} \cdot \ldots \cdot p_l^{s_l}$. Then we have isomorphisms

$$\hat{G}_{\mathbb{Z}/n} \cong \prod \hat{G}_{\mathbb{Z}/p_j} \quad \text{and} \quad G/\gamma_i^{[n]}(G) \cong \prod_j G/\gamma_i^{[p_j^{s_j}]}(G)$$

[6, 12.3]. Let p be one of the prime divisors. Since $\mathbb{Z}/p$ is a field, we have the isomorphism

$$H_2(\hat{G}_{\mathbb{Z}/n}, \mathbb{Z}/p) \cong \bigoplus_{i_1+\dots+i_l=2} \bigotimes_{j=1}^{l} H_{i_j}(\hat{G}_{\mathbb{Z}/p_j}, \mathbb{Z}/p).$$

If $p_j \neq p$, by Proposition 4.14 we have $H_2(\hat{G}_{\mathbb{Z}/p_i}, \mathbb{Z}/p) = 0$ and obviously $H_1(\hat{G}_{\mathbb{Z}/p_i}, \mathbb{Z}/p) = 0$. Thus $H_2(\hat{G}_{\mathbb{Z}/n}, \mathbb{Z}/p) \cong H_2(\hat{G}_{\mathbb{Z}/p}, \mathbb{Z}/p)$. Similarly we get

$$H_2(G/\gamma_i^{[n]}(G), \mathbb{Z}/p) \cong H_2(G/\gamma_i^{[p^s]}(G), \mathbb{Z}/p).$$

Since $\gamma_i^{[p]}(G) \supseteq \gamma_i^{[p^s]}(G) \supseteq \gamma_{i'}^{[p]}(G)$ for any i, we get $\varprojlim H_2(G/\gamma_i^{[p^s]}(G), \mathbb{Z}/p) = H_2(G/\gamma_i^{[p]}(G), \mathbb{Z}/p)$. Then we obtain the isomorphism

$$H_2(\hat{G}_{\mathbb{Z}/n}, \mathbb{Z}/p) \cong \varprojlim H_2(G/\gamma_i^{[n]}(G), \mathbb{Z}/p).$$

Then the theorem holds for $R = \mathbb{Z}/n$ and $K = \mathbb{Z}/p$, where p is a divisor of n. Further, using the short exact sequence $\mathbb{Z}/p^i \rightarrowtail \mathbb{Z}/p^{i+1} \twoheadrightarrow \mathbb{Z}/p$, and the associated long exact sequence of homology $H_*(\hat{G}_{\mathbb{Z}/n}, -)$ and $H_*(G/\gamma_i^{[n]}(G), -)$ by induction we get the theorem for $R = \mathbb{Z}/n$ and $K = \mathbb{Z}/p^m$. Finally, for $K = \mathbb{Z}/n' = \bigoplus \mathbb{Z}/p_i^{m_i}$, we have $H_2(\hat{G}_{\mathbb{Z}/n}, K) = \bigoplus H_2(\hat{G}_{\mathbb{Z}/n}, \mathbb{Z}/p_i^{m_i})$. Therefore, the theorem holds in the general case for $R = \mathbb{Z}/n$. Further, we will assume that in the case of $R = \mathbb{Z}/n$ that $n = p$ is prime.

Let E be a first quadrant homological spectral sequence that converges to $\mathcal{H}_*$. If we are interested only in $\mathcal{H}_m$ for $0 \le m \le 2$ it is convenient to cut off the spectral sequence as follows:

$$\tilde{E}^r_{pq} := \begin{cases} E^r_{pq}, & \text{if } q \in \{0,1\} \text{ or } (p,q) = (0,2) \\ 0, & \text{else} \end{cases}$$

with the obvious differentials. Then $\tilde{E}$ has a limit $\tilde{\mathcal{H}}_*$ such that

$$\tilde{\mathcal{H}}_m \cong \mathcal{H}_m, \quad \text{for } 0 \le m \le 2.$$

Consider the morphism of metabelian decompositions:

$$\begin{tikzcd} 0 \arrow[r] & \hat{M}_S \arrow[r] \arrow[d] & \hat{G}_R \arrow[r] \arrow[d] & \hat{A}_R \arrow[r] \arrow[d] & 1 \\ 0 \arrow[r] & \mathcal{M}^i \arrow[r] & t_i^R(G) \arrow[r] & t_i^R(A) \arrow[r] & 1. \end{tikzcd} \tag{8.1}$$

It induces the morphism of Lyndon-Hochschild-Serre spectral sequences $E(\hat{G}_R) \to E(t_i^R(G))$ for homology with coefficients in K. If we cut off them we get the morphism $\tilde{E}(\hat{G}_R) \to \tilde{E}(t_i^R(G))$. We assume that these spectral sequences start from the second page.

We prove that all the terms of the spectral sequence $E(t_i^R(G))$ are finitely generated K-modules. We only need to prove it for the second page. For first two rows it is obvious and for $(p,q) = (0,2)$ it is the statement of Lemma 7.1.

Therefore $\varprojlim$ is exact on the terms of $\tilde{E}(t_i^R(G))$, and hence we can apply it to all the terms of $\tilde{E}(t_i^R(G))$ and get a new spectral sequence

$$\tilde{E}^{\lim} = \varprojlim \tilde{E}(t_i^R(G)).$$

The morphisms $\tilde{E}(\hat{G}_R) \to \tilde{E}(t_i^R(G))$ induce the morphism

$$\tilde{E}(\hat{G}_R) \longrightarrow \tilde{E}^{\lim}.$$

In order to finish the prove it is sufficient to prove that the morphism $\tilde{E}(\hat{G}_R) \to \tilde{E}^{\lim}$ is an isomorphism of spectral sequences. It is enough to prove it on the second pages. In other words we need to prove that the morphisms

$$H_*(\hat{A}_R, K) \longrightarrow \varprojlim H_*(t_i^R(A), K), \tag{8.2}$$

$$H_*(\hat{A}_R, M_K) \longrightarrow \varprojlim H_*(t_i^R(A), \mathcal{M}_K^i), \tag{8.3}$$

$$H_2(\hat{M}_S, K)_{\hat{A}_R} \longrightarrow \varprojlim H_2(\mathcal{M}^i, K)_{t_i^R(A)} \tag{8.4}$$

are isomorphisms. Note that the homomorphism (8.2) is a special case of (8.3). Then need to prove that the homomorphism (8.3) and the homomorphism (8.4) are isomorphisms. We prove it in Propositions 8.8 and 8.9. □

We denote the natural homomorphisms

$$\tau_A : A \longrightarrow \hat{A}_R, \qquad \tau_M : M \longrightarrow \hat{M}_S$$

and

$$A_1 := \mathrm{Ker}(\tau_A), \qquad A_2 := \mathrm{Coker}(\tau_A).$$

By $\mathsf{t}_p(A)$ we denote the p-power torsion subgroup of A.

Lemma 8.4. (1) $A_1 = \bigoplus_{p \in J} \mathsf{t}_p(A)$.

(2) *If* $R = \mathbb{Z}[J^{-1}]$, *then* $A_2 = (\mathbb{Z}[J^{-1}]/\mathbb{Z}) \otimes A$

(3) *If $R = \mathbb{Z}/p$, then $A_2 = (\mathbb{Z}_p/\mathbb{Z}) \otimes A$. Moreover, if A is finite, then $A_2 = 0$, else $A_2 \cong \mathbb{Q}^{\oplus \mathbf{c}}$.*

Proof. The only non-obvious thing is the last isomorphism. First, we prove that $\mathbb{Z}_p/\mathbb{Z}$ is a divisible abelian group. Let $q \neq p$ be a prime. The group $\mathbb{Z}_p$ is q-divisible, and hence $\mathbb{Z}_p/\mathbb{Z}$ is q-divisible. Then we need to prove that $\mathbb{Z}_p/\mathbb{Z}$ is p-divisible. It follows from the following equality modulo $\mathbb{Z}$: $\sum_{i=0}^{\infty} \alpha_i p^i = \sum_{i=1}^{\infty} \alpha_i p^i = p(\sum_{i=1}^{\infty} \alpha_i p^{i-1})$. Therefore, $\mathbb{Z}_p/\mathbb{Z}$ is a divisible torsion-free group. Then by description of divisible groups [12, IV] we get $\mathbb{Z}_p/\mathbb{Z} \cong \mathbb{Q}^{\oplus \mathbf{c}}$. Then the required statement follows immediately. □

Lemma 8.5.

(1) *If H a finite group such that $|H|^{-1} \in [J^{-1}]$ and L is an $S[H]$-module, then $H_k(H, L) = 0$ for $k > 0$.*
(2) *If L is a $S[A_1]$-module, then $H_k(A_1, L) = 0$ for $k > 0$.*
(3) *If L is a residually nilpotent $K[A_2]$-module, then $H_k(A_2, L) = 0$ for $k > 0$.*

Proof. (1) The trivial $S[H]$-module S is projective because it is isomorphic to the direct summand of $S[H]$ given by the image of the projector $x \mapsto (\sum_{h\in H} xh)/|H|$. Therefore, for any $S[H]$-module L we have $H_k(H, L) = \mathsf{Tor}_k^{S[H]}(S, L) = 0$.

(2) It follows from Lemma 8.4 and (1).

(3) Let $R = \mathbb{Z}[J^{-1}]$. Then $A_2 = (\mathbb{Z}[J^{-1}]/\mathbb{Z}) \otimes A$. Since $\mathbb{Z}[J^{-1}]/\mathbb{Z} = \bigoplus_{q\in J} \mathbb{Z}/q^{\infty}$, we get $A_2 = (\mathbb{Z}[J^{-1}]/\mathbb{Z})^d \oplus B_0$, where B_0 is a finite group such that $|B_0|^{-1} \in \mathbb{Z}[J^{-1}]$. The group $\mathbb{Z}[J^{-1}]/\mathbb{Z}$ is isomorphic to the direct limit $\varinjlim \mathbb{Z}/j_i$, where j_i runs over natural numbers with prime divisors in J such that for any natural number j with prime divisors in J we have $j \mid j_i$ for $i \gg 0$. Therefore, A_2 is isomorphic to the direct limit $\varinjlim B_i$, where B_i is a finite abelian group such that $|B_i|^{-1} \in \mathbb{Z}[J^{-1}]$. Using (1) and the epimorphism $S \twoheadrightarrow K$ we get $H_k(B_i, L) = 0$. Finally, using the formula $H_k(A_2, L) = \varinjlim H_k(B_i, L)$ we get $H_k(A_2, L) = 0$.

Let $R = \mathbb{Z}/p$. If $A_2 = 0$, the statement is obvious, then we can assume $A_2 \cong \mathbb{Q}^{\oplus \mathbf{c}}$ and $K = \mathbb{Z}/p$. Hence for an element a of A_2 there is an element $a_1 \in A_2$ such that $a = a_1^p$. Using the equality $a - 1 = a_1^p - 1 = (a_1 - 1)^p \bmod p$ we get $LI_R = LI_R^p = LI_R^{\infty} = 0$, and hence the action of A_2 on L is trivial. Since A_2 is torsion-free, we have $H_k(A_2) = \wedge^k A_2$. Then $H_k(A_2) \otimes L = 0$ and $\mathsf{Tor}(H_{k-1}(A_2), L) = 0$, and hence by universal coefficient theorem $H_k(A_2, L) = 0$. □

Lemma 8.6. *The homomorphisms τ_A and τ_M induce isomorphisms*

$$H_*(A, M_K) \cong H_*(A, \hat{M}_K) \cong H_*(\hat{A}_R, \hat{M}_K).$$

Proof. The first isomorphism we get by Proposition (5.1). Consider, the short exact sequence $A_1 \rightarrowtail A \twoheadrightarrow \mathrm{Im}(\tau_A)$, and the corresponding Lyndon-Hochschild-Serre spectral sequence $H_i(\mathrm{Im}(\tau_A), H_j(A_1, \hat{M}_K)) \Rightarrow H_{i+j}(A, \hat{M}_K)$. By Lemma 8.5 we get $H_j(A_1, \hat{M}_K) = 0$ for $j > 0$. Moreover, since $\hat{M}_K$ has the natural structure of a $\hat{A}_R$-module that lifts the structure of A-module, then A_1 acts trivially on $\hat{M}_K$, and hence $H_0(A_1, \hat{M}_K) = \hat{M}_K$. It follows that the homomorphism $A \to \mathrm{Im}(\tau_A)$ induce the isomorphism $H_*(A, \hat{M}_K) \cong H_*(\mathrm{Im}(\tau_A), \hat{M}_K)$.

Consider the short exact sequence $\mathrm{Im}(\tau_A) \rightarrowtail \hat{A}_R \twoheadrightarrow A_2$, and the corresponding Lyndon-Hochschild-Serre spectral sequence $H_i(A_2, H_j(\mathrm{Im}(\tau_A), \hat{M}_K)) \Rightarrow H_{i+j}(\hat{A}_R, \hat{M}_K)$. Since M_K is a finitely generated $K[A]$-module, M_K is a finitely generated $K[\mathrm{Im}(\tau_A)]$-module. Using the formula (5.1) we get $H_j(\mathrm{Im}(\tau_A),\hat{M}_K) = H_j(\mathrm{Im}(\tau_A),M_K)$. Using the formula (6.5) and the fact the action of A on $H_j(\mathrm{Im}(\tau_A),\hat{M}_K)$ is trivial, we get

$$H_j(\mathrm{Im}(\tau_A),\hat{M}_K) = H_j(\mathrm{Im}(\tau_A),\hat{M}_K)_A = H_j(\mathrm{Im}(\tau_A),\hat{M}_K)_{\hat{A}_R}.$$

Thus, $H_j(\mathrm{Im}(\tau_A),\hat{M}_K)$ is a trivial $\hat{A}_R$-module. Then by Lemma 8.5 we get $H_i(A_2,H_j(\mathrm{Im}(\tau_A),\hat{M}_K)) = 0$ for $i > 0$ and $H_0(A_2,H_j(\mathrm{Im}(\tau_A),\hat{M}_K)) = H_j(\mathrm{Im}(\tau_A),\hat{M}_K)$. It follows that the homomorphism $\mathrm{Im}(\tau_A) \rightarrowtail \hat{A}_R$ induces the isomorphism $H_*(\mathrm{Im}(\tau_A),\hat{M}_K) \cong H_*(\hat{A}_R,\hat{M}_K)$. It follows that the morphism τ_A induces the isomorphism $H_*(A,\hat{M}_K) \cong H_*(\hat{A}_R,\hat{M}_K)$. □

Lemma 8.7. *If N is a nilpotent $K[t_i^R(A)]$-module such that $NI^{i+1} = 0$, then the homomorphisms $A \to t_i^r(A)$ induce the isomorphism:*

$$H_*(A,N) \cong H_*(t_i^R(A),N).$$

Proof. If $R = \mathbb{Z}[J^{-1}]$, it follows from Lemma 8.6. If $R = \mathbb{Z}/p$, it follows from 5.4. □

Proposition 8.8. *The homomorphism* (8.3) *is an isomorphism.*

Proof. Let $R = \mathbb{Z}[J^{-1}]$. Then $t_i^R(A) = \hat{A}_R = A \otimes R$ and $\mathcal{M}_K^i = M_K/M_K I_K^i$. Then by Lemma 8.6 $H_*(\hat{A}_R,\hat{M}_K) \cong H_*(A,\hat{M}_K)$, by Corollary 2.3 $H_*(A,\hat{M}_K) \cong \varprojlim H_*(A,\mathcal{M}_K^i)$ and again by Lemma 8.6, using that $\mathcal{M}_K^i$ is nilpotent, we get $H_*(A,\mathcal{M}_K^i) \cong H_*(t_i^R(A),\mathcal{M}_K^i)$.

Let $R = \mathbb{Z}/p$. Then by Lemma 5.4 $H_*(t_i^R(A),\mathcal{M}_K^i) \cong H_*(A,\mathcal{M}_K^i)$. Since, the inverse sequence $\mathcal{M}_K^i$ is equivalent to the inverse sequence M/MI^i we have $\varprojlim H_*(A,\mathcal{M}_i) \cong \varprojlim H_*(A,M/MI^i)$. By Proposition 5.1 we get $\varprojlim H_*(A,M/MI^i) \cong H_*(A,\hat{M}_K)$ and by Lemma 8.6 $H_*(A,\hat{M}_K) \cong H_*(\hat{A}_R,\hat{M}_K)$. □

Proposition 8.9. *The homomorphism* (8.4) *is an isomorphism.*

Proof. Consider the morphism of exact sequences
(8.5)

$$\begin{tikzcd}
H_1(\hat{A}_R,\mathsf{Tor}(\hat{M}_S,K)) \arrow[r] \arrow[d, "f_1^i"] & (\wedge^2\hat{M}_K)_{\hat{A}_R} \arrow[r] \arrow[d, "f_2^i"] & H_2(\hat{M}_S,K)_{\hat{A}_R} \arrow[r] \arrow[d, "f^i"] & (M_K)_A \arrow[r] \arrow[d, "f_3^i"] & 0 \\
H_1(t_i^R(A),\mathsf{Tor}(\mathcal{M}^i,K)) \arrow[r] & (\wedge^2\mathcal{M}_K^i)_{t_i^R(A)} \arrow[r] & H_2(\mathcal{M}^i,K)_{t_i^R(A)} \arrow[r] & (\mathcal{M}_K^i)_{t_i^R(A)} \arrow[r] & 0
\end{tikzcd}$$

We need to prove that $\varprojlim f^i$ is an isomorphism. By Lemma 3.1 we have $H_*(\hat{A}_R,\mathsf{Tor}(\hat{M}_S,K)) \cong H_*(\hat{A}_R,\mathsf{Tor}(M_S,K)^\wedge)$. By Lemma 8.6 we get $H_*(\hat{A}_R,\mathsf{Tor}(M_S,K)^\wedge) \cong H_*(A,\mathsf{Tor}(M_S,K)^\wedge) \cong H_*(A,\mathsf{Tor}(\hat{M}_S,K))$. By Propositions 2.4 and 2.6 we obtain $H_*(A,\mathsf{Tor}(\hat{M}_S,K)) \cong H_*(A,\varprojlim \mathsf{Tor}(\mathcal{M}^i,K)) \cong \varprojlim H_*(A,\mathsf{Tor}(\mathcal{M}^i,K))$, and by Lemma 8.7 we obtain $H_*(A,\mathsf{Tor}(\mathcal{M}^i,K)) \cong H_*(t_i^R(A),\mathsf{Tor}(\mathcal{M}^i,K))$. Then $\varprojlim f_1^i$ and $\varprojlim f_3^i$ are isomorphisms. By Proposition 7.2 $\varprojlim f_2^i$ is an isomorphism. Finally, using the five lemma, we get that $\varprojlim f^i$ is an isomorphism. □

9. Bousfield problem for metabelian groups.

We put $\Phi_i^R H_2(G,K) = \mathrm{Ker}(H_2(G,K) \to H_2(G/\gamma_{i+1}^R(G),K))$. Then $\Phi_i H_2(G,K) = \Phi_i^{\mathbb{Z}} H_2(G,K)$ is the Dwyer filtration on $H_2(G,K)$ (see [10]).

Theorem 9.1. *Let G be a finitely presented metabelian group, $R = \mathbb{Z}[J^{-1}]$ or $R = \mathbb{Z}/n$ and K be an Artinian quotient ring of R. Then for $i \gg 0$ there is a short exact sequence*

$$0 \longrightarrow \Phi_i^R H_2(G,K) \longrightarrow H_2(G,K) \longrightarrow H_2(\hat{G}_R,K) \longrightarrow 0,$$

where the epimorphism is induced by the homomorphism $G \to \hat{G}_R$.

The following corollary is the answer on the Bousfield problem for the class of metabelian groups.

Corollary 9.2. *Let G be a finitely presented metabelian group. Then the homomorphisms $G \to \hat{G}_{\mathbb{Q}}$ and $G \to \hat{G}_{\mathbb{Z}/n}$ induce the epimorphisms*

$$H_2(G,\mathbb{Q}) \twoheadrightarrow H_2(\hat{G}_{\mathbb{Q}},\mathbb{Q}), \qquad H_2(G,\mathbb{Z}/n) \twoheadrightarrow H_2(\hat{G}_{\mathbb{Z}/n},\mathbb{Z}/n).$$

Corollary 9.3. *Let G be a finitely presented metabelian group and $\hat{G}$ be the pronilpotent completion. Then for $m \gg 0$ there is a short exact sequence*

$$0 \longrightarrow \Phi_m H_2(G,\mathbb{Z}/n) \longrightarrow H_2(G,\mathbb{Z}/n) \longrightarrow H_2(\hat{G},\mathbb{Z}/n) \longrightarrow 0,$$

where the epimorphism is induced by the homomorphism $G \to \hat{G}$.

Proof of Theorem 9.1. For the sake of simplicity we put $\gamma_i^R = \gamma_i^R(G)$ Consider the short exact sequence

$$1 \longrightarrow \gamma_i^R \longrightarrow G \longrightarrow G/\gamma_i^R \longrightarrow 1,$$

and the associated five term exact sequence

$$H_2(G,K) \to H_2(G/\gamma_i^R,K) \to H_1(\gamma_i^R,K)_G \to H_1(G,K) \twoheadrightarrow H_1(G/\gamma_i^R,K).$$

Note that $H_1(G,K) \cong H_1(G/\gamma_i^R,K) \cong (G/\gamma_2^R) \otimes K$ and the morphism $H_1(G,K) \to H_1(G/\gamma_i^R,K)$ is the isomorphism. Moreover,

$$H_1(\gamma_i^R,K)_G = (\gamma_i^R/[\gamma_i^R,G]) \otimes K = (\gamma_i^R/\gamma_{i+1}^R) \otimes K.$$

Hence, we get the exact sequence:

$$0 \longrightarrow \Phi_{i-1}^R H_2(G,K) \longrightarrow H_2(G,K) \xrightarrow{\xi_i} H_2(G/\gamma_i^R,K) \longrightarrow (\gamma_i^R/\gamma_{i+1}^R) \otimes K \longrightarrow 0.$$

The inclusion $\gamma_{i+1}^R \hookrightarrow \gamma_i^R$ induce zero homomorphism $(\gamma_{i+1}^R/\gamma_{i+2}^R) \otimes K \to (\gamma_i^R/\gamma_{i+1}^R) \otimes K$. Therefore $\varprojlim (\gamma_i^R/\gamma_{i+1}^R) \otimes K = 0$. It follows that $\varprojlim \mathrm{Im}(\xi_i) \cong \varprojlim H_2(G/\gamma_i^R,K)$. By Theorem 8.1 we get $\varprojlim \mathrm{Im}(\xi_i) \cong H_2(\hat{G}_R,K)$. Since $H_2(G,K)$ is an Artinian K-module, the sequence $\Phi_j^R H_2(G,K)$ stabilizes and we get $\varprojlim \Phi_i^R H_2(G,K) = \bigcap_j \Phi_j^R H_2(G,K) = \Phi_i^R H_2(G,K)$ for $i \gg 0$. It follows that the image $\mathrm{Im}(\xi_i)$ stabilizes. Hence $\mathrm{Im}(\xi_i) \cong H_2(\hat{G}_R,K)$ for $i \gg 0$ and we have the short exact sequences.

$$0 \longrightarrow H_2(\hat{G}_R,K) \longrightarrow H_2(G/\gamma_i^R,K) \longrightarrow (\gamma_i^R/\gamma_{i+1}^R) \otimes K \longrightarrow 0$$
$$0 \longrightarrow \Phi_i^R H_2(G,K) \longrightarrow H_2(G,K) \longrightarrow H_2(\hat{G}_R,K) \longrightarrow 0,$$

for $i \gg 0$. □

Remark 9.4. In the proof of Theorem 9.1 we get the short exact sequence

$$0 \longrightarrow H_2(\hat{G}_R, K) \longrightarrow H_2(G/\gamma_i^R, K) \longrightarrow (\gamma_i^R/\gamma_{i+1}^R) \otimes K \longrightarrow 0$$

for $i \gg 0$. Then informally the group $H_2(\hat{G}_R, K)$ can be considered as 'the biggest part' of $H_2(G/\gamma_i^R, K)$ independent of i.

Next we give an example of a polycyclic metabelian residually nilpotent group H, such that the intersection of Dwyer filtration $\cap_i \Phi_i^{\mathbb{Z}}(H)$ is nonzero (see [17] for detailed study of this group and its localizations).

Example 9.5. Let $H = \langle a, b \mid a^{b^2} = aa^{3b},\ [a, a^b] = 1\rangle$. The group H is the semidirect product $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}$, where the cyclic group $\mathbb{Z} = \langle b\rangle$ acts on $\mathbb{Z} \oplus \mathbb{Z}$ as the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$. For the group H,

$$\cap_i \Phi_i^{\mathbb{Z}}(H) = H_2(H) = \mathbb{Z}/2.$$

10. The Telescope.

In this section we assume that $R = \mathbb{Z}$ and $\hat{G} = \hat{G}_R$ is the pronilpotent completion of G. Moreover, we assume that $A = G_{ab}$ and $M = [G, G]$. In [15] and [16], J.P. Levine defines closely related groups, his algebraic closure of G, whose image in the pronilpotent completion has important properties. In the case of metabelian group G this image is called the Telescope of G and denoted by $\bar{G}$. It was proved in [2] that the metabelian decomposition $M \rightarrowtail G \twoheadrightarrow A$ induces the following metabelian decomposition

$$0 \longrightarrow M^{\ell} \longrightarrow \bar{G} \longrightarrow A \longrightarrow 1,$$

where $M^{\ell} = M[(1+I)^{-1}]$ is the localization of M with respect to the multiplicative set $1+I$.

Proposition 10.1. *Let G be a finitely generated metabelian group and $n \geq 1$. Then the inclusion $\bar{G} \to \hat{G}$ induces an isomorphism*

$$H_2(\bar{G}, \mathbb{Z}/n) \cong H_2(\hat{G}, \mathbb{Z}/n).$$

Proof. The morphism of the metabelian decompositions

$$\begin{tikzcd} 0 \arrow[r] & M^{\ell} \arrow[r] \arrow[d] & \bar{G} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathsf{id}"] & 1 \\ 0 \arrow[r] & \hat{M} \arrow[r] & \hat{G} \arrow[r] & A \arrow[r] & 1. \end{tikzcd} \tag{10.1}$$

gives the morphism of the Lyndon-Hochschild-Serre spectral sequences $E(\bar{G}) \to E(\hat{G})$. First, we note that $E^2_{i,0}(\bar{G}) = E^2_{i,0}(\hat{G}) = H_i(A, \mathbb{Z}/n)$. By Proposition 5.1 we have the isomorphisms $H_i(A, M \otimes \mathbb{Z}/n) \cong H_i(A, M^{\ell} \otimes \mathbb{Z}/n) \cong H_i(A, \hat{M} \otimes \mathbb{Z}/n)$. Therefore, the morphism $E^2_{ij}(\bar{G}) \to E^2_{ij}(\hat{G})$ is an isomorphism for $j \in \{0, 1\}$. Then, it sufficient to prove that the morphism $E^2_{0,2}(\bar{G}) = H_2(M^{\ell}, \mathbb{Z}/n)_A \to H_2(\hat{M}, \mathbb{Z}/n)_A = E^2_{0,2}(\hat{G})$ is an isomorphism.

By Proposition 5.1, we get $H_i(A, \mathsf{Tor}(M^\ell, \mathbb{Z}/n)) \cong H_i(A, \mathsf{Tor}(\hat{M}, \mathbb{Z}/n))$ and by Proposition 7.2 we get $(\wedge^2 M^\ell)_A \cong (\wedge^2 \hat{M})_A$. Consider the morphism of exact sequences
(10.2)

$$\begin{tikzcd}
H_1(A, \mathsf{Tor}(M^\ell, \mathbb{Z}/n)) \arrow[r] \arrow[d, "\cong"] & (\wedge^2 M^\ell)_A \arrow[r] \arrow[d, "\cong"] & H_2(M^\ell, \mathbb{Z}/n) \arrow[r, two heads] \arrow[d] & \mathsf{Tor}(M^\ell, \mathbb{Z}/n)_A \arrow[d, "\cong"] \\
H_1(A, \mathsf{Tor}(\hat{M}, \mathbb{Z}/n)) \arrow[r] & (\wedge^2 \hat{M})_A \arrow[r] & H_2(\hat{M}, \mathbb{Z}/n) \arrow[r, two heads] & \mathsf{Tor}(\hat{M}, \mathbb{Z}/n)_A.
\end{tikzcd}$$

Using the five lemma, we obtain that the morphism $H_2(M^\ell, \mathbb{Z}/n) \to H_2(\hat{M}, \mathbb{Z}/n)$ is an isomorphism. □

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