ON THE HOCHSCHILD COHOMOLOGY RING OF THE QUATERNION GROUP OF ORDER EIGHT IN CHARACTERISTIC TWO

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ABSTRACT. Let k be an algebraically closed field of characteristic two and let Q_8 be the quaternion group of order 8. We determine the Gerstenhaber Lie algebra structure and the Batalin-Vilkovisky structure on the Hochschild cohomology ring of the group algebra kQ_8.

INTRODUCTION

Let A be an associative algebra over a field k. The Hochschild cohomology $HH^*(A)$ of A has a very rich structure. It is a graded commutative algebra via the cup product or the Yoneda product, and it has a graded Lie bracket of degree -1 so that it becomes a graded Lie algebra; these make $HH^*(A)$ a Gerstenhaber algebra [5].

During several decades, a new structure in Hochschild theory has been extensively studied in topology and mathematical physics, and recently this was introduced into algebra, the so-called Batalin-Vilkovisky structure. Roughly speaking a Batalin-Vilkovisky (aka. BV) structure is an operator on Hochschild cohomology which squares to zero and which, together with the cup product, can express the Lie bracket. A BV structure exists only on Hochschild cohomology of certain special classes of algebras. T. Tradler first found that the Hochschild cohomology algebra of a finite dimensional symmetric algebra, such as a group algebra of a finite group, is a BV algebra [19]; for later proofs, see e.g. [3, 17].

One of the value of BV structure is that it gives a method to compute the Gerstenhaber Lie bracket which is usually out of reach in practice. This paper deals with a concrete example. Let k be an algebraically closed field of characteristic two and let Q_8 be the quaternion group of order 8. In this paper, we compute explicitly the Gerstenhaber Lie algebra structure and the Batalin-Vilkovisky structure on the Hochschild cohomology ring of the group algebra kQ_8. The Hochschild cohomology ring of kQ_8 was calculated by A. I. Generalov in [4] using a minimal projective bimodule resolution of kQ_8. Since the Gerstenhaber Lie bracket is defined using the bar resolution, one needs to find the comparison morphisms between the (normalized) bar resolution and the resolution of Generalov. To this end, we use an effective method employing the notion of weak self-homotopy, recently popularized by J. Le and the fourth author ([11]).

1. HOCHSCHILD (CO)HOMOLOGY

The cohomology theory of associative algebras was introduced by Hochschild ([8]). The Hochschild cohomology ring of a k-algebra is a Gerstenhaber algebra, which was first discovered by Gerstenhaber in [5]. Let us recall his construction here. Given a k-algebra A, its Hochschild cohomology

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groups are defined as $HH^n(A) \cong \mathrm{Ext}^n_{A^e}(A,A)$ for $n \geq 0$, where $A^e = A \otimes A^{\mathrm{op}}$ is the enveloping algebra of A. There is a projective resolution of A as an A^e-module

$$\mathrm{Bar}_*(A)\colon \cdots \to A^{\otimes(r+2)} \xrightarrow{d_r} A^{\otimes(r+1)} \to \cdots \to A^{\otimes 3} \xrightarrow{d_1} A^{\otimes 2} (\overset{d_0=\mu}{\to} A),$$

where $\mathrm{Bar}_r(A) := A^{\otimes(r+2)}$ for $r \geq 0$, the map $\mu : A \otimes A \to A$ is the multiplication of A, and d_r is defined by

$$d_r(a_0 \otimes a_1 \otimes \cdots \otimes a_{r+1}) = \sum_{i=0}^{r} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{r+1}$$

for all $a_0, \cdots, a_{r+1} \in A$. This is usually called the (unnormalized) bar resolution of A. The normalized version $\overline{\mathrm{Bar}}_*(A)$ is given by $\overline{\mathrm{Bar}}_r(A) = A \otimes \overline{A}^{\otimes r} \otimes A$, where $\overline{A} = A/(k \cdot 1_A)$, and with the induced differential from that of $\mathrm{Bar}_*(A)$.

The complex which is used to compute the Hochschild cohomology is $C^*(A) = \mathrm{Hom}_{A^e}(\mathrm{Bar}_*(A), A)$. Note that for each $r \geq 0$, $C^r(A) = \mathrm{Hom}_{A^e}(A^{\otimes(r+2)}, A) \cong \mathrm{Hom}_k(A^{\otimes r}, A)$. If $f \in C^r(A)$, then the expression $f(a)$ makes sense for $a \in A^{\otimes(r+2)}$ and $a \in A^{\otimes r}$ simultaneously. We identify $C^0(A)$ with A. Thus $C^*(A)$ has the following form:

$$C^*(A)\colon A \xrightarrow{\delta^0} \mathrm{Hom}_k(A,A) \to \cdots \to \mathrm{Hom}_k(A^{\otimes r}, A) \xrightarrow{\delta^r} \mathrm{Hom}_k(A^{\otimes(r+1)}, A) \to \cdots .$$

Given f in $\mathrm{Hom}_k(A^{\otimes r}, A)$, the map $\delta^r(f)$ is defined by sending $a_1 \otimes \cdots \otimes a_{r+1}$ to

$$a_1 \cdot f(a_2 \otimes \cdots \otimes a_{r+1}) + \sum_{i=1}^{r} (-1)^i f(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{r+1}) + (-1)^{r+1} f(a_1 \otimes \cdots \otimes a_r) \cdot a_{r+1}.$$

There is also a normalized version $\overline{C}^*(A) = \mathrm{Hom}_{A^e}(\overline{\mathrm{Bar}}_*(A), A) \cong \mathrm{Hom}_k(\overline{A}^{\otimes *}, A)$.

The cup product $\alpha \smile \beta \in C^{n+m}(A) = \mathrm{Hom}_k(A^{\otimes(n+m)}, A)$ for $\alpha \in C^n(A)$ and $\beta \in C^m(A)$ is given by

$$(\alpha \smile \beta)(a_1 \otimes \cdots \otimes a_{n+m}) := \alpha(a_1 \otimes \cdots \otimes a_n) \cdot \beta(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

This cup product induces a well-defined product in Hochschild cohomology

$$\smile\colon HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m}(A)$$

which turns the graded k-vector space $HH^*(A) = \bigoplus_{n \geq 0} HH^n(A)$ into a graded commutative algebra ([5, Corollary 1]).

The Lie bracket is defined as follows. Let $\alpha \in C^n(A)$ and $\beta \in C^m(A)$. If $n, m \geq 1$, then for $1 \leq i \leq n$, set $\alpha \circ_i \beta \in C^{n+m-1}(A)$ by

$$(\alpha \circ_i \beta)(a_1 \otimes \cdots \otimes a_{n+m-1}) := \alpha(a_1 \otimes \cdots \otimes a_{i-1} \otimes \beta(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1});$$

if $n \geq 1$ and $m = 0$, then $\beta \in A$ and for $1 \leq i \leq n$, set

$$(\alpha \circ_i \beta)(a_1 \otimes \cdots \otimes a_{n-1}) := \alpha(a_1 \otimes \cdots \otimes a_{i-1} \otimes \beta \otimes a_i \otimes \cdots \otimes a_{n-1});$$

for any other case, set $\alpha \circ_i \beta$ to be zero. Now define

$$\alpha \circ \beta := \sum_{i=1}^{n} (-1)^{(m-1)(i-1)} \alpha \circ_i \beta$$

and

$$[\alpha, \beta] := \alpha \circ \beta - (-1)^{(n-1)(m-1)} \beta \circ \alpha.$$

Note that $[\alpha, \beta] \in C^{n+m-1}(A)$. The above $[\ ,\ ]$ induces a well-defined Lie bracket in Hochschild cohomology

$$[\ ,\ ]\colon HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m-1}(A)$$

such that $(HH^*(A), \smile, [\ ,\ ])$ is a Gerstenhaber algebra ([5]).

The complex used to compute the Hochschild homology $HH_*(A)$ is $C_*(A) = A \otimes_{A^e} \mathrm{Bar}_*(A)$. Notice that $C_r(A) = A \otimes_{A^e} A^{\otimes(r+2)} \simeq A^{\otimes(r+1)}$ and the differential $\partial_r : C_r(A) = A^{\otimes(r+1)} \to$

$C_{r-1}(A) = A^{\otimes r}$ sends $a_0 \otimes \cdots \otimes a_r$ to $\sum_{i=0}^{r-1}(-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_r + (-1)^r a_r a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1}$.

There is a Connes' $\mathfrak{B}$-operator in the Hochschild homology theory which is defined as follows. For $a_0 \otimes \cdots \otimes a_r \in C_r(A)$, let $\mathfrak{B}(a_0 \otimes \cdots \otimes a_r) \in C_{r+1}(A)$ be

$$\sum_{i=0}^{r}(-1)^{ir} 1 \otimes a_i \otimes \cdots \otimes a_r \otimes a_0 \otimes \cdots \otimes a_{i-1} + \sum_{i=0}^{r}(-1)^{ir} a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_r \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

It is easy to check that $\mathfrak{B}$ is a chain map satisfying $\mathfrak{B} \circ \mathfrak{B} = 0$, which induces an operator $\mathfrak{B} : HH_r(A) \to HH_{r+1}(A)$.

All the above constructions, the cup product, the Lie bracket, the Connes' $\mathfrak{B}$-operator, carry over to normalized complexes.

Definition 1.1. *A* Batalin–Vilkovisky algebra *(BV algebra for short) is a Gerstenhaber algebra* $(A^\bullet, \smile, [\ ,\ ])$ *together with an operator* $\Delta\colon A^\bullet \to A^{\bullet -1}$ *of degree* -1 *such that* $\Delta \circ \Delta = 0$ *and*

$$[a, b] = -(-1)^{(|a|-1)|b|}(\Delta(a \smile b) - \Delta(a) \smile b - (-1)^{|a|} a \smile \Delta(b))$$

for homogeneous elements $a, b \in A^\bullet$.

Tradler noticed that the Hochschild cohomology algebra of a symmetric algebra is a BV algebra [19], see also [17, 3]. For a symmetric algebra A, he showed that the Δ-operator on the Hochschild cohomology corresponds to the Connes' $\mathfrak{B}$-operator on the Hochschild homology via the duality between the Hochschild cohomology and the Hochschild homology.

Recall that a finite dimensional k-algebra A is called symmetric if A is isomorphic to its dual $DA = \mathrm{Hom}_k(A, k)$ as A^e-module, or equivalently, if there exists a symmetric associative non-degenerate bilinear form $\langle\ ,\ \rangle\colon A \times A \to k$. This bilinear form induces a duality between the Hochschild cohomology and the homology. In fact,

$$\begin{aligned} \mathrm{Hom}_k(C_*(A), k) &= \mathrm{Hom}_k(A \otimes_{A^e} \mathrm{Bar}_*(A), k) \\ &\cong \mathrm{Hom}_{A^e}(\mathrm{Bar}_*(A), \mathrm{Hom}_k(A, k)) \\ &\cong \mathrm{Hom}_{A^e}(\mathrm{Bar}_*(A), A) = C^*(A). \end{aligned}$$

Via this duality, for $n \geq 1$ we obtain an operator $\Delta\colon HH^n(A) \to HH^{n-1}(A)$ which is the dual of Connes' operator.

We recall the following theorem by Tradler.

Theorem 1.2. [19, Theorem 1] *With the notation above, together with the cup product, the Lie bracket and the* Δ*-operator defined above, the Hochschild cohomology of* A *is a BV algebra. More precisely, for* $\alpha \in C^n(A) = \mathrm{Hom}_k(A^{\otimes n}, A)$, $\Delta(\alpha) \in C^{n-1}(A) = \mathrm{Hom}_k(A^{\otimes(n-1)}, A)$ *is given by the equation*

$$\langle \Delta(\alpha)(a_1 \otimes \cdots \otimes a_{n-1}),\ a_n \rangle = \sum_{i=1}^{n}(-1)^{i(n-1)} \langle \alpha(a_i \otimes \cdots \otimes a_{n-1} \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{i-1}),\ 1 \rangle$$

for $a_1, \cdots, a_n \in A$. *The same formula holds also for the normalized complex* $\overline{C}^*(A)$.

2. Constructing comparison morphisms

Let k be a field and let B be a k-algebra. Given two left B-modules M and N, let P_* (resp. Q_*) be a projective resolutions of M (resp. N). Then given a homomorphism of B-modules $f : M \to N$, it is well known that there exists a chain map $f_* : P_* \to Q_*$ lifting f (and different lifts are equivalent up to homotopy). However, sometimes in practice we need the actual construction of this chain map, called comparison morphism, to perform actual computations. This section presents a method to construct them. The method is not new and it is explained in the book of Mac Lane; see [14, Chapter IX Theorem 6.2].

Our setup is the following. Suppose that

$$\cdots \longrightarrow P_n \xrightarrow{d_n^P} P_{n-1} \xrightarrow{d_{n-1}^P} \cdots \xrightarrow{d_1^P} P_0 (\xrightarrow{d_0^P} M \to 0)$$

is a projective resolution of M. Then for each $n \geq 0$ there are sets $\{e_{n,i}\}_{i\in X_n} \subset P_n$ and $\{f_{n,i}\}_{i\in X_n} \subset \mathrm{Hom}_B(P_n, B)$ such that $x = \sum_{i\in X_n} f_{n,i}(x)e_{n,i}$ for all $x \in P_n$. Suppose that the second projective resolution

$$\cdots \longrightarrow Q_n \xrightarrow{d_n^Q} Q_{n-1} \xrightarrow{d_{n-1}^Q} \cdots \xrightarrow{d_1^Q} Q_0 (\xrightarrow{d_0^Q} N \to 0)$$

has a weak self-homotopy in the sense of the following definition.

Definition 2.1. [1] *Let*

$$\cdots Q_n \overset{d_n^Q}{\to} Q_{n-1} \overset{d_{n-1}^Q}{\to} \cdots \overset{d_2^Q}{\to} Q_1 \overset{d_1^Q}{\to} Q_0 \overset{d_0^Q}{\to} N \to 0$$

be a complex. A weak self-homotopy of this complex is a collection of k-linear maps $t_n : Q_n \to Q_{n+1}$ for each $n \geq 0$ and $t_{-1} : M \to Q_0$ such that for $n \geq 0$, $t_{n-1}d_n^Q + d_{n+1}^Q t_n = Id_{Q_n}$ and $d_0^Q t_{-1} = Id_N$.

Now we construct a chain map $f_n : P_n \to Q_n$ for $n \geq 0$ lifting $f_{-1} = f$. We need to specify the value of f_n on the elements $e_{n,i}$ for all $i \in X_n$.

For $n = 0$, define $f_0(e_{0,i}) = t_{-1} f d_0^P(e_{0,i})$. Then $d_0^Q f_0(e_{0,i}) = d_0^Q t_{-1} f d_0^P(e_{0,i}) = f d_0^P(e_{0,i})$.

Suppose that we have constructed $f_0, \cdots, f_{n-1}$ such that for $0 \leq i \leq n-1$, $d_i^Q f_i = f_{i-1} d_i^P$. Define $f_n(e_{n,i}) = t_{n-1} f_{n-1} d_n^P(e_{n,i})$. It is easy to check that

$$d_n^Q f_n(e_{n,i}) \quad = \quad f_{n-1} d_n^P(e_{n,i}).$$

This proves the following

Proposition 2.2. *The maps f_* constructed above form a chain map from P_* to Q_* lifting $f : M \to N$.*

This result reduces the computation of comparison morphisms to the construction of weak self-homotopies. It is easy to see that the complex Q_* is exact if and only if there exists a weak self-homotopy of it. In fact, we can obtain more. Denote $Z_n = \mathrm{Ker}(d_n)$ for $n \geq 0$ and $Z_{-1} = N$. As vector spaces, one can fix a decomposition of $Q_n = Z_n \oplus Z_{n-1}$ for $n \geq 0$. Under these identifications, the differential d_n is equal to $\begin{pmatrix} 0 & Id \\ 0 & 0 \end{pmatrix} : Z_n \oplus Z_{n-1} \to Z_{n-1} \oplus Z_{n-2}$ and we can define $t_{-1} = \begin{pmatrix} 0 \\ Id \end{pmatrix} : Z_{-1} \to Z_0 \oplus Z_{-1}$ and for $n \geq 0$, $t_n : Z_n \oplus Z_{n-1} \to Z_{n+1} \oplus Z_n$ to be the map $\begin{pmatrix} 0 & 0 \\ Id & 0 \end{pmatrix}$. Note that our construction has an additional property:

Lemma 2.3. *For an exact complex of modules over a k-algebra, one can always find a weak self-homotopy $\{t_i, i \geq -1\}$ such that $t_{i+1}t_i = 0$ for any $i \geq -1$.*

We are interested in computing Hochschild cohomology of algebras. Let A be a k-algebra. In order to compute Hochschild (co)homology of A, one needs a projective resolution of A as a bimodule. Since this resolution splits as complexes of one-sided modules, one can even choose a weak self-homotopy which are right module homomorphisms and which satisfies the additional property in Lemma 2.3.

Now let P_* be an A^e-projective resolution of A. Denote now $Q_* = \mathrm{Bar}_*(A)$ (or $Q_* = \overline{\mathrm{Bar}}_*(A)$). Let us consider the construction of comparison morphisms $\Psi_* : Q_* \to P_*$ and $\Phi_* : P_* \to Q_*$.

Suppose now that $Q_* = \overline{\mathrm{Bar}}_*(A)$. In this case Q_* has a weak self-homotopy s_* defined by the formula

$$s_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) = 1 \otimes a_0 \otimes \cdots \otimes a_n \otimes 1.$$

Note that $s_{n+1}s_n = 0$ for $n \geq -1$, as we are working with the normalized Bar resolution. Suppose that the homomorphism d_n^P is defined by the formula

$$d_n^P(e_{n,i}) = \sum_{j\in X_{n-1}} \sum_{p\in T_{n,i,j}} a_p e_{n-1,j} b_p + \sum_{j\in X_{n-1}} \sum_{q\in T'_{n,i,j}} e_{n-1,j} b'_q,$$

where $a_p \in J_A$ (here J_A is the Jacobson radical of A), $b_p, b'_q \in A$, $T_{n,i,j}$ and $T'_{n,i,j}$ are certain index sets and $e_{n,i}$ as above.

Lemma 2.4. *If $\Phi_* : P_* \to Q_*$ is the chain map constructed using s_*, then*

$$\Phi_n(e_{n,i}) = 1 \otimes \sum_{j \in X_{n-1}} \sum_{p \in T_{n,i,j}} a_p \Phi_{n-1}(e_{n-1,j}) b_p. \tag{1}$$

Proof By construction, $\Phi_n(e_{n,i}) = s_{n-1}\Phi_{n-1}d_n(e_{n,i})$. Note that

$$\Phi_{n-1}(e_{n-1,j}) = s_{n-2}\Phi_{n-2}d^P_{n-1}(e_{n-1,j})$$

and thus

$$s_{n-1}\Phi_{n-1}(e_{n-1,j}b'_q) = s_{n-1}s_{n-2}\Phi_{n-2}d^P_{n-1}(e_{n-1,j})b'_q = 0.$$

Therefore,

$$\begin{aligned} \Phi_n(e_{n,i}) &= s_{n-1}\Phi_{n-1}d_n(e_{n,i}) \\ &= s_{n-1}\Phi_{n-1}(\sum_{j \in X_{n-1}} \sum_{p \in T_{n,i,j}} a_p e_{n-1,j} b_p) \\ &= 1 \otimes \sum_{j \in X_{n-1}} \sum_{p \in T_{n,i,j}} a_p \Phi_{n-1}(e_{n-1,j}) b_p. \end{aligned}$$

□

Let $\mathcal{B}$ be some k-basis of A (or $\overline{A}$ in the case $Q_* = \overline{\mathrm{Bar}}_*(A)$). Then the set $Y_n = \{1 \otimes b_1 \otimes \cdots \otimes b_n \otimes 1 \mid b_1, \ldots, b_n \in \mathcal{B}\}$ is a basis for Q_n as free A^e-module. Suppose that we have constructed a weak self-homotopy t_* of P_* such that $t_{n+1}t_n = 0$ and that t_n is a homomorphism of right A-modules for all $n \geq -1$.

Lemma 2.5. *If $\Psi_* : Q_* \to P_*$ is the chain map constructed using t_*, then*

$$\Psi_n(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) = t_{n-1}(a_1 \Psi_{n-1}(1 \otimes a_2 \otimes \cdots \otimes a_n \otimes 1)) \tag{2}$$

for $n \geq 1$ and $a_i \in A$ $(1 \leq i \leq n)$

Proof Denote

$$y = \sum_{i=1}^{n-1} (-1)^i 1 \otimes a_1 \otimes \ldots a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n \otimes 1 + (-1)^n 1 \otimes a_1 \otimes \cdots \otimes a_n.$$

As $t_{n-1}t_{n-2} = 0$, we have

$$\begin{aligned} &\Psi_n(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \\ =\ & t_{n-1}\Psi_{n-1}d^Q_n(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) = t_{n-1}\Psi_{n-1}(a_1 \otimes \cdots \otimes a_n \otimes 1 + y) \\ =\ & t_{n-1}(a_1\Psi_{n-1}(1 \otimes a_2 \otimes \cdots \otimes a_n \otimes 1)) + t_{n-1}t_{n-2}\Psi_{n-2}d^Q_{n-1}(y) \\ =\ & t_{n-1}(a_1\Psi_{n-1}(1 \otimes a_2 \otimes \cdots \otimes a_n \otimes 1)). \end{aligned}$$

□

Let $V = \oplus_{i=1}^n kx_i$ be a k-vector space with basis $\{x_i, 1 \leq i \leq n\}$. Let $A = T(V)/I = k\langle x_1, \cdots, x_n\rangle / I$ be an algebra given by generators and relations. Then the minimal projective bimodule resolution of A begins with

$$\cdots \to A \otimes R \otimes A \xrightarrow{d_2} A \otimes V \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{d_0} A \to 0, \tag{3}$$

where

- $V = \oplus_{i=1}^n kx_i$, R is a k-complement of $JI + IJ$ in I (thus R is a set of minimal relations), where J is the ideal of $k\langle x_1, \cdots, x_n\rangle$ generated by $x_1, \cdots, x_n$;
- d_0 is the multiplication of A;
- d_1 is induced by $d_1(1 \otimes x_i \otimes 1) = x_i \otimes 1 - 1 \otimes x_i$ for $1 \leq i \leq n$;
- d_2 is induced by the restriction to R of the bimodule derivation $\mathsf{C} : TV \to TV \otimes V \otimes TV$ sending a path $x_{i_1}x_{i_2}\cdots x_{i_r}$ (with $1 \leq i_1, \cdots, i_r \leq n$) to $\sum_{j=1}^r x_{i_1}\cdots x_{i_{j-1}} \otimes x_{i_j} \otimes x_{i_{j+1}}\cdots x_{i_r}$.

We shall construct the first three maps of a weak self-homotopy of this projective resolution, which are moreover right module homomorphisms. Let $\mathcal{B}$ be the basis of A formed by monomials in $x_1, \cdots, x_n$.

The first two are easy. We define $t_{-1} = 1 \otimes 1$ and $t_0(b \otimes 1) = \mathsf{C}(b)$ for $b \in$ B.

For $t_1 : A \otimes V \otimes A \to A \otimes R \otimes A$, we first fix a vector space decomposition $TV/I^2 = A \oplus I/I^2$. The space R, identified with $I/(JI + IJ)$, generates I/I^2 considered as A-A-bimodule. For $b \in \mathcal{B}$, consider $bx_i \in TV/I^2$, then we can write $bx_i = \sum_{b' \in \mathcal{B}} \lambda_{b'} b' + \sum_j p_j r_j q_j$ with $r_j \in R$ via the vector space decomposition $TV/I^2 = A \oplus I/I^2$. We define

$$t_1(b \otimes x_i \otimes 1) = \sum_j p_j \otimes r_j \otimes q_j.$$

Proposition 2.6. *The above defined maps t_{-1}, t_0, t_1 form the first three maps of a weak self-homotopy of the minimal projective bimodule resolution (3).*

Proof We have $d_0 t_{-1}(1) = d_0(1 \otimes 1) = 1$ and thus $d_0 t_{-1} = Id$.

For $b = x_{i_1} x_{i_2} \cdots x_{i_r} \in \mathcal{B}$, $t_{-1} d_0(b \otimes 1) = t_{-1}(b) = 1 \otimes b$, and

$$\begin{array}{rcl} d_1 t_0(b \otimes 1) & = & d_1 \mathsf{C}(b) \\ & = & d_1(\sum_{j=1}^r x_{i_1} \cdots x_{i_{j-1}} \otimes x_{i_j} \otimes x_{i_{j+1}} \cdots x_{i_r}) \\ & = & \sum_{j=1}^r x_{i_1} \cdots x_{i_{j-1}} x_{i_j} \otimes x_{i_{j+1}} \cdots x_{i_r} - \sum_{j=1}^r x_{i_1} \cdots x_{i_{j-1}} \otimes x_{i_j} x_{i_{j+1}} \cdots x_{i_r} \\ & = & b \otimes 1 - 1 \otimes b. \end{array}$$

Therefore, $(d_1 t_0 + t_{-1} d_0)(b \otimes 1) = b \otimes 1 - 1 \otimes b + 1 \otimes b = b \otimes 1$.

Now for $b \in \mathcal{B}$ and $1 \leq i \leq n$, $t_0 d_1(b \otimes x_i \otimes 1) = t_0(bx_i \otimes 1 - b \otimes x_i)$. Recall that via the decomposition $TV/I^2 = A \oplus I/I^2$, $bx_i = \sum_{b' \in \mathcal{B}} \lambda_{b'} b' + \sum_j p_j r_j q_j$, so

$$t_0 d_1(b \otimes x_i \otimes 1) = t_0(bx_i \otimes 1 - b \otimes x_i) = \sum_{b' \in \mathcal{B}} \lambda_{b'} \mathsf{C}(b') - \mathsf{C}(b) x_i.$$

We have also

$$d_2 t_1(b \otimes x_i \otimes 1) = d_2(\sum_j p_j \otimes r_j \otimes q_j) = \sum_j p_j \mathsf{C}(r_j) q_j.$$

Recall that the bimodule derivation $\mathsf{C} : TV \to TV \otimes V \otimes TV$ composed with the sujection $TV \otimes V \otimes TV \to A \otimes V \otimes A$ vanishes on I^2 and thus induces a well-defined map $\mathsf{C} : TV/I^2 \to A \otimes V \otimes A$. Furthermore, C restricted to I/I^2 is a homomorphism of A-A-bimodules. This shows that $\sum_j p_j \mathsf{C}(r_j) q_j + \sum_{b' \in \mathcal{B}} \lambda_{b'} \mathsf{C}(b') = \mathsf{C}(bx_i)$ and since $\mathsf{C}(bx_i) = \mathsf{C}(b) x_i + b \otimes x_i \otimes 1$, we obtain that $(t_0 d_1 + d_2 t_1)(b \otimes x_i \otimes 1) = b \otimes x_i \otimes 1$.

This completes the proof.

□

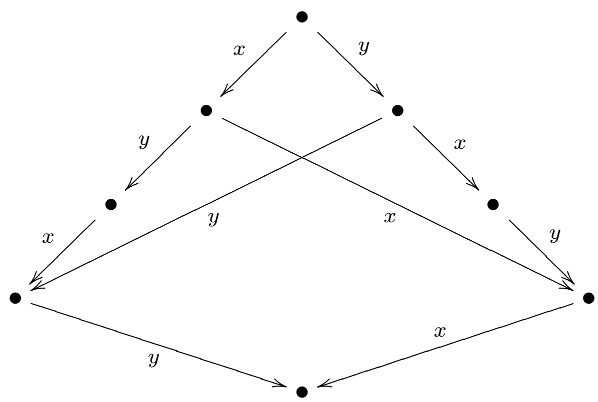
3. Weak self-homotopy for kQ_8

Let k be an algebrically closed field of characteristic two. Let Q_8 be the quaternion group of order 8. Denote by $A = kQ_8$ its group algebra. It is well known that A is isomorphic to the following bounded quiver algebra kQ/I:

with relations

$$x^2 + yxy, y^2 + xyx, x^4, y^4.$$

The structure of A can be visualised as follows:



A basis of A is given by $\mathcal{B} = \{1, x, y, xy, yx, xyx, yxy, xyxy\}$. Notice that $\mathcal{B}$ contains a basis of the socle of A.

The group algebra A is a symmetric algebra, with respect to the symmetrising form

$$\langle b_1, b_2 \rangle = \begin{cases} 1 & \text{if } b_1 b_2 \in Soc(A) \\ 0 & \text{otherwise} \end{cases}$$

with $b_1, b_2 \in \mathcal{B}$. The correspondance between elements of $\mathcal{B}$ and its dual basis $\mathcal{B}^*$ is given by

$$\begin{array}{ccccccccc} b \in \mathcal{B} & 1 & x & y & xy & yx & xyx & yxy & xyxy \\ b^* \in \mathcal{B}^* & xyxy & yxy & xyx & xy & yx & y & x & 1 \end{array}$$

Since A is an algebra with a DTI-family of relations, there is a minimal projective resolution constructed by the second author in [9]. Let us recall the concrete construction of this resolution.

After ([4]), there is an exact sequences of bimodules as follows:

$$(0 \to A \overset{\rho}{\to}) A \otimes A \overset{d_3}{\to} A \otimes kQ_1^* \otimes A \overset{d_2}{\to} A \otimes kQ_1 \otimes A \overset{d_1}{\to} A \otimes A \overset{d_0}{\to} A \to 0$$

where

- $Q_1 = \{x, y\}$ and $Q_1^* = \{r_x, r_y\}$ with $r_x = x^2 + yxy$ and $r_y = y^2 + xyx$;
- the map d_0 is the multiplication of A;
- $d_1(1 \otimes x \otimes 1) = x \otimes 1 + 1 \otimes x$ and $d_1(1 \otimes y \otimes 1) = y \otimes 1 + 1 \otimes y$;
- $d_2(1 \otimes r_x \otimes 1) = 1 \otimes x \otimes x + x \otimes x \otimes 1 + 1 \otimes y \otimes xy + y \otimes x \otimes y + yx \otimes y \otimes 1$
  and
  $d_2(1 \otimes r_y \otimes 1) = 1 \otimes y \otimes y + y \otimes y \otimes 1 + 1 \otimes x \otimes yx + x \otimes y \otimes x + xy \otimes x \otimes 1$;
- $d_3(1 \otimes 1) = x \otimes r_x \otimes 1 + 1 \otimes r_x \otimes x + y \otimes r_y \otimes 1 + 1 \otimes r_y \otimes y$;
- $\rho(1) = \sum_{b \in \mathcal{B}} b^* \otimes b$.

Using this exact sequence, one can construct a minimal projective bimodule resolution of A which is periodic of period 4:

- $P_0 = A \otimes A = P_3$, $P_1 = A \otimes kQ_1 \otimes A$ and $P_2 = A \otimes kQ_1^* \otimes A$;
- $P_4 = P_0 = A \otimes A$ and $d_4 = \rho \circ d_0 : P_4 \to P_3$;
- for $n \geq 1$ and $i \in \{0, 1, 2, 3\}$, we have $P_{4n+i} = P_i$ and $d_{4n+i+1} = d_{i+1}$.

We shall establish a weak self-homotopy $\{t_i : P_i \to P_{i+1}; t_{-1} : A \to P_0\}$ over this periodic resolution which are right module homomorphisms.

The first two are easy which are $t_{-1} = 1 \otimes 1$ and $t_0(b \otimes 1) = \mathsf{C}(b)$ for $b \in \mathrm{B}$, where $\mathsf{C} : kQ \to kQ \otimes kQ_1 \otimes kQ$ is the bimodule derivation sending a path $\alpha_1 \cdots \alpha_n$ with $\alpha_1, \cdots, \alpha_n \in Q_1$ to $\sum_{i=1}^n \alpha_1 \cdots \alpha_{i-1} \otimes \alpha_i \otimes \alpha_{i+1} \cdots \alpha_n$.

The map $t_1 : P_1 \to P_2$ is given by

$$\begin{array}{lcl}
t_1(1\otimes x\otimes 1) & = & 0,\\
t_1(x\otimes x\otimes 1) & = & 1\otimes r_x\otimes 1,\\
t_1(y\otimes x\otimes 1) & = & 0,\\
t_1(xy\otimes x\otimes 1) & = & 0,\\
t_1(yx\otimes x\otimes 1) & = & y\otimes r_x\otimes 1+xy\otimes r_x\otimes y+1\otimes r_y\otimes xy,\\
t_1(xyx\otimes x\otimes 1) & = & xy\otimes r_x\otimes 1+x\otimes r_y\otimes xy,\\
t_1(yxy\otimes x\otimes 1) & = & 1\otimes r_y\otimes y+y\otimes r_y\otimes 1,\\
t_1(xyxy\otimes x\otimes 1) & = & 1\otimes r_x\otimes yxy-x\otimes r_x\otimes x+yxy\otimes r_x\otimes 1+yx\otimes r_y\otimes xy\\
t_1(1\otimes y\otimes 1) & = & 0,\\
t_1(x\otimes y\otimes 1) & = & 0,\\
t_1(y\otimes y\otimes 1) & = & 1\otimes r_y\otimes 1,\\
t_1(xy\otimes y\otimes 1) & = & 1\otimes r_x\otimes yx+x\otimes r_y\otimes 1+yx\otimes r_y\otimes x,\\
t_1(yx\otimes y\otimes 1) & = & 0,\\
t_1(xyx\otimes y\otimes 1) & = & 0,\\
t_1(yxy\otimes y\otimes 1) & = & y\otimes r_x\otimes yx+yx\otimes r_y\otimes 1,\\
t_1(xyxy\otimes y\otimes 1) & = & xy\otimes r_x\otimes yx+xyx\otimes r_y\otimes 1.
\end{array}$$

Notice that $t_1(b_1\otimes b_2\otimes 1)=0$ for $b_1,b_2\in\mathcal{B}$ with $b_1b_2\in\mathcal{B}$. This observation will simplify very much some computations.

The map $t_2 : P_2 \to P_3$ is given by

$$\begin{array}{lcl}
t_2(1\otimes r_x\otimes 1) & = & 0,\\
t_2(x\otimes r_x\otimes 1) & = & 1\otimes 1,\\
t_2(y\otimes r_x\otimes 1) & = & 0,\\
t_2(xy\otimes r_x\otimes 1) & = & 0,\\
t_2(yx\otimes r_x\otimes 1) & = & y\otimes 1,\\
t_2(xyx\otimes r_x\otimes 1) & = & xy\otimes 1+x\otimes y,\\
t_2(yxy\otimes r_x\otimes 1) & = & 1\otimes x,\\
t_2(xyxy\otimes r_x\otimes 1) & = & 1\otimes yxy+yxy\otimes 1+y\otimes xy+yx\otimes y\\
t_2(1\otimes r_y\otimes 1) & = & 0,\\
t_2(x\otimes r_y\otimes 1) & = & 0,\\
t_2(y\otimes r_y\otimes 1) & = & 0,\\
t_2(xy\otimes r_y\otimes 1) & = & x\otimes 1,\\
t_2(yx\otimes r_y\otimes 1) & = & 0,\\
t_2(xyx\otimes r_y\otimes 1) & = & 0,\\
t_2(yxy\otimes r_y\otimes 1) & = & y\otimes x+yx\otimes 1,\\
t_2(xyxy\otimes r_y\otimes 1) & = & x\otimes yx+xy\otimes x+xyx\otimes 1.
\end{array}$$

We define $\tau : P_3 = A\otimes A\to A$ as follows: $\tau(xyxy\otimes 1)=1$ and $\tau(b\otimes 1)=0$ for $b\in\mathcal{B}-\{xyxy\}$. We impose $t_3=t_{-1}\circ\tau : P_3\to P_4$ and define $t_{4n+i}=t_i$ for $n\geq 0$ and $i\in\{0,1,2,3\}$.

Proposition 3.1. *The above defined maps $\{t_i\}_{i\geq -1}$ form a weak self-homotopy over P_*.*

Proof Since the resolution is periodic of period 4, it suffices to prove that

$$\begin{cases}
d_0t_{-1} & = & Id,\\
d_{p+1}t_p+t_{p-1}d_p & = & Id, \text{ for } 0\leq p\leq 2,\\
t_2d_3+\rho\tau & = & Id,\\
\tau\rho & = & Id.
\end{cases}$$

The first two maps t_{-1} and t_0 are given at the end of Section 2.

The map t_1 can be computed using the formula given at the end of Section 2. For instance, for $t_1(xyxy\otimes x\otimes 1)$, one can write

$$xyxyx = xr_xx+yxyr_x+1r_xyxy+yxr_yxy+r_x^2\in TV.$$

As $r_x^2 \in I^2$, we have

$$t_1(xyxy \otimes x \otimes 1) = x \otimes r_x \otimes x + yxy \otimes r_x \otimes 1 + 1 \otimes r_x \otimes yxy + yx \otimes r_y \otimes xy.$$

Another expression is

$$xyxyx = r_y yx + y r_y x + yxy r_x + yx r_y xy + yxxyxxy \in TV,$$

Notice that $yxx \in I$ and $yxxyxxy \in I^2$, which give

$$t_1(xyxy \otimes x \otimes 1) = 1 \otimes r_y \otimes yx + y \otimes r_y \otimes x + yxy \otimes r_x \otimes 1 + yx \otimes r_y \otimes xy.$$

The maps t_2 and τ are computed by direct inspection. The details are tedious and long, but not difficult.

□

4. Comparison morphisms for kQ_8

For an algebra A, denote by $\overline{A} = A/(k \cdot 1)$. The normalized bar resolution is a quotient complex of the usual bar resolution whose p-th term is $B_p(A) = A \otimes \overline{A}^{\otimes p} \otimes A$ and whose differential is induced from that of the usual bar resolution. It is easy to see that this complex is well-defined.

Using the method from Section 2, one can compute comparison morphsims between the minimal resolution P_* and the normalized bar resolution $\mathrm{Bar}_*(A)$, denoted by $\Phi_* : P_* \to \mathrm{Bar}_*(A)$ and $\Psi_* : \mathrm{Bar}_*(A) \to P_*$.

The chain map $\Phi_* : P_* \to B_* := \mathrm{Bar}_*(A)$ can be computed by applying Lemma 2.4. Let us give the formulas for Φ_i with $i \leq 5$.

- $\Phi_0 = Id : P_0 = A \otimes A \to B_0 = A \otimes A$;
- $\Phi_1 : P_1 = A \otimes kQ_1 \otimes A \to B_1 = A \otimes \overline{A} \otimes A$ is induced by the inclusion $kQ_1 \hookrightarrow \overline{A}$;
- $\Phi_2 : P_2 = A \otimes kQ_1^* \otimes A \to B_2 = A \otimes \overline{A}^{\otimes 2} \otimes A$ is given by
$$\Phi_2(1 \otimes r_x \otimes 1) = 1 \otimes x \otimes x \otimes 1 + 1 \otimes y \otimes x \otimes y + 1 \otimes yx \otimes y \otimes 1$$
and
$$\Phi_2(1 \otimes r_y \otimes 1) = 1 \otimes y \otimes y \otimes 1 + 1 \otimes x \otimes y \otimes x + 1 \otimes xy \otimes x \otimes 1;$$
- $\Phi_3 : P_3 = A \otimes A \to B_3 = A \otimes \overline{A}^{\otimes 3} \otimes A$ is given by
$$\begin{aligned}\Phi_3(1 \otimes 1) =& 1 \otimes x \otimes x \otimes x \otimes 1 + 1 \otimes x \otimes y \otimes x \otimes y + 1 \otimes x \otimes yx \otimes y \otimes 1 \\ &+ 1 \otimes y \otimes y \otimes y \otimes 1 + 1 \otimes y \otimes x \otimes y \otimes x + 1 \otimes y \otimes xy \otimes x \otimes 1;\end{aligned}$$
- $\Phi_4 : P_4 = A \otimes A \to B_4 = A \otimes \overline{A}^{\otimes 4} \otimes A$ is given by
$$\Phi_4(1 \otimes 1) = \sum_{b \in \mathcal{B} \setminus \{1\}} 1 \otimes b \Phi_3(1 \otimes 1) b^*;$$
- $\Phi_5 : P_5 = A \otimes kQ_1 \otimes A \to B_5 = A \otimes \overline{A}^{\otimes 5} \otimes A$ is given by
$$\Phi_5(1 \otimes x \otimes 1) = 1 \otimes x \Phi_4(1 \otimes 1)$$
and
$$\Phi_5(1 \otimes y \otimes 1) = 1 \otimes y \Phi_4(1 \otimes 1).$$

The chain map $\Psi_* : \mathrm{Bar}_*(A) \to P_*$ can be computed by applying the method of Section 2 to t_*. But the dimension of $\mathrm{Bar}_n(A)$ grows very fast. We have to specify the value of Ψ_n on 7^n elements to fully describe it. So we give the full description only for Ψ_0 and Ψ_1.

- $\Psi_0 = Id : B_0 = A \otimes A \to P_0 = A \otimes A$;
- $\Psi_1 : B_1 = A \otimes \overline{A} \otimes A \to P_1 = A \otimes kQ_1 \otimes A$ is given by $\Psi_1(1 \otimes b \otimes 1) = \mathsf{C}(b)$ for $b \in \mathcal{B} - \{1\}$.

5. BV-structure on $HH^*(kQ_8)$

Generalov proved the following result in [4].

Theorem 5.1. [4, Theorem 1.1, case 1b)]*Let k be an algebrically closed field of characteristic two. Let Q_8 be the quaternion group of order 8. We have $HH^*(kQ_8) \simeq k[\mathcal{X}]/I$ where*

- $\mathcal{X} = \{p_1, p_2, p_2', p_3, u_1, u_1', v_1, v_2, v_2', z\}$ *with*

$$\begin{cases} |p_1| = |p_2| = |p_2'| = |p_3| = 0, |u_1| = |u_1'| = 1, \\ |v_1| = |v_2| = |v_2'| = 2, |z| = 4; \end{cases}$$

- *the ideal I is generated by the following relations*

of degree 0

$$\begin{cases} p_1^2, p_2^2, (p_1')^2, p_1p_2, p_1p_2', p_2p_2', \\ p_3^2, p_1p_3, p_2p_3, p_2'p_3; \end{cases}$$

of degree 1

$$p_2u_1 - p_2'u_1', p_2'u_1 - p_1u_1', p_1u_1 - p_2u_1';$$

of degree 2

$$\begin{cases} p_1v_1, p_2v_2, p_2'v_2', p_3v_1, p_3v_2, p_3v_2', u_1u_1', \\ p_2v_1 - p_1v_2', p_2v_1 - p_2'v_2, p_2v_1 - p_3u_1^2, \\ p_2'v_1 - p_1v_2, p_2'v_1 - p_2v_2', p_2'v_1 - p_3(u_1')^2; \end{cases}$$

of degree 3

$$u_1'v_2 - u_1v_2', u_1'v_1 - u_1v_2, u_1v_1 - u_1'v_2', u_1^3 - (u_1')^3;$$

of degree 4

$$v_1^2, v_2^2, (v_2')^2, v_1v_2, v_1v_2', v_2v_2'.$$

Remark 5.2. *Let P be one of the members of the minimal resolution P_*. We use the following notion for the elements of $\mathrm{Hom}_{A^e}(P, A)$. If $P = A \otimes A$ and $a \in A$, then we denote by a the map which sends $1 \otimes 1$ to a. If $P = A \otimes Q_1 \otimes A$ ($P = A \otimes Q_1^* \otimes A$), $a, b \in A$, then we denote by (a, b) the the map which sends $1 \otimes x \otimes 1$ and $1 \otimes y \otimes 1$ ($1 \otimes r_x \otimes 1$ and $1 \otimes r_y \otimes 1$) to a and b respectively. Moreover, we use the same notation for the corresponding cohomology classes. It follows from the work* [4] *that $p_1 = xy + yx$, $p_2 = xyx$, $p_2' = yxy$, $p_3 = xyxy$, $u_1 = (1 + xy, x)$, $u_1' = (y, 1 + yx)$, $v_1 = (y, x)$, $v_2 = (x, 0)$, $v_2' = (0, y)$ and $z = 1$ in this notation. By* [4, Remarks 3.0.3, 3.1.18] *we have that*

$$(xy + yx, 0), (0, xy + yx), (xyx, yxy) \in B^1(A)$$

and

$$(xy + yx, yxy), (xyx, xy + yx), (xyxy, yxy), (xyx, xyxy) \in B^2(A).$$

We want to compute the Lie bracket and BV structure on $HH^*(kQ_8)$. By definition 1.1 and the Poisson rule,

$$[a \smile b, c] = [a, c] \smile b + (-1)^{|a|(|c|-1)} a \smile [b, c],$$

we have an equality (in characteristic 2)

$$\Delta(abc) = \Delta(ab)c + \Delta(ac)b + \Delta(bc)a + \Delta(a)bc + \Delta(b)ac + \Delta(c)ab. \tag{4}$$

So we need to compute $\Delta(x)$ only for $x \in \mathcal{X}$ and $x = a \cup b$ where $a, b \in \mathcal{X}$. Suppose that $a \in HH^n(kQ_8)$ is given by a cocycle $f : P_n \to A$, then we compute $\Delta(a)$ using the following formula

$$\Delta(a) = \Delta(f \circ \Psi_n) \circ \Phi_{n-1}.$$

It is clear that $\Delta(a) = 0$ for $a \in \{p_1, p_2, p_2', p_3\}$ because Δ is a map of degree -1.

For $b, c \in \mathcal{B}$ we have

$$\langle b, c \rangle = \begin{cases} 1, & \text{if } c = b^*, \\ 0 & \text{overwise.} \end{cases}$$

Then it follows from Theorem 1.2 that

$$\Delta(\alpha)(a_1\otimes\cdots\otimes a_{n-1})=\sum_{b\in\mathcal{B}\setminus\{1\}}\left\langle\sum_{i=1}^{n}(-1)^{i(n-1)}\alpha(a_i\otimes\cdots\otimes a_{n-1}\otimes b\otimes a_1\otimes\cdots\otimes a_{i-1}),\ 1\right\rangle b^*$$

for $\alpha\in C^n(A)$, $a_1,\dots,a_{n-1}\in A$.

Lemma 5.3.

$$\begin{array}{ll}\Delta(u_1)=\Delta(u_1')=0, & \Delta(p_1u_1)=\Delta(p_3u_1)=\Delta(p_2u_1')=p_2',\\ \Delta(p_2u_1)=\Delta(p_2'u_1')=p_1, & \Delta(p_2'u_1)=\Delta(p_1u_1')=\Delta(p_3u_1')=p_2.\end{array}$$

Proof We have

$$p_1u_1=p_2u_1'=(xyxy,xyx),\ p_2u_1=p_2'u_1'=(xyx,0),\ p_2'u_1=p_1u_1'=(yxy,xyxy),$$
$$p_3u_1=(xyxy,0),\ p_3u_1=(0,xyxy)$$

in $HH^1(A)$ (see Remark 5.2).

For $a\in\mathrm{HH}^1(A)$ we have

$$\Delta(a)(1\otimes 1)=\Delta(a\circ\Psi_1)\Phi_0(1\otimes 1)=\sum_{b\in\mathcal{B}\setminus\{1\}}\big\langle a\big(\mathsf{C}(b)\big),\ 1\big\rangle\, b^*.$$

It is easy to check that

$$\big\langle a\big(\mathsf{C}(b)\big),\ 1\big\rangle=\begin{cases}0, & \text{if } a\in\{u_1,u_1'\},\ b\in\mathcal{B},\text{ or } a\in\{p_1u_1,p_3u_1\},\ b\in\mathcal{B}\setminus\{x\},\\ & \text{or } a=p_2u_1,\ b\in\mathcal{B}\setminus\{xy,yx\},\text{ or } a\in\{p_2'u_1,p_3u_1'\},\ b\in\mathcal{B}\setminus\{y\},\\ 1, & \text{if } a\in\{p_1u_1,p_3u_1\},\ b=x,\text{ or } a=p_2u_1,\ b\in\{xy,yx\},\\ & \text{or } a\in\{p_2'u_1,p_3u_1'\},\ b=y.\end{cases}$$

Lemma follows from this formula.

□

Lemma 5.4.

$$\Delta(ab)=0.$$

for $a\in\{v_1,v_2,v_2'\},b\in\{1,p_1,p_2,p_2',p_3\}$.

Proof For $a\in\mathrm{HH}^2(A)$ we have

$$\Delta(a)(1\otimes x\otimes 1)=\Delta(a\circ\Psi_2)\Phi_1(1\otimes x\otimes 1)=\sum_{b\in\mathcal{B}\setminus\{1\}}\langle(a\circ\Psi_2)(b\otimes x+x\otimes b),\ 1\rangle\, b^*,$$

$$\Delta(a)(1\otimes y\otimes 1)=\Delta(a\circ\Psi_2)\Phi_1(1\otimes y\otimes 1)=\sum_{b\in\mathcal{B}\setminus\{1\}}\langle(a\circ\Psi_2)(b\otimes y+y\otimes b),\ 1\rangle\, b^*.$$

Direct calculations show that

$$\Psi_2(b\otimes x+x\otimes b)=t_1(b\otimes x\otimes 1+x\mathsf{C}(b))$$
$$=\begin{cases}0, & \text{if } b\in\{x,y\},\\ +1\otimes r_x\otimes y+y\otimes r_x\otimes yx+yx\otimes r_y\otimes 1, & \text{if } b=xy,\\ y\otimes r_x\otimes 1+xy\otimes r_x\otimes y+1\otimes r_y\otimes xy, & \text{if } b=yx,\\ xy\otimes r_x\otimes 1+x\otimes r_y\otimes xy+1\otimes r_x\otimes yx+yx\otimes r_y\otimes x, & \text{if } b=xyx,\\ 1\otimes r_y\otimes y+y\otimes r_y\otimes 1, & \text{if } b=yxy,\\ x\otimes r_x\otimes x+yxy\otimes r_x\otimes 1, & \text{if } b=xyxy;\end{cases}$$

$$\Psi_2(b\otimes y+y\otimes b)=t_1(b\otimes y\otimes 1+y\mathsf{C}(b))$$
$$=\begin{cases}0, & \text{if } b\in\{x,y\},\\ 1\otimes r_x\otimes yx+x\otimes r_y\otimes 1+yx\otimes r_y\otimes x, & \text{if } b=xy,\\ +1\otimes r_y\otimes x+xy\otimes r_x\otimes 1+x\otimes r_y\otimes xy, & \text{if } b=yx,\\ 1\otimes r_y\otimes y+y\otimes r_y\otimes 1, & \text{if } b=xyx,\\ y\otimes r_x\otimes yx+yx\otimes r_y\otimes 1+1\otimes r_y\otimes xy+xy\otimes r_x\otimes y, & \text{if } b=yxy,\\ 1\otimes r_y\otimes xyx+y\otimes r_y\otimes y, & \text{if } b=xyxy.\end{cases}$$

It follows from Remark 5.2 and the formulas above that $\Delta(v_1)=\Delta(v_2)=\Delta(v_2')=0$ in $HH^1(A)$.

The remaining formulas of lemma can be deduced in the same way. But there is an easier way. By Theorem 5.1 it is enough to prove that $\Delta(p_3u_1^2)=\Delta(p_3(u_1')^2)=0$. And this equalities can be easily deduced from Lemma 5.3 and the formula (4).

□

Lemma 5.5.

$$\Delta(u_1v_1)=\Delta(u_1'v_2')=(u_1')^2+v_2, \Delta(u_1'v_1)=\Delta(u_1v_2)=u_1^2+v_2', \Delta(u_1'v_2)=\Delta(u_1v_2')=v_1$$

in $HH^2(A)$.

Proof For $a\in \mathrm{HH}^3(A)$ we have

$$\begin{aligned}\Delta(a)(1\otimes r_x\otimes 1)&=\Delta(a\circ\Psi_3)\Phi_2(1\otimes r_x\otimes 1)\\ &=\sum_{b\in\mathcal{B}\setminus\{1\}}\langle(a\circ\Psi_3)(b\otimes x\otimes x+x\otimes b\otimes x+x\otimes x\otimes b),\ 1\rangle\, b^*\\ &+\sum_{b\in\mathcal{B}\setminus\{1\}}\langle(a\circ\Psi_3)(b\otimes y\otimes x+x\otimes b\otimes y+y\otimes x\otimes b),\ 1\rangle\, b^*y\\ &+\sum_{b\in\mathcal{B}\setminus\{1\}}\langle(a\circ\Psi_3)(b\otimes yx\otimes y+y\otimes b\otimes yx+yx\otimes y\otimes b),\ 1\rangle\, b^*,\\ \Delta(a)(1\otimes r_y\otimes 1)&=\Delta(a\circ\Psi_3)\Phi_2(1\otimes r_y\otimes 1)\\ &=\sum_{b\in\mathcal{B}\setminus\{1\}}\langle(a\circ\Psi_3)(b\otimes y\otimes y+y\otimes b\otimes y+y\otimes y\otimes b),\ 1\rangle\, b^*\\ &+\sum_{b\in\mathcal{B}\setminus\{1\}}\langle(a\circ\Psi_3)(b\otimes x\otimes y+y\otimes b\otimes x+x\otimes y\otimes b),\ 1\rangle\, b^*x\\ &+\sum_{b\in\mathcal{B}\setminus\{1\}}\langle(a\circ\Psi_3)(b\otimes xy\otimes x+x\otimes b\otimes xy+xy\otimes x\otimes b),\ 1\rangle\, b^*.\end{aligned}$$

Direct calculations (see also the proof of Lemma 5.4) show that

$$\Psi_3(b\otimes x\otimes x+x\otimes b\otimes x+x\otimes x\otimes b)=t_2(b\otimes r_x\otimes 1+xt_1(b\otimes x\otimes 1+x\mathsf{C}(b)))$$
$$=\begin{cases}1\otimes 1, & \text{if } b=x,\\ 0, & \text{if } b=y,\\ 1\otimes y, & \text{if } b=xy,\\ y\otimes 1, & \text{if } b=yx,\\ xy\otimes 1+x\otimes y+yx\otimes xy+1\otimes yx, & \text{if } b=xyx,\\ 1\otimes x+x\otimes 1, & \text{if } b=yxy,\\ 1\otimes yxy, & \text{if } b=xyxy;\end{cases}$$

$$\Psi_3(b\otimes y\otimes x+x\otimes b\otimes y+y\otimes x\otimes b)=t_2(xt_1(b\otimes y\otimes 1)+yt_1(x\mathsf{C}(b)))$$
$$=\begin{cases}0, & \text{if } b\neq xy,\\ 1\otimes yx+y\otimes x+yx\otimes 1+xy\otimes yx, & \text{if } b=xy;\end{cases}$$

$$\Psi_3(b\otimes yx\otimes y+y\otimes b\otimes yx+yx\otimes y\otimes b)=t_2(yt_1(b\otimes y\otimes x+by\otimes x\otimes 1)+yxt_1(y\mathsf{C}(b)))$$
$$=\begin{cases}0, & \text{if } b\in\{x,xy,yx,yxy\},\\ 1\otimes x, & \text{if } b=y,\\ yx\otimes 1, & \text{if } b=xyx,\\ xy\otimes yxy+xyx\otimes x, & \text{if } b=xyxy;\end{cases}$$
$$\Psi_3(b\otimes y\otimes y+y\otimes b\otimes y+y\otimes y\otimes b)=t_2(b\otimes r_y\otimes 1+yt_1(b\otimes y\otimes 1+y\mathsf{C}(b)))$$
$$=\begin{cases}0, & \text{if } b\in\{x,y,xyx\},\\ x\otimes 1, & \text{if } b=xy,\\ 1\otimes x, & \text{if } b=yx,\\ y\otimes x+yx\otimes 1+xy\otimes yx+1\otimes xy, & \text{if } b=yxy,\\ x\otimes yx+xy\otimes x+xyx\otimes 1, & \text{if } b=xyxy;\end{cases}$$
$$\Psi_3(b\otimes x\otimes y+y\otimes b\otimes x+x\otimes y\otimes b)=t_2(yt_1(b\otimes x\otimes 1)+xt_1(y\mathsf{C}(b)))$$
$$=\begin{cases}0, & \text{if } b\in\{x,y,xy,yxy\},\\ xy\otimes 1+x\otimes y+1\otimes xy+yx\otimes xy, & \text{if } b=yx,\\ 1\otimes x+x\otimes 1, & \text{if } b=xyx,\\ y\otimes x, & \text{if } b=xyxy;\end{cases}$$
$$\Psi_3(b\otimes xy\otimes x+x\otimes b\otimes xy+xy\otimes x\otimes b)=t_2(xt_1(b\otimes x\otimes y+bx\otimes y\otimes 1)+xyt_1(x\mathsf{C}(b)))$$
$$=\begin{cases}1\otimes y, & \text{if } b=x,\\ 0, & \text{if } b\in\{y,xy,yx,xyx\},\\ x\otimes y, & \text{if } b=yxy,\\ yxy\otimes y+yx\otimes xyx, & \text{if } b=xyxy.\end{cases}$$

By Theorem 5.1 it is enough to calculate Δ on $u_1'v_2'$, u_1v_2 and $u_1'v_2$. By [4, Lemmas 4.1.2, 4.1.8] and Remark 5.2 we have

$$u_1'v_2'=u_1'T^1(v_2')=(y,1+yx)\begin{pmatrix}0\\ y\otimes 1\end{pmatrix}=y,\ u_1v_2=u_1T^1(v_2)=(1+xy,x)\begin{pmatrix}x\otimes 1\\ 0\end{pmatrix}=x,$$
$$u_1'v_2=u_1T^1(v_2)=(y,1+yx)\begin{pmatrix}x\otimes 1\\ 0\end{pmatrix}=xy,$$
$$u_1^2=u_1\mathrm{T}^1(u_1)=(1+xy,x)\begin{pmatrix}1\otimes(1+xy+yx) & (y+yxy)\otimes 1\\ 1\otimes y+x\otimes x+yxy\otimes 1 & x\otimes 1+1\otimes x+x\otimes yx\end{pmatrix}=(1,y),$$
$$(u_1')^2=u_1'\mathrm{T}^1(u_1')=(y,1+yx)\begin{pmatrix}y\otimes 1+1\otimes y+y\otimes xy & 1\otimes x+y\otimes y+xyx\otimes 1\\ (x+xyx)\otimes 1 & 1\otimes(1+yx+xy)\end{pmatrix}=(x,1).$$

From the formulas above we obtain

$$\Delta(u_1'v_2')=(0,1)=(u_1')^2+v_2,\ \Delta(u_1v_2)=(1,0)=u_1^2+v_2',\ \Delta(u_1'v_2)=(y,x)=v_1.$$

□

Lemma 5.6.

$$\Delta(az)=0.$$

for $a\in\{a,p_1,p_2,p_2',p_3\}$.

Proof It follows from the formula for Φ_3 that we have to calculate Ψ_4 on four kinds of elements:
1) $b\otimes a_1\otimes a_2\otimes a_3$;
2) $a_3\otimes b\otimes a_1\otimes a_2$;
3) $a_2\otimes a_3\otimes b\otimes a_1$;
4) $a_1\otimes a_2\otimes a_3\otimes b$.

In all points $b \in \mathcal{B}$,

$$(a_1, a_2, a_3) \in \mathcal{A} = \{(x,x,x), (x,y,x), (x,yx,y), (y,y,y), (y,x,y), (y,xy,x)\}.$$

1) Note that

$$\Psi_3(a_1 \otimes a_2 \otimes a_3) = t_2(a_1 t_1(a_2 \otimes a_3 \otimes 1)) = \begin{cases} 1 \otimes 1, & \text{if } a_1 = a_2 = a_3 = x, \\ 0 & \text{if } (a_1, a_2, a_3) \in \mathcal{A} \setminus (x,x,x). \end{cases}$$

So if $(a_1, a_2, a_3) \in \mathcal{A}$, then

$$\Psi_4(b \otimes a_1 \otimes a_2 \otimes a_3) = t_3(b\Psi_3(a_1 \otimes a_2 \otimes a_3)) = \begin{cases} 1 \otimes 1, & \text{if } b = xyxy,\ a_1 = a_2 = a_3 = x, \\ 0 & \text{overwise.} \end{cases}$$

2) If $(a_1, a_2, a_3) \in \mathcal{A} \setminus \{(x,x,x), (y,y,y)\}$, then $t_1(a_1 \mathsf{C}(a_2)) = 0$ and so $\Psi_4(a_3 \otimes b \otimes a_1 \otimes a_2) = 0$. For the remaining cases we have

$$\Psi_4(x \otimes b \otimes x \otimes x) = t_3(xt_2(b \otimes r_x \otimes 1)) = \begin{cases} 1 \otimes 1, & \text{if } b = xyxy, \\ 0 & \text{overwise;} \end{cases}$$

$$\Psi_4(y \otimes b \otimes y \otimes y) = t_3(yt_2(b \otimes r \otimes 1)) = \begin{cases} 1 \otimes 1, & \text{if } b = xyxy, \\ 0 & \text{overwise.} \end{cases}$$

Let $b \in \mathcal{B}$, $r \in \{r_x, r_y\}$. Note that $t_3(xt_2(b \otimes r \otimes 1))$ can be nonzero only for $(b, r) = (xyxy, r_x)$. Analogously $t_3(yt_2(b \otimes r \otimes 1))$ can be nonzero only for $(b, r) = (xyxy, r_y)$. Also note that for $b \in \mathcal{B}$, $a \in \{x, y\}$ the element $t_1(b \otimes a \otimes 1)$ is a sum of elements of the form $u \otimes r \otimes v$, where $u, v \in \mathcal{B}$, $r \in \{r_x, r_y\}$ and $(u, r) \notin \{(xyx, r_x), (yx, r_x), (yxy, r_y), (xy, r_y)\}$. So we have the equalities

$$t_3(xt_2(yt_1(A)) = t_3(xt_2(yxt_1(A)) = t_3(yt_2(xt_1(A)) = t_3(yt_2(xyt_1(A)) = 0$$

for any $A \in P_1$. In the same way the equalities

$$t_3(yxt_2(yt_1(A)) = t_3(xyt_2(xt_1(A)) = 0$$

can be proved. Then Ψ_4 can be nonzero in points 3) and 4) only for $a_1 = a_2 = a_3 = x$ and $a_1 = a_2 = a_3 = y$. The same arguments show that

$$t_3(xt_2(xt_1(b \otimes a \otimes 1)) = 0 \ \ ((b, a) \in (\mathcal{B} \times \{x, y\}) \setminus \{xyxy, x\})$$

and

$$t_3(yt_2(yt_1(b \otimes a \otimes 1)) = 0 \ \ ((b, a) \in (\mathcal{B} \times \{x, y\}) \setminus \{xyxy, y\}).$$

So we obtain equalities

$$\Psi_4(x \otimes x \otimes b \otimes x) = \begin{cases} 1 \otimes 1, & \text{if } b = xyxy, \\ 0 & \text{overwise;} \end{cases} \ \Psi_4(x \otimes x \otimes x \otimes b) = 0;$$

$$\Psi_4(y \otimes y \otimes b \otimes y) = \Psi_4(y \otimes y \otimes y \otimes b) = \begin{cases} 1 \otimes 1, & \text{if } b = xyxy, \\ 0 & \text{overwise.} \end{cases}$$

We set

$$\begin{aligned} S(a_1, a_2, a_3, b) &:= \Psi_4(b \otimes a_1 \otimes a_2 \otimes a_3 + a_3 \otimes b \otimes a_1 \otimes a_2 + a_2 \otimes a_3 \otimes b \otimes a_1 + a_1 \otimes a_2 \otimes a_3 \otimes b) \\ &= \begin{cases} 1 \otimes 1, & \text{if } b = xyxy,\ (a_1, a_2, a_3) \in \{(x,x,x), (y,y,y)\}, \\ 0 & \text{if } b \in \mathcal{B} \setminus \{xyxy\} \text{ or } (a_1, a_2, a_3) \in \mathcal{A} \setminus \{(x,x,x), (y,y,y)\}. \end{cases} \end{aligned}$$

Then for $a \in HH^4(A)$ we have

$$\begin{aligned}\Delta(a)(1\otimes 1) &= \Delta(a\circ\Psi_4)\Phi_3(1\otimes 1)\\ &= \sum_{b\in\mathcal{B}\setminus\{1\},(a_1,a_2,a_3)\in\mathcal{A}\setminus\{(x,y,x),(y,x,y)\}} \langle a(S(a_1,a_2,a_3,b)),\ 1\rangle\, b^*\\ &+ \sum_{b\in\mathcal{B}\setminus\{1\}} \langle a(S(x,y,x,b)),\ 1\rangle\, b^*y + \sum_{b\in\mathcal{B}\setminus\{1\}} \langle a(S(y,x,y,b)),\ 1\rangle\, b^*x\\ &= \langle a(1\otimes 1+1\otimes 1),\ 1\rangle = 0.\end{aligned}$$

□

If we know the values of $\Delta(a)$ and $\Delta(b)$, then it is enough to calculate $[a,b]$ to find $\Delta(ab)$. Sometimes it is easier than calculate $\Delta(ab)$ directly. Suppose that a and b are given by cocycles $f: P_n \to A$ and $g: P_m \to A$, then we compute $[a,b]$ using the following formula

$$[a,b] = [f\circ\Psi_n, g\circ\Psi_m]\circ\Phi_{n+m-1}.$$

Lemma 5.7.

$$\Delta(u_1z) = \Delta(u_1'z) = 0.$$

Proof It is enough to prove that $[u_1,z] = [u_1',z] = 0$. For $a \in \{u_1,u_1'\}$ we have

$$[a,z](1\otimes 1) = \big((a\circ\Psi_1)\circ(z\circ\Psi_4)\big)\Phi_4(1\otimes 1) + \big((z\circ\Psi_4)\circ(a\circ\Psi_1)\big)\Phi_4(1\otimes 1).$$

Let prove that $\Psi_4\Phi_4 = Id$. Direct calculations show that $\Psi_3\Phi_3 = Id$ (see the proof of Lemma 5.6). Then

$$(\Psi_4\Phi_4)(1\otimes 1) = \sum_{b\in\mathcal{B}\setminus\{1\}} t_3(b\Psi_3\Phi_3(1\otimes 1))b^* = \sum_{b\in\mathcal{B}\setminus\{1\}} t_3(b\otimes 1)b^* = 1\otimes 1.$$

If $|a| = 1$, we have

$$\big((a\circ\Psi_1)\circ(z\circ\Psi_4)\big)\Phi_4(1\otimes 1) = (a\circ\Psi_1)\big(z(\Psi_4\Phi_4)(1\otimes 1)\big) = (a\circ\Psi_1)(1) = 0.$$

It remains to prove that

$$\big((z\circ\Psi_4)\circ(a\circ\Psi_1)\big)\Phi_4(1\otimes 1) = 0$$

for $a \in \{u_1,u_1'\}$. Let us introduce the notation

$$f = (z\circ\Psi_4)\circ(u_1\circ\Psi_1),\ f' = (z\circ\Psi_4)\circ(u_1'\circ\Psi_1).$$

In this proof we need to know the values of $u_1\circ\Psi_1$ and $u_1'\circ\Psi_1$ on elements of $\mathcal{B}$. Direct calculations show that

$$(u_1\circ\Psi_1)(b) = \begin{cases} 1+xy, & \text{if } b=x,\\ x, & \text{if } b=y,\\ y+yxy, & \text{if } b=xy,\\ y, & \text{if } b=yx,\\ xy+yx, & \text{if } b=xyx,\\ xyx, & \text{if } b=yxy,\\ yxy, & \text{if } b=xyxy;\end{cases} \quad (u_1'\circ\Psi_1)(b) = \begin{cases} y, & \text{if } b=x,\\ 1+yx, & \text{if } b=y,\\ x, & \text{if } b=xy,\\ x+xyx, & \text{if } b=yx,\\ yxy, & \text{if } b=xyx,\\ xy+yx, & \text{if } b=yxy,\\ xyx, & \text{if } b=xyxy.\end{cases}$$

We want to calculate the values of $(z\circ\Psi_4)\circ(a\circ\Psi_1)$ on elements of the form $b\otimes a_1\otimes a_2\otimes a_3$, where $b\in\mathcal{B}\setminus\{1\}$ and $(a_1,a_2,a_3)\in\mathcal{A}$ (see the proof of Lemma 5.6 for notation). Let consider each element of $\mathcal{A}$ separately.

1) Let $a_1 = a_2 = a_3 = x$. Let $a\in\{u_1,u_1'\}$. Then

$$(z\circ\Psi_4)\circ_1(a\circ\Psi_1)(b\otimes x\otimes x\otimes x) = zt_3((a\circ\Psi_1)(b)\otimes 1) = 0,$$

because $(a\circ\Psi_1)(b)$ is a sum of elements of $\mathcal{B}\setminus\{xyxy\}$. Further we have

$$\begin{aligned}
\Psi_3((u_1\circ\Psi_1)(x)\otimes x\otimes x) &= \Psi_3(xy\otimes x\otimes x) = t_2(xy\otimes r_x\otimes 1) = 0;\\
\Psi_2((u_1\circ\Psi_1)(x)\otimes x) &= t_1(xy\otimes x\otimes 1) = 0;\\
\Psi_3(x\otimes x\otimes (u_1\circ\Psi_1)(x)) &= \Psi_3(x\otimes x\otimes xy) = t_2(xt_1(x\otimes x\otimes y + yxy\otimes y\otimes 1)) = 1\otimes y;\\
\Psi_3(y\otimes x\otimes x) &= \Psi_3(x\otimes y\otimes x) = \Psi_3(x\otimes x\otimes y) = 0.
\end{aligned}$$

Consequently,

$$f(b\otimes x\otimes x\otimes x) = \begin{cases} y, & \text{if } b = xyxy,\\ 0 & \text{overwise,}\end{cases} \quad f'(b\otimes x\otimes x\otimes x) = 0.$$

2) $(a_1,a_2,a_3) = (x,y,x)$. We have

$$(z\circ\Psi_4)\circ_1 (a\circ\Psi_1)(b\otimes x\otimes y\otimes x) = (z\circ\Psi_4)\circ_2 (a\circ\Psi_1)(b\otimes x\otimes y\otimes x) = 0$$

for $a\in\{u_1,u_1'\}$, because $\Psi_2(y\otimes x) = 0$. Further we have

$$\begin{aligned}
\Psi_3(x\otimes (u_1\circ\Psi_1)(y)\otimes x) &= \Psi_3(x\otimes x\otimes x) = 1\otimes 1;\\
\Psi_2(y\otimes (u_1\circ\Psi_1)(x)) &= \Psi_2(y\otimes xy) = t_1(y\otimes x\otimes y + yx\otimes y\otimes 1) = 0;\\
\Psi_3(x\otimes (u_1'\circ\Psi_1)(y)\otimes x) &= \Psi_3(x\otimes yx\otimes x) = t_2(xt_1(yx\otimes x))\\
&= t_2(xy\otimes r_x\otimes 1 + x\otimes r_y\otimes xy) = 0;\\
\Psi_3(x\otimes y\otimes (u_1'\circ\Psi_1)(x)) &= \Psi_3(x\otimes y\otimes y) = t_2(x\otimes r_y\otimes 1) = 0.
\end{aligned}$$

Consequently,

$$f(b\otimes x\otimes y\otimes x) = \begin{cases} 1, & \text{if } b = xyxy,\\ 0 & \text{overwise,}\end{cases} \quad f'(b\otimes x\otimes y\otimes x) = 0.$$

3) $(a_1,a_2,a_3) = (x,yx,y)$. We have

$$(z\circ\Psi_4)\circ_1 (a\circ\Psi_1)(b\otimes x\otimes yx\otimes y) = (z\circ\Psi_4)\circ_2 (a\circ\Psi_1)(b\otimes x\otimes yx\otimes y) = 0$$

for $a\in\{u_1,u_1'\}$, because $\Psi_2(yx\otimes y) = 0$. Further we have

$$\begin{aligned}
\Psi_3(x\otimes (u_1\circ\Psi_1)(yx)\otimes y) &= \Psi_3(x\otimes y\otimes y) = t_2(x\otimes r_y\otimes 1) = 0;\\
\Psi_3(x\otimes yx\otimes (u_1\circ\Psi_1)(y)) &= \Psi_3(x\otimes yx\otimes x) = t_2(xy\otimes r_x\otimes 1 + x\otimes r_y\otimes xy) = 0;\\
\Psi_2((u_1'\circ\Psi_1)(yx)\otimes y) &= \Psi_2((x+xyx)\otimes y) = t_1(x\otimes y\otimes 1 + xyx\otimes y\otimes 1) = 0;\\
\Psi_3(x\otimes yx\otimes (u_1\circ\Psi_1)(y)) &= \Psi_3(x\otimes yx\otimes yx) = t_2(xt_1(yx\otimes y\otimes x + yxy\otimes x\otimes 1))\\
&= t_2(x\otimes r_y\otimes y + xy\otimes r_y\otimes 1) = x\otimes 1.
\end{aligned}$$

Consequently,

$$f(b\otimes x\otimes yx\otimes y) = 0,\ f'(b\otimes x\otimes yx\otimes y) = \begin{cases} 1, & \text{if } b = yxy,\\ 0 & \text{overwise.}\end{cases}$$

4) Let $a_1 = a_2 = a_3 = y$. Let $a\in\{u_1,u_1'\}$. Then

$$(z\circ\Psi_4)\circ_1 (a\circ\Psi_1)(b\otimes y\otimes y\otimes y) = 0,$$

because $\Psi_3(y\otimes y\otimes y) = 0$. Further we have

$$\begin{aligned}
\Psi_3(x\otimes y\otimes y) &= \Psi_3(y\otimes x\otimes y) = \Psi_3(y\otimes y\otimes x) = 0,\\
\Psi_3((u_1'\circ\Psi_1)(y)\otimes y\otimes y) &= \Psi_3(yx\otimes y\otimes y) = t_2(yx\otimes r_y\otimes 1) = 0;\\
\Psi_2((u_1'\circ\Psi_1)(y)\otimes y) &= t_1(yx\otimes y\otimes 1) = 0;\\
\Psi_3(y\otimes y\otimes (u_1'\circ\Psi_1)(y)) &= \Psi_3(y\otimes y\otimes yx) = t_2(yt_1(y\otimes y\otimes x + xyx\otimes x\otimes 1)) = 1\otimes x.
\end{aligned}$$

Consequently,

$$f(b\otimes y\otimes y\otimes y)=0,\ f'(b\otimes y\otimes y\otimes y)=\begin{cases}x, & \text{if } b=xyxy,\\ 0 & \text{overwise.}\end{cases}.$$

5) $(a_1,a_2,a_3)=(y,x,y)$. We have

$$(z\circ\Psi_4)\circ_1(a\circ\Psi_1)(b\otimes y\otimes x\otimes y)=(z\circ\Psi_4)\circ_2(a\circ\Psi_1)(b\otimes y\otimes x\otimes y)=0$$

for $a\in\{u_1,u_1'\}$, because $\Psi_2(x\otimes y)=0$. Further we have

$$\begin{aligned}
\Psi_3(y\otimes(u_1\circ\Psi_1)(x)\otimes y)&=t_2(yt_1(xy\otimes y\otimes 1))=t_2(y\otimes r_x\otimes yx+yx\otimes r_y\otimes 1)=0;\\
\Psi_3(y\otimes x\otimes(u_1\circ\Psi_1)(y))&=\Psi_3(y\otimes x\otimes x)=0;\\
\Psi_3(y\otimes(u_1'\circ\Psi_1)(x)\otimes y)&=\Psi_3(y\otimes y\otimes y)=0;\\
\Psi_2(x\otimes(u_1'\circ\Psi_1)(y))&=\Psi_2(x\otimes yx)=t_1(x\otimes y\otimes x+xy\otimes x\otimes 1)=0.
\end{aligned}$$

Consequently,

$$f(b\otimes y\otimes x\otimes y)=f'(b\otimes y\otimes x\otimes y)=0.$$

6) $(a_1,a_2,a_3)=(y,xy,x)$. We have

$$(z\circ\Psi_4)\circ_1(a\circ\Psi_1)(b\otimes y\otimes xy\otimes x)=(z\circ\Psi_4)\circ_2(a\circ\Psi_1)(b\otimes y\otimes xy\otimes x)=0$$

for $a\in\{u_1,u_1'\}$, because $\Psi_2(xy\otimes x)=0$. Further we have

$$\begin{aligned}
\Psi_3(y\otimes(u_1\circ\Psi_1)(xy)\otimes x)&=\Psi_3(y\otimes(y+yxy)\otimes x)=t_2(y\otimes r_y\otimes y-xyx\otimes r_y\otimes 1)=0;\\
\Psi_2(xy\otimes(u_1\circ\Psi_1)(x))&=\Psi_2(xy\otimes xy)=t_1(xy\otimes x\otimes y+xyx\otimes y\otimes 1)=0;\\
\Psi_3(y\otimes(u_1'\circ\Psi_1)(xy)\otimes x)&=\Psi_3(y\otimes x\otimes x)=0;\\
\Psi_3(y\otimes xy\otimes(u_1'\circ\Psi_1)(x))&=\Psi_3(y\otimes xy\otimes y)=t_2(y\otimes r_x\otimes yx+yx\otimes r_y\otimes 1)=0.
\end{aligned}$$

Consequently,

$$f(b\otimes y\otimes xy\otimes x)=f'(b\otimes y\otimes xy\otimes x)=0.$$

Thus we obtain

$$\begin{aligned}
f\Phi_4(1\otimes 1)=&\sum_{b\in\mathcal{B}\setminus\{1\}}f(b\otimes x\otimes x\otimes x+b\otimes x\otimes yx\otimes y+b\otimes y\otimes y\otimes y+b\otimes y\otimes xy\otimes x)b^*\\
&+\sum_{b\in\mathcal{B}\setminus\{1\}}f(b\otimes x\otimes y\otimes x)yb^*+\sum_{b\in\mathcal{B}\setminus\{1\}}f(b\otimes y\otimes x\otimes y)xb^*=y+y=0;\\
f'\Phi_4(1\otimes 1)=&\sum_{b\in\mathcal{B}\setminus\{1\}}f'(b\otimes x\otimes x\otimes x+b\otimes x\otimes yx\otimes y+b\otimes y\otimes y\otimes y+b\otimes y\otimes xy\otimes x)b^*\\
&+\sum_{b\in\mathcal{B}\setminus\{1\}}f'(b\otimes x\otimes y\otimes x)yb^*+\sum_{b\in\mathcal{B}\setminus\{1\}}f'(b\otimes y\otimes x\otimes y)xb^*=x+x=0;
\end{aligned}$$

□

Lemma 5.8.

$$\Delta(v_1z)=\Delta(v_2z)=\Delta(v_2'z)=0.$$

Proof Firstly note that it is enough to prove that $[v_2,z]=0$. Indeed, by Jacoby identity and Lemmas 5.3–5.7 we have

$$\begin{aligned}
\Delta(v_1z)&=[v_1,z]=[[u_1',v_2],z]=[v_2,[u_1',z]]+[u_1',[v_2,z]]=[u_1',[v_2,z]],\\
\Delta(v_2'z)&=[v_2',z]=[[u_1,v_2]+u_1^2,z]=[v_2,[u_1,z]]+[u_1,[v_2,z]]+2[u_1,z]=[u_1,[v_2,z]]
\end{aligned}$$

and $\Delta(v_2z)=[v_2,z]$. For $a\in\{x,y\}$ we have

$$[v_2,z](1\otimes a\otimes 1)=\big((v_2\circ\Psi_2)\circ(z\circ\Psi_4)\big)\Phi_5(1\otimes a\otimes 1)+\big((z\circ\Psi_4)\circ(v_2\circ\Psi_2)\big)\Phi_5(1\otimes a\otimes 1).$$

Note that if $(a_1,a_2,a_3)\in\mathcal{A}\setminus\{(x,x,x),(y,y,y)\}$, then $\Psi_2(1\otimes a_1\otimes a_2\otimes 1)=\Psi_2(1\otimes a_2\otimes a_3\otimes 1)=0$ (see the proof of Lemma 5.6 for the notion of $\mathcal{A}$). It follows from this that

$$\big((v_2\circ\Psi_2)\circ_i(z\circ\Psi_4)\big)(a\otimes b\otimes a_1\otimes a_2\otimes a_3)=0$$

for $i\in\{1,2\}$, $a\in\{x,y\}$, $b\in\mathcal{B}$, $(a_1,a_2,a_3)\in\mathcal{A}\setminus\{(x,x,x),(y,y,y)\}$. Further we have

$$(z\circ\Psi_4)(b\otimes y\otimes y\otimes y)=zt_3(b\Psi_3(y\otimes y\otimes y))=0,$$

$$(z\circ\Psi_4)(a\otimes b\otimes y\otimes y)=zt_3(at_2(b\otimes r_y\otimes 1))=\begin{cases}1, & \text{if } a=y,\ b=xyxy,\\ 0, & \text{if } a=x \text{ or } b\in\mathcal{B}\setminus\{xyxy\}.\end{cases}$$

Because of $\Psi_2(1\otimes 1\otimes y\otimes 1)=0$, we have

$$\big((v_2\circ\Psi_2)\circ(z\circ\Psi_4)\big)(a\otimes b\otimes y\otimes y\otimes y)=0.$$

Further we have

$$(z\circ\Psi_4)(b\otimes x\otimes x\otimes x)=zt_3(b\otimes 1)=\begin{cases}1, & \text{if } b=xyxy,\\ 0, & \text{if } b\in\mathcal{B}\setminus\{xyxy\};\end{cases}$$

$$(z\circ\Psi_4)(a\otimes b\otimes x\otimes x)=zt_3(at_2(b\otimes r_x\otimes 1))=\begin{cases}1, & \text{if } a=x,\ b=xyxy,\\ 0, & \text{if } a=y \text{ or } b\in\mathcal{B}\setminus\{xyxy\}.\end{cases}$$

Because of $\Psi_2(1\otimes a\otimes 1\otimes 1)=\Psi_2(1\otimes 1\otimes x\otimes 1)=0$, we have

$$\big((v_2\circ\Psi_2)\circ(z\circ\Psi_4)\big)(a\otimes b\otimes x\otimes x\otimes x)=0.$$

It remains to prove that

$$\big((z\circ\Psi_4)\circ(v_2\circ\Psi_2)\big)\Phi_5=0$$

in $HH^5(A)$. If $(a_1,a_2,a_3)\in\mathcal{A}\setminus\{(x,x,x),(y,y,y)\}$, then

$$\big((z\circ\Psi_4)\circ_i(v_2\circ\Psi_2)\big)(a\otimes b\otimes a_1\otimes a_2\otimes a_3)=0$$

for $1\le i\le 4$. It follows from the formulas $(v_2\circ\Psi_2)(a_1\otimes a_2)=(v_2\circ\Psi_2)(a_2\otimes a_3)=0$ and the fact that

$$(z\circ\Psi_4)(u\otimes v\otimes a_2\otimes a_3)=0$$

for all $u,v\in A$. We have

$$\big((z\circ\Psi_4)\circ_i(v_2\circ\Psi_2)\big)(a\otimes b\otimes y\otimes y\otimes y)=0$$

for $i=3$ and $i=4$ because $(v_2\circ\Psi_2)(y\otimes y)=v_2(1\otimes r_y\otimes 1)=0$. Moreover

$$(z\circ\Psi_4)\big((v_2\circ\Psi_2)(a\otimes b)\otimes y\otimes y\otimes y\big)=zt_3\big((v_2\circ\Psi_2)(a\otimes b)\Psi_3(y\otimes y\otimes y)\big)=0.$$

Also we have

$$(z\circ\Psi_4)\big(a\otimes(v_2\circ\Psi_2)(b\otimes y)\otimes y\otimes y\big)=zt_3\Big(at_2\big(v_2t_1(b\otimes y\otimes 1)\otimes r_y\otimes 1\big)\Big)$$

$$=\begin{cases}zt_3(at_2(xyx\otimes r_y\otimes 1)), & \text{if } b=xy,\\ zt_3(at_2(xyxy\otimes r_y\otimes 1)), & \text{if } b=yxy,\\ 0, & \text{if } b\in\mathcal{B}\setminus\{xy,yxy\}\end{cases}=\begin{cases}1, & \text{if } a=y,\ b=yxy,\\ 0, & \text{if } a=x \text{ or } b\in\mathcal{B}\setminus\{yxy\}.\end{cases}$$

Thus

$$\big((z\circ\Psi_4)\circ(v_2\circ\Psi_2)\big)(a\otimes b\otimes y\otimes y\otimes y)=\begin{cases}1, & \text{if } a=y,\ b=yxy,\\ 0, & \text{if } a=x \text{ or } b\in\mathcal{B}\setminus\{yxy\}.\end{cases}$$

We have

$$\big((z\circ\Psi_4)\circ_i(v_2\circ\Psi_2)\big)(a\otimes b\otimes x\otimes x\otimes x)=(z\circ\Psi_4)(a\otimes b\otimes x\otimes x)$$

for $i=3$ and $i=4$. So

$$\big((z\circ\Psi_4)\circ_3(v_2\circ\Psi_2)\big)(a\otimes b\otimes x\otimes x\otimes x)+\big((z\circ\Psi_4)\circ_4(v_2\circ\Psi_2)\big)(a\otimes b\otimes x\otimes x\otimes x)=0.$$

Further we have

$$(z\circ\Psi_4)\big((v_2\circ\Psi_2)(a\otimes b)\otimes x\otimes x\otimes x\big)=zt_3(v_2t_1(a\mathsf{C}(b))\otimes 1).$$

Direct calculations show that

$$v_2t_1(a\mathsf{C}(b))=\begin{cases}x, & \text{if } a=x,\, b=x,\\ 0, & \text{if } a=x,\, b\in\{y,yx,yxy\} \text{ or } a=y,\, b\in\{x,y,xy,xyx,xyxy\},\\ xy+xyxy, & \text{if } a=x,\, b=xy,\\ xyx, & \text{if } a=x,\, b=xyx \text{ or } a=y,\, b=yx,\\ xyxy, & \text{if } a=x,\, b=xyxy \text{ or } a=y,\, b=yxy.\end{cases}$$

Finely we have

$$(z\circ\Psi_4)\big(a\otimes(v_2\circ\Psi_2)(b\otimes x)\otimes x\otimes x\big)=zt_3(at_2(v_2t_1(b\otimes x\otimes 1)\otimes r_x\otimes 1)).$$

Note that $t_3(yt_2(u\otimes r_x\otimes 1))=0$ for any $u\in\mathcal{B}$. Direct calculations show that

$$v_2t_1(b\otimes x\otimes 1)=\begin{cases}x, & \text{if } b=x,\\ 0, & \text{if } b\in\{y,xy,yxy\},\\ yx+xyxy, & \text{if } b=yx,\\ xyx, & \text{if } b=xyx,\\ xyxy, & \text{if } b=xyxy.\end{cases}$$

Then

$$\begin{aligned}&((z\circ\Psi_4)\circ(v_2\circ\Psi_2)\big)(a\otimes b\otimes x\otimes x\otimes x)\\ &=\begin{cases}1, & \text{if } a=x,\, b=xy \text{ or } a=x,\, b=yx \text{ or } a=y,\, b=yxy\\ 0, & \text{if } a=x,\, b\in\mathcal{B}\setminus\{xy,yx\} \text{ or } a=y,\, b\in\mathcal{B}\setminus\{yxy\}.\end{cases}\end{aligned}$$

Thus we have

$$\big((z\circ\Psi_4)\circ(v_2\circ\Psi_2)\big)\Phi_5(1\otimes x\otimes 1)=xy+yx,$$
$$\big((z\circ\Psi_4)\circ(v_2\circ\Psi_2)\big)\Phi_5(1\otimes y\otimes 1)=x+x=0.$$

So $[v_2,z]=(xy+yx,0)=0$, because $(xy+yx,0)\in B^1(A)=B^5(A)$ by Remark 5.2 and the 4-periodicity of the resolution P_*.

□

We now can prove a theorem which describes the BV structure on $HH^*(A)$.

Theorem 5.9. *Let $A=kQ_8$, $\mathrm{char}k=2$ and Δ be the BV differential from Theorem 1.2. Then*
1) *Δ is equal 0 on the generators of $HH^*(A)$ from Theorem 5.1;*
2) *$\Delta(ab)=0$ for $a\in\{v_1,v_2,v_2'\}$, $b\in\{p_1,p_2,p_2',p_3\}$;*
3) *$\Delta(az)=0$ if a is a generator of $HH^*(A)$ from Theorem 5.1;*
4) *Δ satisfies the equalities*

$$\Delta(p_1u_1)=\Delta(p_3u_1)=\Delta(p_2u_1')=p_2',\ \Delta(p_2u_1)=\Delta(p_2'u_1')=p_1,$$
$$\Delta(p_2'u_1)=\Delta(p_1u_1')=\Delta(p_3u_1')=p_2,\ \Delta(u_1v_1)=\Delta(u_1'v_2')=(u_1')^2+v_2,$$
$$\Delta(u_1'v_1)=\Delta(u_1v_2)=u_1^2+v_2',\ \Delta(u_1'v_2)=\Delta(u_1v_2')=v_1.$$

Points 1)–4) with Theorem 5.1 determines BV algebra structure (and in particular Gerstenhaber algebra structure) on $HH^*(A)$.

Proof Points 1)–4) follow from Lemmas 5.3–5.8. To determine BV algebra structure we need the value of Δ on generators and all their pairwise products. Point 1) determines Δ on generators. Points 2)–4) determine Δ on all pairwise products of generators except zero products (see Theorem 5.1) and squares of generators. All the listed products are zero in characteristic two. So BV structure is fully determined.

□

Corollary 5.10. *Let $A = kQ_8$, $\mathrm{char} k = 2$ and $[,]$ be the Gerstehaber bracket from Theorem 1.2. Then the bracket is zero for all pairs of generators of $HH^*(A)$ from Theorem 5.1 exept:*

$$[p_1, u_1] = [p_3, u_1] = [p_2, u_1'] = p_2',\ [p_2, u_1] = [p_2', u_1'] = p_1,$$
$$[p_2', u_1] = [p_1, u_1'] = [p_3, u_1'] = p_2,\ [u_1, v_1] = [u_1', v_2'] = (u_1')^2 + v_2,$$
$$[u_1', v_1] = [u_1, v_2] = u_1^2 + v_2',\ [u_1', v_2] = [u_1, v_2'] = v_1.$$

This completely determines Gerstehaber algebra structure on $HH^*(A)$.

Proof From the Theorem 5.9 we know, that BV-differential Δ equals zero on any generator of BV-algebra $HH^*(A)$. Then using formula from the definition 1.1 one immediately has $[a, b] = \Delta(a \smile b)$ for any a, b from the set of generators of $HH^*(A)$.

□

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