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## Abstract convex approximations of nonsmooth functions

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In the article we use abstract convexity theory in order to unify and generalize many different concepts of nonsmooth analysis. We introduce the concepts of abstract codifferentiability, abstract quasidifferentiability and abstract convex (concave) approximations of a nonsmooth function mapping a topological vector space to an order complete topological vector lattice. We study basic properties of these notions, construct elaborate calculus of abstract codifferentiable functions and discuss continuity of abstract codifferential. We demonstrate that many classical concepts of nonsmooth analysis, such as subdifferentiability and quasidifferentiability, are particular cases of the concepts of abstract codifferentiability and abstract quasidifferentiability. We also show that abstract convex and abstract concave approximations are a very convenient tool for the study of nonsmooth extremum problems. We use these approximations in order to obtain various necessary optimality conditions for nonsmooth nonconvex optimization problems with the abstract codifferentiable or abstract quasidifferentiable objective function and constraints. Then, we demonstrate how these conditions can be transformed into simpler and more constructive conditions in some particular cases.

**Keywords:** nonsmooth analysis; abstract codifferentiability; abstract quasidifferentiability; abstract convex approximation

**AMS Subject Classifications:** 49K27; 90C30; 90C48

### 1. Introduction

One of the first ideas in the study of the local behaviour of a function was to approximate the function under consideration in a neighbourhood of a point by a very simple function, namely linear function, and to use this linear function in order to study some properties of the initial function. This simple idea gave rise to the concept of derivative and eventually led to the development of classical differential calculus. In the twentieth century, various generalizations of the derivative were proposed in nonsmooth analysis. Most of these generalizations are just modifications of the directional derivative or the subgradient and the subdifferential of a convex function (see, e.g. [1–5]). Although these generalizations are effective tools for solving various nonsmooth problems, they are discontinuous, as well as the subdifferential of a convex function in the nonsmooth case. A lack of continuity causes

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a lot of complications and makes the construction of effective and convenient numerical methods very difficult.

However, there is a different way of a generalization of the classical derivative. In order to study a more broad class of functions than the class of differentiable functions, one should simply approximate a function in a neighbourhood of a point by more broad (and inevitably more complicated) set of functions than the set of linear functions. From the point of view of optimization, a natural candidate on the role of the set of approximating functions is the set of convex (concave or the sum of convex and concave) functions, since the class of convex functions is the simplest and the most profoundly studied class of functions in optimization. For a long time this simple idea had not been fulfilled in nonsmooth analysis, until in 1988 Demyanov introduced the concept of codifferentiable function [6] (that implicitly carried out this idea) in order to construct a continuous approximation of a nonsmooth function. Since usually a continuous approximation of a nonsmooth function must be nonhomogeneous, (see the introduction in [7]), then we naturally come to the following definition of codifferentiable function, which is a generalization of the concept of quasidifferentiable function.[7] Let  $\Omega \subset \mathbb{R}^d$  be an open set. A function  $f: \Omega \rightarrow \mathbb{R}$  is called codifferentiable at a point  $x \in \Omega$ , if there exist convex compact sets  $\underline{d}f(x), \bar{d}f(x) \subset \mathbb{R}^{d+1}$  such that for any admissible  $\Delta x \in \mathbb{R}^d$  (i.e. such that  $\text{co}\{x, x + \Delta x\} \subset \Omega$ ) one has

$$f(x + \Delta x) - f(x) = \max_{(a,v) \in \underline{d}f(x)} (a + \langle v, \Delta x \rangle) + \min_{(b,w) \in \bar{d}f(x)} (b + \langle w, \Delta x \rangle) + o(\Delta x, x),$$

where  $o(\alpha \Delta x, x)/\alpha \rightarrow 0$  as  $\alpha \rightarrow +0$ , and  $\langle \cdot, \cdot \rangle$  is the inner product in  $\mathbb{R}^d$ . In actuality, the previous definition is equivalent to the following one (cf. Example 3.10 below): a function  $f: \Omega \rightarrow \mathbb{R}$  is said to be codifferentiable at a point  $x \in \Omega$  if there exist a finite convex function  $\Phi: \mathbb{R}^d \rightarrow \mathbb{R}$  and a finite concave function  $\Psi: \mathbb{R}^d \rightarrow \mathbb{R}$  such that for any admissible  $\Delta x \in \mathbb{R}^d$

$$f(x + \Delta x) - f(x) = \Phi(\Delta x) + \Psi(\Delta x) + o(\Delta x, x),$$

where  $o(\alpha \Delta x, x)/\alpha \rightarrow 0$  as  $\alpha \rightarrow +0$ . The concept of codifferentiability appeared to be amazingly effective tool for solving various optimization problems.[7–10] Let us mention here the marvellous ability of the method of codifferential descent [7] to ‘jump over’ some points of local minimum.[10]

The aim of this article is to take the next step and to utilize some ideas of abstract convexity in nonsmooth analysis. Namely, we introduce and study the concepts of abstract codifferentiability, abstract quasidifferentiability, and abstract convex and abstract concave approximations of a nonsmooth function mapping a topological vector space to an order complete topological vector lattice. These concepts are based on the idea of an approximation of a function in a neighbourhood of a point by an abstract convex function (or an abstract concave function, or the sum of abstract convex and abstract concave functions). Actually, many well-known notions of nonsmooth analysis, such as subdifferentiability, quasidifferentiability, codifferentiability, exhauster and coexhauster, are just a particular cases of the concepts of abstract quasidifferentiability and abstract codifferentiability. Thus, the theory presented in the article gives us a new understanding of these notions and allows one to present many different concepts and results of nonsmooth analysis in a unified and convenient framework. Moreover, the theory of abstract codifferentiability furnishes one with a useful approach to the construction and study of continuous approximations of nonsmooth functions. Therefore, we pay a lot of attention to the problem of continuity of

an abstract codifferential and thoroughly develop the calculus of abstract codifferentiable functions.

In the article, we also derive necessary optimality conditions for various nonsmooth nonconvex optimization problems with the use of the abstract convex and abstract concave approximations of a nonsmooth function. The author thinks that the abstract convex and abstract concave approximations are a very effective tool for the study of various nonsmooth constrained extremum problems, since they allow one to obtain necessary optimality conditions for these problems in a very simple manner. As applications of the general theory, we give new characterizations of some classes of nonsmooth functions and obtain new necessary optimality conditions for these classes of functions.

## 2. Preliminaries

In this section, we recall some basic notions from abstract convexity, and introduce several specific sets and operations on these sets, which will simplify the exposition of the main results in the article. We assume that the reader is familiar with some basic definitions and facts from the theory of topological vector lattices [11,12] and abstract convex analysis.[13–15]

### 2.1. Abstract convexity

We recall some definitions from abstract convexity, that is used in subsequent. Let  $X$  be an arbitrary nonempty set,  $E$  be a complete lattice,  $f: X \rightarrow E$  be an arbitrary function, and let  $H$  be a nonvoid set of mappings  $h: X \rightarrow E$ . If  $h \in H$  and  $h(x) \leq f(x)$  for all  $x \in X$ , then we write  $h \leq f$  (or  $f \geq h$ ).

*Definition 2.1* The function  $f$  is called abstract convex with respect to  $H$  (or  $H$ -convex) if there exists a nonempty set  $U \subset H$  such that  $f(x) = \sup_{h \in U} h(x)$  for all  $x \in X$ . In this case one says that the abstract convex function  $f$  is generated by  $U$ .

The function  $f$  is called abstract concave with respect to  $H$  (or  $H$ -concave) if there exists a nonempty set  $V \subset H$  such that  $f(x) = \inf_{h \in V} h(x)$  for any  $x \in X$ . In the latter case, the abstract concave function  $f$  is said to be generated by  $V$ .

The set  $\text{supp}^+(f, H) = \{h \in H \mid f \leq h\}$  is called an upper support set of  $f$  with respect to  $H$  and the set  $\text{supp}^-(f, H) = \{h \in H \mid f \geq h\}$  is referred to as a lower support set of  $f$  with respect to  $H$ . The set  $\underline{\partial}_H f(x) = \{h \in \text{supp}^-(f, H) \mid h(x) = f(x)\}$  is called an  $H$ -subdifferential of  $f$  at  $x$  and the set  $\overline{\partial}_H f(x) = \{h \in \text{supp}^+(f, H) \mid h(x) = f(x)\}$  is referred to as an  $H$ -superdifferential of  $f$  at  $x$ .

Note an obvious condition for the global minimum (maximum) of the function  $f$  via abstract convex structures. Suppose that the set  $H$  contains all constant functions. Then, it is easy to check that for the function  $f$  to have a global minimum (maximum) value at a point  $x^*$  it is necessary and sufficient that

$$f(x^*) \in \underline{\partial}_H f(x^*) \quad (f(x^*) \in \overline{\partial}_H f(x^*)). \tag{1}$$

One can suppose that only the constant function  $h \equiv f(x^*)$  belongs to  $H$  in order to get (1).

## 2.2. Special sets

Let  $X$  be an arbitrary nonvoid set and  $E$  be an order complete vector lattice. We add the improper elements  $+\infty$  and  $-\infty$  to the vector lattice  $E$ , where, as usual,  $+\infty$  is considered as a greatest element and  $-\infty$  is considered as a least element. Denote  $\bar{E} = E \cup \{-\infty\} \cup \{+\infty\}$ . It is clear that  $\bar{E}$  endowed with an obvious order relation is a complete lattice. Set

$$\begin{aligned} x + (+\infty) &= (+\infty) + x = +\infty, & x + (-\infty) &= (-\infty) + x = -\infty, \\ \alpha(+\infty) &= +\infty, & \alpha(-\infty) &= -\infty & \text{if } \alpha > 0, \\ \alpha(+\infty) &= -\infty, & \alpha(-\infty) &= +\infty & \text{if } \alpha < 0. \end{aligned}$$

We will not consider such expressions as  $+\infty + (-\infty)$  or  $0(+\infty)$ . For an arbitrary function  $F: X \rightarrow \bar{E}$  denote  $\text{dom } F = \{x \in X \mid F(x) \neq -\infty, F(x) \neq +\infty\}$ . The sum  $l = p + q$  of functions  $p, q: X \rightarrow \bar{E}$  is said to be well-defined if  $p^{-1}(e) \cap q^{-1}(-e) = \emptyset$  when  $e \in \{+\infty, -\infty\}$ . Here,  $p^{-1}(e)$  is the preimage of  $e$  under  $p$ .

Let  $H$  be a nonempty set of functions  $h: X \rightarrow \bar{E}$ . The set  $H$  is said to be closed under addition if for any  $h_1, h_2 \in H$  the sum  $h_1 + h_2$  is well-defined and belongs to  $H$ .

Let  $\mathfrak{F}$  be a filter on  $X$ . Denote by  $PF(X, \mathfrak{F}, \bar{E}, H)$  the set consisting of all pairs of functions  $(\Phi, \Psi)$  such that  $\Phi: X \rightarrow \bar{E}$  is  $H$ -convex,  $\Psi: X \rightarrow \bar{E}$  is  $H$ -concave, and there exists  $S \in \mathfrak{F}$  such that  $S \subset \text{dom } \Phi \cap \text{dom } \Psi$ .

*Remark 1* We will only consider values of the sum  $\Phi + \Psi$  in a ‘neighbourhood’ of a given point  $x$ , since the sum  $\Phi + \Psi$  will serve as an approximation of the increment of a function in this ‘neighbourhood’. Therefore, it is natural to demand that the sum  $\Phi + \Psi$  is well-defined and finite only in a ‘neighbourhood’ of  $x$ . Thus, the filter  $\mathfrak{F}$  will usually be the filter of neighbourhoods of a point  $x$ .

In subsequent we will consider an approximation of the increment of a function by the sum of  $H$ -convex and  $H$ -concave functions. Different pairs of  $H$ -convex and  $H$ -concave functions could define the same approximation. Therefore, it is convenient to introduce the set of equivalence classes of pairs of  $H$ -convex and  $H$ -concave functions that define the same approximation.

Let us introduce a binary relation  $\sigma$  on the set  $PF(X, \mathfrak{F}, \bar{E}, H)$ . We say that  $((\Phi_1, \Psi_1), (\Phi_2, \Psi_2)) \in \sigma$ , where  $(\Phi_i, \Psi_i) \in PF(X, \mathfrak{F}, \bar{E}, H)$ ,  $i \in \{1, 2\}$ , if and only if there exists  $S \in \mathfrak{F}$  such that  $S \subset \text{dom } \Phi_i \cap \text{dom } \Psi_i$ ,  $i \in \{1, 2\}$  and

$$\Phi_1(x) + \Psi_1(x) = \Phi_2(x) + \Psi_2(x) \quad \forall x \in S.$$

It is easy to see that  $\sigma$  is an equivalence relation on  $PF(X, \mathfrak{F}, \bar{E}, H)$ . The quotient set of  $PF(X, \mathfrak{F}, \bar{E}, H)$  by  $\sigma$  is denoted by  $EPPF(X, \mathfrak{F}, \bar{E}, H)$ . If  $(\Phi, \Psi) \in PF(X, \mathfrak{F}, \bar{E}, H)$ , then the equivalence class of  $(\Phi, \Psi)$  under  $\sigma$  is denoted by  $[\Phi, \Psi]$ .

Since an  $H$ -convex (or  $H$ -concave) function is defined by a subset of the set  $H$ , then one can consider the set  $PS(H, \mathfrak{F})$  instead of  $PF(X, \mathfrak{F}, \bar{E}, H)$ , where  $PS(H, \mathfrak{F})$  is the set consisting of all pairs  $(U, V)$  of nonempty sets  $U, V \subset H$  such that  $(\sup_{h \in U} h, \inf_{p \in V} p) \in PF(X, \mathfrak{F}, \bar{E}, H)$ . Let us introduce a binary relation  $\hat{\sigma}$  on the set  $PS(H, \mathfrak{F})$ , which is similar to the relation  $\sigma$ . Define  $((U_1, V_1), (U_2, V_2)) \in \hat{\sigma}$ , where  $(U_i, V_i) \in PS(H, \mathfrak{F})$ ,  $i \in \{1, 2\}$ , if and only if

$$\left( \left( \sup_{h_1 \in U_1} h_1, \inf_{p_1 \in V_1} p_1 \right), \left( \sup_{h_2 \in U_2} h_2, \inf_{p_2 \in V_2} p_2 \right) \right) \in \sigma$$

It is obvious that  $\widehat{\sigma}$  is an equivalence relation on  $PS(H, \mathfrak{F})$ . The quotient set of  $PS(H, \mathfrak{F})$  by  $\widehat{\sigma}$  is denoted by  $EPS(H, \mathfrak{F})$ . If  $(U, V) \in PS(H, \mathfrak{F})$ , then the equivalence class of the element  $(U, V)$  under  $\widehat{\sigma}$  is denoted by  $[U, V]$ .

Introduce the operations of addition and scalar multiplication on the set  $EPF(X, \mathfrak{F}, \overline{E}, H)$ . Let  $\alpha \in \mathbb{R}$  be arbitrary. Suppose that  $0 \in H$  in the case  $\alpha = 0$ , and the set  $H$  is a cone (i.e. for any  $h \in H$  and for all  $\lambda > 0$  one has  $p = \lambda h \in H$ ) in the case  $\alpha \neq 0$ . Denote  $(-H) = \{-h \mid h \in H\}$ .

Let  $(\Phi, \Psi) \in PF(X, \mathfrak{F}, \overline{E}, H)$ . Define

$$\alpha[\Phi, \Psi] = \begin{cases} [\alpha\Phi, \alpha\Psi] \in EPF(X, \mathfrak{F}, \overline{E}, H), & \text{if } \alpha > 0, \\ [\alpha\Psi, \alpha\Phi] \in EPF(X, \mathfrak{F}, \overline{E}, -H), & \text{if } \alpha < 0, \\ [0, 0], & \text{if } \alpha = 0. \end{cases}$$

It is easy to check that the previous definition is correct in the sense that if  $(\Phi_1, \Psi_1), (\Phi_2, \Psi_2) \in [\Phi, \Psi]$ , then  $[\alpha\Phi_1, \alpha\Psi_1] = [\alpha\Phi_2, \alpha\Psi_2]$  in the case  $\alpha > 0$  and  $[\alpha\Psi_1, \alpha\Phi_1] = [\alpha\Psi_2, \alpha\Phi_2]$  in the case  $\alpha < 0$ .

Suppose now that the set  $H$  is closed under addition. Let  $(\Phi_1, \Psi_1), (\Phi_2, \Psi_2) \in PF(X, \mathfrak{F}, \overline{E}, H)$ . Then we set  $[\Phi_1, \Psi_1] + [\Phi_2, \Psi_2] = [\Phi_1 + \Phi_2, \Psi_1 + \Psi_2]$ . It is easy to verify that the given definition of the sum is correct.

The operations of addition and scalar multiplication on the set  $EPS(H, \mathfrak{F})$  are defined in a similar way.

*Remark 2* (i) The construction of the sets  $EPS(H, \mathfrak{F})$  and  $EPF(X, \mathfrak{F}, \overline{E}, H)$  is similar to the construction of the space of convex sets [16] and the set of the differences of sublinear functions.[7,17]

(ii) Let  $X$  be a topological vector space and  $\mathfrak{F}$  be the filter of neighbourhoods of the origin. Then, we write  $PF(X, \overline{E}, H)$  instead of  $PF(X, \mathfrak{F}, \overline{E}, H)$  and use analogous abbreviations for  $EPF(X, \mathfrak{F}, \overline{E}, H)$ ,  $PS(H, \mathfrak{F})$  and  $EPS(H, \mathfrak{F})$ .

We need to introduce other equivalence relations on the set  $PF(X, \mathfrak{F}, \overline{E}, H)$  in order to avoid ambiguity in the definition of abstract codifferentiable function.

Let  $X$  be a topological vector space (normed space) over the field of real or complex numbers, and  $E$  be an order complete Hausdorff topological vector lattice. Define a binary relation  $\sigma_w$  (and  $\sigma_s$ ) on the set  $PF(X, \overline{E}, H)$ . Let  $(\Phi_i, \Psi_i) \in PF(X, \overline{E}, H)$ ,  $i \in \{1, 2\}$  be arbitrary. Set

$$((\Phi_1, \Psi_1), (\Phi_2, \Psi_2)) \in \sigma_w \quad (((\Phi_1, \Psi_1), (\Phi_2, \Psi_2)) \in \sigma_s)$$

if and only if  $\Phi_1(0) + \Psi_1(0) = \Phi_2(0) + \Psi_2(0)$  and for any  $x \in X$

$$\lim_{\alpha \downarrow 0} \frac{1}{\alpha} (\Phi_1(\alpha x) + \Psi_1(\alpha x) - \Phi_2(\alpha x) - \Psi_2(\alpha x)) = 0$$

$$\left( \lim_{x \rightarrow 0} \frac{1}{\|x\|} (\Phi_1(x) + \Psi_1(x) - \Phi_2(x) - \Psi_2(x)) = 0 \right).$$

Hereafter, we write  $\alpha \downarrow 0$  instead of  $\alpha \in \mathbb{R}, \alpha \rightarrow +0$ . It is easy to see that  $\sigma_w$  and  $\sigma_s$  are equivalence relations on the set  $PF(X, \overline{E}, H)$ . The quotient set of  $PF(X, \overline{E}, H)$  by  $\sigma_w$  is denoted by  $EPF_w(X, \overline{E}, H)$ , and the quotient set of  $PF(X, \overline{E}, H)$  by  $\sigma_s$  is denoted by  $EPF_s(X, \overline{E}, H)$ . If  $(\Phi, \Psi) \in PF(X, \overline{E}, H)$ , then the equivalence class of  $(\Phi, \Psi)$  under  $\sigma_w$  is denoted by  $[\Phi, \Psi]_w$ , and the equivalence class of  $(\Phi, \Psi)$  under  $\sigma_s$  is denoted by  $[\Phi, \Psi]_s$ .

One can introduce similar equivalence relations  $\widehat{\sigma}_w$  and  $\widehat{\sigma}_s$  on the set  $PS(H)$  and the quotient sets  $EPS_w(H)$  and  $EPS_s(H)$ . Also, it is easy to define the operations of addition and scalar multiplication on the sets  $EPF_w(X, \overline{E}, H)$ ,  $EPF_s(X, \overline{E}, H)$ ,  $EPS_w(H)$  and  $EPS_s(H)$  in the same way as we defined these operations on the sets  $EPF(X, \overline{\mathfrak{F}}, \overline{E}, H)$  and  $EPS(H, \overline{\mathfrak{F}})$ .

Let us give several definitions that is useful for the study of continuity. Let, as earlier,  $X$  be a nonvoid set and  $f: X \rightarrow EPS(H, \overline{\mathfrak{F}})$  be an arbitrary mapping (one can also consider  $f: X \rightarrow EPS_w(H)$  or  $f: X \rightarrow EPS_s(H)$ ). A mapping  $\varphi = (\varphi_1, \varphi_2): X \rightarrow PS(H, \overline{\mathfrak{F}})$ , where  $\varphi_i: X \rightarrow S(H), i \in \{1, 2\}$ , is said to be a selection of the mapping  $f$  if  $\varphi(x) \in f(x)$  for all  $x \in X$ . Here  $S(H)$  is the set of all nonempty subsets of  $H$ .

Let  $X$  and  $H$  be equipped with topologies and let  $\Omega$  be a neighbourhood of a point  $x \in X$ .

*Definition 2.2* A mapping  $f: \Omega \rightarrow EPS(H, \overline{\mathfrak{F}})$  is called lower semicontinuous (upper semicontinuous, continuous) at the point  $x$  if there exists a selection  $\varphi = (\varphi_1, \varphi_2): \Omega \rightarrow PS(H, \overline{\mathfrak{F}})$  of  $f$  such that the set-valued mappings  $\varphi_1, \varphi_2$  are lower semicontinuous (upper semicontinuous, continuous) at the point  $x$ . If  $H$  is a metric space, then the mapping  $f$  is called Hausdorff continuous at the point  $x$  if there exists a selection  $\varphi = (\varphi_1, \varphi_2): \Omega \rightarrow PS(H, \overline{\mathfrak{F}})$  of  $f$  such that the set-valued mappings  $\varphi_1, \varphi_2$  are Hausdorff continuous at this point.

### 3. Abstract codifferentiable functions

In the following subsections we give definitions of  $H$ -codifferentiable and  $H$ -quasi-differentiable functions and discuss related notions. Also, we show that many well-known classes of nonsmooth functions are, in fact,  $H$ -codifferentiable or  $H$ -quasidifferentiable for particular sets  $H$ .

#### 3.1. A definition of abstract codifferentiable functions

Hereafter, let  $X$  be a Hausdorff topological vector space over the field of real or complex numbers,  $E$  be an order complete Hausdorff topological vector lattice,  $H$  be a nonempty set of functions  $h: X \rightarrow \overline{E}$  and  $\Omega \subset X$  be an open set. Denote the closure of a subset  $A \subset T$  of a topological space  $T$  by  $\text{cl } A$  and the convex hull of a subset  $A \subset L$  of a linear space  $L$  by  $\text{co } A$ .

*Definition 3.1* A function  $F: \Omega \rightarrow E$  is said to be weakly  $H$ -codifferentiable (or Gâteaux  $H$ -codifferentiable, or weakly abstract codifferentiable with respect to  $H$ ) at a point  $x \in \Omega$  if there exists an element  $\delta F_H[x] \in EPF_w(X, \overline{E}, H)$  for which there exists a pair  $(\Phi, \Psi) \in \delta F_H[x]$  such that  $\Phi(0) + \Psi(0) = 0$  and for any admissible argument increment  $\Delta x \in X$



(i.e.  $\text{co}\{x, x + \Delta x\} \subset (\Omega \cap \text{dom } \Phi \cap \text{dom } \Psi)$ ) the following holds

$$F(x + \Delta x) - F(x) = \Phi(\Delta x) + \Psi(\Delta x) + o(\Delta x, x),$$

where  $o(\alpha \Delta x, x)/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ . The element  $\delta F_H[x]$  is called a weak  $H$ -derivative (or Gâteaux  $H$ -derivative) of the function  $F$  at the point  $x$ .

It is clear that if a function  $F$  is weakly  $H$ -codifferentiable at a point  $x$ , then any pair  $(\Phi, \Psi) \in \delta F_H[x]$  satisfies all assumptions of the previous definition, i.e. the definition of weakly  $H$ -codifferentiable function does not depend on the choice of a pair  $(\Phi, \Psi) \in \delta F_H(x)$ .

*Definition 3.2* Let  $X$  be a normed space. A function  $F: \Omega \rightarrow E$  is said to be strongly  $H$ -codifferentiable (or Fréchet  $H$ -codifferentiable) at a point  $x \in \Omega$  if there exists an element  $F'_H[x] \in E P F_s(X, \bar{E}, H)$  for which there exists a pair  $(\Phi, \Psi) \in F'_H[x]$  such that  $\Phi(0) + \Psi(0) = 0$  and for any admissible  $\Delta x \in X$

$$F(x + \Delta x) - F(x) = \Phi(\Delta x) + \Psi(\Delta x) + o(\Delta x, x),$$

where  $o(\Delta x, x)/\|\Delta x\| \rightarrow 0$  as  $\Delta x \rightarrow 0$ . The element  $F'_H[x]$  is called a strong  $H$ -derivative (or Fréchet  $H$ -derivative) of the function  $F$  at the point  $x$ .

It is easy to check that the weak (strong)  $H$ -derivative of a function  $F: \Omega \rightarrow E$  at a point  $x \in \Omega$  is uniquely defined. Also, it is clear that if a function  $F: \Omega \rightarrow E$  is strongly  $H$ -codifferentiable at a point  $x$ , then  $F$  is weakly  $H$ -codifferentiable at this point and for any  $(\Phi, \Psi) \in F'_H[x]$  one has  $(\Phi, \Psi) \in \delta F_H[x]$  (the opposite inclusion does not hold true in the general case).

*Remark 1* One can consider the definition of  $H$ -codifferentiation in a more general framework. Indeed, let  $X$  be a vector space,  $E$  be a complete vector lattice,  $\Omega \subset X$  be an arbitrary set. Denote by

$$\text{core } \Omega = \{x \in \Omega \mid \forall g \in X \exists \alpha_g > 0: x + \alpha g \in \Omega \quad \forall \alpha \in (0, \alpha_g)\}.$$

the algebraic interior of the set  $\Omega$ . Let  $\mathfrak{F} = \{S \subset X \mid 0 \in \text{core } S\}$ , and suppose that  $\text{core } \Omega \neq \emptyset$ .

A function  $F: \Omega \rightarrow E$  is said to be order  $H$ -codifferentiable at a point  $x \in \text{core } \Omega$  if there exists an element  $\delta_o F_H[x] \in E P F(X, \mathfrak{F}, \bar{E}, H)$  for which there exists a pair  $(\Phi, \Psi) \in \delta_o F_H[x]$  such that  $\Phi(0) + \Psi(0) = 0$  and for any argument increment  $\Delta x \in E$  such that  $\text{co}\{x, x + \Delta x\} \subset \text{core}(\Omega \cap \text{dom } \Phi \cap \text{dom } \Psi)$  the following holds

$$\text{o-lim}_{\alpha \downarrow 0} |F(x + \Delta x) - F(x) - \Phi(\Delta x) - \Psi(\Delta x)|/\alpha = 0,$$

where o-lim stands for the order limit in the lattice  $E$ .

Let  $X$  be a topological vector space. One can also consider the notion of  $H$ -codifferentiability for a function defined on the set  $\Omega \cap \mathcal{K}$ , where  $\mathcal{K} \subset X$  is a cone, or on a closed set  $M \subset X$ . In these cases, the  $H$ -derivative of a function is an element of  $E P F(\mathcal{K}, \mathfrak{F}, \bar{E}, H)$ , where  $\mathfrak{F} = \{S \subset \mathcal{K} \mid 0 \in \text{int}_{\mathcal{K}} S\}$ ,  $\text{int}_{\mathcal{K}}$  stands for the interior of a set in the topological subspace  $\mathcal{K}$  of the space  $X$ , and  $\mathcal{K} \subset X$  is either an arbitrary cone or some kind of a tangent cone to the set  $M$ .



We will not consider the generalization of  $H$ -codifferentiability suggested above. The interested reader can transfer main results obtained in the article to these more general cases.

Let a function  $F: \Omega \rightarrow E$  be weakly  $H$ -codifferentiable at a point  $x \in \Omega$ , and let  $(\Phi, \Psi) \in \delta F_H[x]$  be arbitrary. Then, by the definitions of abstract convex and abstract concave functions, there exist nonempty sets  $U, V \subset H$  such that

$$\Phi(y) = \sup_{h \in U} h(y), \quad \Psi(y) = \inf_{p \in V} p(y) \quad \forall y \in X. \quad (2)$$

We denote the equivalence class  $[U, V]_w \in EPS_w(H)$  by  $D_H^w F(x)$ . The set  $D_H^w F(x)$  is called a weak  $H$ -codifferential (or Gâteaux  $H$ -codifferential) of the function  $F$  at the point  $x$ . It is easy to check that  $D_H^w F(x)$  does not depend on the choice of  $(\Phi, \Psi) \in \delta F_H[x]$  and the choice of the sets  $U, V \subset H$  satisfying (2). Hence the weak  $H$ -codifferential of the function  $F$  at the point  $x$  is unique. One can analogously define a strong  $H$ -codifferential (or Fréchet  $H$ -codifferential)  $D_H^s F(x)$  of the function  $F$  at the point  $x$ .

*Definition 3.3* Let a function  $F: \Omega \rightarrow E$  be weakly (strongly)  $H$ -codifferentiable at a point  $x \in \Omega$ , and suppose that  $0 \in H$ . The function  $F$  is said to be weakly (strongly)  $H$ -hypodifferentiable at  $x$  if there exists an  $H$ -convex function  $\Phi: X \rightarrow \bar{E}$  such that  $\delta F_H[x] = [\Phi, 0]_w$  ( $F'_H[x] = [\Phi, 0]_s$ ). The function  $F$  is said to be weakly (strongly)  $H$ -hyperdifferentiable at  $x$  if there exists an  $H$ -concave function  $\Psi$  such that  $\delta F_H[x] = [0, \Psi]_w$  ( $F'_H[x] = [0, \Psi]_s$ ).

Although the  $H$ -derivative of a function is unique, in the general case there exist  $(\Phi_i, \Psi_i) \in F'_H[x]$ ,  $i \in \{1, 2\}$  such that  $[\Phi_1, \Psi_1] \neq [\Phi_2, \Psi_2]$ . The following example shows the difference between equivalence relations  $\sigma$  and  $\sigma_s$ .

*Example 3.4* Let  $X = E = \mathbb{R}$ ,  $H$  be the set of all affine functions, i.e.

$$H = \{h: \mathbb{R} \rightarrow \mathbb{R} \mid h(x) = ax + b, \text{ where } a, b, x \in \mathbb{R}\},$$

and  $F(x) = x^4$  for all  $x \in \mathbb{R}$ . It is clear that  $F$  is strongly  $H$ -codifferentiable at the point  $x = 0$ , and  $F'_H[0] = [0, 0]_s$ . Define  $\Phi(x) = x^2$ ,  $x \in \mathbb{R}$ . It is easy to verify that  $\Phi$  is  $H$ -convex and  $[\Phi, 0] \neq [0, 0]$ , despite the fact that  $(\Phi, 0) \in F'_H[0]$ , i.e.  $[\Phi, 0]_s = [0, 0]_s$ .

Let us introduce the important concept of continuously  $H$ -codifferentiable functions. Let  $H$  be endowed with a topology.

*Definition 3.5* A function  $F: \Omega \rightarrow E$  is said to be continuously (upper semicontinuously, lower semicontinuously or, in the case when  $H$  is equipped with a metric, Hausdorff continuously) weakly  $H$ -codifferentiable at a point  $x \in \Omega$  if the function  $F$  is weakly  $H$ -codifferentiable in a neighbourhood  $\mathcal{O}$  of  $x$ , and the mapping  $y \rightarrow D_H^w F(y)$ ,  $y \in \mathcal{O}$  is continuous (upper semicontinuous, lower semicontinuous, Hausdorff continuous) at  $x$ . Continuously strongly  $H$ -codifferentiable functions are defined in the same way.

*Definition 3.6* Let  $0 \in H$ . A function  $F: \Omega \rightarrow E$  is said to be continuously weakly  $H$ -hypodifferentiable at a point  $x \in \Omega$  if the function  $F$  is weakly  $H$ -hypodifferentiable in a neighbourhood  $\mathcal{O}$  of  $x$  and there exists a continuous mapping  $\varphi: \mathcal{O} \rightarrow S(H)$  such

that  $(\varphi(y), 0) \in D_H^w F(y)$  for all  $y \in \mathcal{O}$ . Other types of continuity (semicontinuity) of  $H$ -hypodifferentiable and  $H$ -hyperdifferentiable functions are defined in a similar way.

*Remark 2* It is to be mentioned that the theory of continuously  $H$ -codifferentiable functions is closely related to the theory of continuous approximations of nonsmooth functions.[18,19]

Let us give an auxiliary definition that will be useful in subsequent.

*Definition 3.7* Suppose that  $X$  is a normed space, and  $E$  is an order complete normed lattice. Let a function  $F: \Omega \rightarrow E$  be weakly (strongly)  $H$ -codifferentiable at a point  $x \in \Omega$ . The weak (strong)  $H$ -derivative of  $F$  at  $x$  is said to be Lipschitz continuous in a neighbourhood of zero (or to satisfy the Lipschitz condition in a neighbourhood of zero) if there exists  $(\Phi, \Psi) \in \delta F_H[x]$  ( $(\Phi, \Psi) \in F'_H[x]$ ) such that the functions  $\Phi(\cdot)$  and  $\Psi(\cdot)$  are Lipschitz continuous in a neighbourhood of zero.

Note an obvious property of an  $H$ -codifferentiable function which  $H$ -derivative is Lipschitz continuous in a neighbourhood of zero.

**PROPOSITION 3.8** *Let  $X$  be a normed space,  $E$  be an order complete normed lattice, and a function  $F: \Omega \rightarrow E$  be weakly  $H$ -codifferentiable at a point  $x \in \Omega$ . Suppose that  $\delta F_H[x]$  is Lipschitz continuous in a neighbourhood of zero. Then there exists  $L > 0$  such that for any admissible argument increment  $\Delta x \in X$  there exists  $\alpha_0 > 0$  such that*

$$\|F(x + \alpha \Delta x) - F(x)\| \leq L\alpha \|\Delta x\| \quad \forall \alpha \in (0, \alpha_0).$$

*Moreover, if  $F$  is strongly  $H$ -codifferentiable at  $x$  and  $F'_H[x]$  is Lipschitz continuous in a neighbourhood of zero, then there exists  $L > 0$  and  $r > 0$  such that*

$$\|F(x + \Delta x) - F(x)\| \leq L\|\Delta x\| \quad \forall \Delta x \in X, \|\Delta x\| \leq r,$$

*and, in particular, the function  $F$  is continuous and calm at the point  $x$ .*

### 3.2. Examples of abstract codifferentiable functions

In this subsection we show that some well-known classes of nonsmooth functions are  $H$ -codifferentiable for particular sets  $H$ .

*Example 3.9* Let  $X$  be a normed space,  $E$  be an order complete normed lattice, and let  $\mathcal{B}(X, E) \subset H$ , i.e.  $H$  includes the space of all bounded linear operators mapping  $X$  to  $E$ . Then it is clear that if a function  $F: \Omega \rightarrow E$  is Gâteaux (Fréchet) differentiable at a point  $x \in \Omega$ , then  $F$  is weakly (strongly)  $H$ -codifferentiable at this point. Moreover, if  $\delta F[x]$  ( $F'[x]$ ) is the Gâteaux (Fréchet) gradient of the function  $F$  at the point  $x$ , then

$$\begin{aligned} \delta F_H[x] &= [\delta F[x], 0]_w = [0, \delta F[x]]_w, & D_H^w F(x) &= [\{\delta F[x]\}, \{0\}]_w = [\{0\}, \{\delta F[x]\}]_w \\ (F'_H[x] &= [F'[x], 0]_s = [0, F'[x]]_s, & D_H^s F(x) &= [\{F'[x]\}, \{0\}]_s = [\{0\}, \{F'[x]\}]_s). \end{aligned}$$

The space  $H$  can be equipped with the standard operator norm. Then it is easy to see that if the function  $F: \Omega \rightarrow E$  is continuously Gâteaux (Fréchet) differentiable at a point  $x \in \Omega$ , then  $F$  is Hausdorff continuously weakly (strongly)  $H$ -codifferentiable at this point.

*Example 3.10* Let  $X$  be a real normed space,  $E = \mathbb{R}$ , and let  $H$  be the set of all continuous affine functions mapping  $X$  to  $\mathbb{R}$ , i.e.

$$H = \{h: X \rightarrow \mathbb{R} \mid h(\cdot) = a + p(\cdot), a \in \mathbb{R}, p \in X^*\},$$

where, as usual,  $X^*$  is the topological dual space of  $X$ . The set  $H$  can be identified with the space  $\mathbb{R} \times X^*$ . Thus,  $H$  is a linear space that can be endowed with the norm

$$\|h\|_r = (|a|^r + \|p\|^r)^{\frac{1}{r}}, \quad h = (a, p) \in H = \mathbb{R} \times X^*,$$

where  $1 \leq r < \infty$ , or  $\|h\|_\infty = \max\{|a|, \|p\|\}$ .

It is well-known (cf. [20], Proposition I.3.1) that a function  $\Phi: X \rightarrow \overline{\mathbb{R}}$  is abstract convex (abstract concave) with respect to the set  $H$  under consideration if and only if  $\Phi$  is a proper lower semicontinuous convex function (proper upper semicontinuous concave function). Hence, a function  $F: \Omega \rightarrow \mathbb{R}$  is weakly  $H$ -codifferentiable at a point  $x \in \Omega$  if and only if there exist a proper lower semicontinuous (l.s.c.) convex function  $\Phi: X \rightarrow \overline{\mathbb{R}}$  and a proper upper semicontinuous (u.s.c.) concave function  $\Psi: X \rightarrow \overline{\mathbb{R}}$  such that  $0 \in \text{int}(\text{dom } \Phi \cap \text{dom } \Psi)$ ,  $\Phi(0) + \Psi(0) = 0$ , and for any admissible argument increment  $\Delta x \in X$

$$F(x + \Delta x) - F(x) = \Phi(\Delta x) + \Psi(\Delta x) + o(\Delta x, x),$$

where  $o(\alpha \Delta x, x)/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ .

We need the following proposition in order to give another characterization of  $H$ -codifferentiability for the set  $H$  under consideration. Let  $x \in X$  and  $r > 0$ . Denote  $\mathcal{O}(x, r) = \{y \in X \mid \|x - y\| < r\}$  and  $B(x, r) = \{y \in X \mid \|x - y\| \leq r\}$ .

**PROPOSITION 3.11** *Let  $X$  be a real Banach space and  $f: X \rightarrow \overline{\mathbb{R}}$  be a proper l.s.c. convex function such that  $0 \in \text{int dom } f$ . Then there exist  $r > 0$  and a convex bounded set  $A \subset \mathbb{R} \times X^*$  that is compact in the topological product  $(\mathbb{R}, \tau) \times (X^*, w^*)$  and such that*

$$f(x) = \max_{(a,p) \in A} (a + p(x)) \quad \forall x \in B(x, r). \quad (3)$$

Here  $\tau$  is the standard topology on  $\mathbb{R}$  and  $w^*$  is the weak\* topology on  $X^*$ .

*Proof* Since the space  $X$  is complete,  $0 \in \text{int dom } f$  and  $f$  is a proper l.s.c. convex function, then  $f$  is continuous on  $\text{int dom } f$  ([20], Corollary I.2.5), and for any  $x \in \text{int dom } f$  one has  $\partial f(x) \neq \emptyset$  ([20], Proposition I.5.2), where  $\partial f(x)$  is the subdifferential of the convex function  $f$  at a point  $x$ . Thus, there exist  $r > 0$  and  $C > 0$  such that

$$|f(x)| \leq C \quad \forall x \in \mathcal{O}(0, 4r). \quad (4)$$

With the use of the definition of the subgradient of a convex function it is easy to show that there exists  $M > 0$  ( $M \leq C/r$ ) such that for all  $x \in \mathcal{O}(0, 2r)$

$$\|p\| \leq M \quad \forall p \in \partial f(x), \quad (5)$$

i.e. the subdifferential of  $f$  is bounded on  $\mathcal{O}(0, 2r)$ .

Since  $\underline{\partial}f(x) \neq \emptyset$  for all  $x \in B(0, r)$ , then (by the axiom of choice) there exists a mapping  $B(0, r) \ni x \rightarrow p[x] \in X^*$  such that  $p[x] \in \underline{\partial}f(x)$ . Introduce the set

$$A = \text{cl co}\{(a, p) \in \mathbb{R} \times X^* \mid a = f(x) - p[x](x), p = p[x], x \in B(0, r)\}.$$

Here, the closure is taken in the topology  $\tau \times w^*$ . The set  $A$  is obviously convex. Taking into account (4) and (5) one has that

$$A \subset [-C - rM, C + rM] \times \{p \in X^* \mid \|p\| \leq M\}. \tag{6}$$

Therefore, the set  $A$  is bounded and compact in the topology  $\tau \times w^*$ , since the set  $\{p \in X^* \mid \|p\| \leq M\}$  is weak\* compact by the Banach–Alaoglu theorem, and the set on the right-hand side of (6) is compact in the topology  $\tau \times w^*$  as the direct product of two compact sets.

By the definition of the subgradient of a convex function one has that

$$f(y) \geq f(x) - p[x](x) + p[x](y) \quad \forall y \in X, \forall x \in B(0, r)$$

and the last inequality turns into an equality when  $y = x$ . Hence the validity of (3) follows from the definition of the set  $A$ . □

**COROLLARY 3.12** *Let  $X$  be a Banach space and  $\{f_\lambda\}, \lambda \in \Lambda$  be a family of proper l.s.c. convex functions mapping  $X$  to  $\overline{\mathbb{R}}$ . Suppose that there exist  $\rho > 0$  and  $C_\lambda > 0, \lambda \in \Lambda$  such that  $|f_\lambda(x)| \leq C_\lambda$  for all  $x \in \mathcal{O}(0, \rho)$  and  $\lambda \in \Lambda$ . Then there exist  $r > 0$  (depending only on  $\rho$ ) and a family  $\{A_\lambda\}, \lambda \in \Lambda$  of subsets of the space  $\mathbb{R} \times X^*$  such that for any  $\lambda \in \Lambda$  the set  $A_\lambda$  is nonempty, convex, bounded and compact in the topology  $\tau \times w^*$ , and the following holds*

$$f_\lambda(x) = \max_{(a,p) \in A_\lambda} (a + p(x)) \quad \forall x \in B(0, r).$$

Let us give a description of  $H$ -codifferentiable functions for the set  $H$  under considerations. Suppose that the normed space  $X$  is complete. By virtue of the previous proposition one has that a function  $F: \Omega \rightarrow \mathbb{R}$  is weakly  $H$ -codifferentiable at a point  $x \in \Omega$  if and only if there exist bounded convex sets  $A, B \subset \mathbb{R} \times X^*$  that are compact in the topology  $\tau \times w^*$  and such that for any admissible argument increment  $\Delta x \in X$

$$F(x + \Delta x) - F(x) = \max_{(a,p) \in A} (a + p(\Delta x)) + \min_{(b,q) \in B} (b + q(\Delta x)) + o(\Delta x, x),$$

where  $o(\alpha \Delta x, x)/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ . Thus, the function  $F$  is weakly  $H$ -codifferentiable at a point  $x \in \Omega$  if and only if it is codifferentiable at this point (cf. [7,19,21,22]). Also it is easy to show that the function  $F$  is Hausdorff continuously weakly  $H$ -codifferentiable at a point  $x \in \Omega$  if and only if  $F$  is continuously codifferentiable at this point. Moreover,  $F$  is strongly  $H$ -codifferentiable if and only if  $F$  is codifferentiable uniformly in directions (see [7,21]). If  $F$  is strongly  $H$ -codifferentiable at  $x$ , then we will call it Fréchet (or strongly) codifferentiable at  $x$ .

*Remark 3* The concept of codifferentiability in Banach lattices [19] is, in fact, the particular case of  $H$ -codifferentiability, when the set  $H$  consists of all affine functions  $h: X \rightarrow E, h(x) = a + Ax$ , where  $a \in E$  and  $A: X \rightarrow E$  is a linear operator.

*Example 3.13* Let  $X$  be a real Banach space,  $E = \mathbb{R}$ , and let the set  $H$  consist of all proper l.s.c. convex functions  $h: X \rightarrow \overline{\mathbb{R}}$  such that  $0 \in \text{int dom } h$ . In this example we only consider  $H$ -hyperdifferentiable functions, since the set of all  $H$ -hyperdifferentiable functions contains a certain class of nonsmooth functions.

Suppose that a function  $F: \Omega \rightarrow \mathbb{R}$  is weakly  $H$ -hyperdifferentiable at a point  $x \in \Omega$ , i.e. there exists a set  $U \subset H$  such that for any admissible argument increment  $\Delta x \in X$

$$F(x + \Delta x) - F(x) = \inf_{h \in U} h(\Delta x) + o(\Delta x, x),$$

where  $o(\alpha \Delta x, x)/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ . Suppose also that there exist  $\rho > 0$  and  $C_h > 0$ ,  $h \in U$  such that

$$|h(x)| \leq C_h \quad \forall x \in \mathcal{O}(0, \rho), \forall h \in U. \quad (7)$$

Then, applying Corollary 3.12 one gets that there exists a family of convex bounded sets  $A_h \subset \mathbb{R} \times X^*$ ,  $h \in U$ , which are compact in the topology  $\tau \times w^*$  and such that for any admissible argument increment  $\Delta x \in X$

$$F(x + \Delta x) - F(x) = \inf_{h \in U} \max_{(a, p) \in A_h} (a + p(\Delta x)) + o(\Delta x, x),$$

where  $o(\alpha \Delta x, x)/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ . Thus, the family  $\overline{E}(x) = \{A_h \subset \mathbb{R} \times X^* \mid h \in U\}$ , that is said to be generated by  $U$ , is a Dini upper coexhauster of the function  $F$  at the point  $x$  (cf. [23]). Therefore, as it is easy to check, a function  $F$  has a Dini upper coexhauster at a point  $x$  if and only if  $F$  is weakly  $H$ -hyperdifferentiable at this point and there exist  $(U, \{0\}) \in D_H^w F[x]$ ,  $\rho > 0$  and  $C_h > 0$ ,  $h \in U$  such that (7) holds true. The notion of coexhauster of a nonsmooth function was introduced by Aban'kin [24], where the functions having upper coexhauster were called  $H$ -hyperdifferentiable (cf. also, [23]).

We will say that a family of nonempty convex, bounded and compact in the topology  $\tau \times w^*$  subsets  $\overline{E}(x)$  of  $\mathbb{R} \times X^*$  is a Fréchet upper coexhauster of  $F$  at  $x$  if  $F$  is Fréchet  $H$ -hyperdifferentiable at this point and there exists  $(U, \{0\}) \in D_H^s F(x)$  such that  $\overline{E}(x)$  is generated by  $U$ .

*Remark 4* One can also consider an example, that is similar to the previous one, where the set  $H$  coincides with the set of all proper u.s.c. concave functions  $h: X \rightarrow \overline{\mathbb{R}}$  such that  $0 \in \text{int dom } h$ . In this case, if a function  $F: \Omega \rightarrow \mathbb{R}$  has a Dini lower coexhauster at a point  $x \in \Omega$  then  $F$  is weakly  $H$ -hypodifferentiable at this point.

### 3.3. Abstract quasidifferentiable functions

It is easy to verify that the following proposition about the directional derivative of an  $H$ -codifferentiable function holds true.

**PROPOSITION 3.14** *Let  $X$  be a topological vector space (normed space), a function  $F: \Omega \rightarrow E$  be weakly (strongly)  $H$ -codifferentiable at a point  $x \in \Omega$ . Suppose also that there exists  $(\Phi, \Psi) \in \delta F_H[x]$  ( $(\Phi, \Psi) \in F'_H[x]$ ) such that the functions  $\Phi$  and  $\Psi$  are Dini (Hadamard) directionally differentiable at the origin. Then the function  $F$  is Dini (Hadamard) directionally differentiable at the point  $x$  and*

$$F'(x, g) = \Phi'(0, g) + \Psi'(0, g) \quad \forall g \in X.$$

Here  $F'(x, \cdot)$ ,  $\Phi'(0, \cdot)$  and  $\Psi'(0, \cdot)$  are the Dini (Hadamard) directional derivatives of the functions  $F$ ,  $\Phi$  and  $\Psi$ , respectively.

**COROLLARY 3.15** *Let  $X$  be a topological vector space, a function  $F: \Omega \rightarrow E$  be weakly  $H$ -codifferentiable at a point  $x \in \Omega$ . Suppose that any function  $h \in H$  is positively homogeneous of degree one (p.h.). Then the function  $F$  is Dini directionally differentiable at the point  $x$  and for any  $(\Phi, \Psi) \in \delta F_H[x]$  one has*

$$F'(x, g) = \Phi(g) + \Psi(g) \quad \forall g \in X.$$

*Remark 5* About Dini and Hadamard directional derivatives see, e.g. [7,23].

The previous corollary motivates us to introduce the definition of  $H$ -quasidifferentiable (or abstract quasidifferentiable with respect to  $H$ ) function. Suppose that any function  $h \in H$  is p.h. (then any  $H$ -convex or  $H$ -concave function is also p.h. and the equivalence relations  $\sigma$ ,  $\sigma_w$  and  $\sigma_s$  coincide).

**Definition 3.16** A function  $F: X \rightarrow E$  is said to be Dini (Hadamard)  $H$ -quasidifferentiable at a point  $x \in \Omega$  if  $F$  is Dini (Hadamard) directionally differentiable at this point and there exists an element  $\mathcal{D}_H F(x) \in EPF(X, \overline{E}, H)$  such that for any  $(p, q) \in \mathcal{D}_H F(x)$

$$F'(x, g) = p(g) + q(g) \quad \forall g \in X,$$

where  $F'(x, \cdot)$  is the Dini (Hadamard) directional derivative of  $F$  at  $x$ .

The element  $\mathcal{D}_H F(x)$  from the definition of Dini (Hadamard)  $H$ -quasidifferentiable function is called a Dini (Hadamard)  $H$ -quasidifferential of the function  $F$  at the point  $x$ . It is clear that  $\mathcal{D}_H F(x)$  is uniquely defined.

**Definition 3.17** Let a function  $F: \Omega \rightarrow E$  be Dini (Hadamard)  $H$ -quasidifferentiable at a point  $x \in \Omega$ , and suppose that  $0 \in H$ . The function  $F$  is said to be Dini (Hadamard)  $H$ -subdifferentiable at the point  $x$  if there exists an  $H$ -convex function  $p: X \rightarrow E$  such that  $\mathcal{D}_H F(x) = [p, 0]$ . The function  $F$  is said to be Dini (Hadamard)  $H$ -superdifferentiable at the point  $x$  if there exists an  $H$ -concave function  $q: X \rightarrow E$  such that  $\mathcal{D}_H F(x) = [0, q]$ .

Note a connection between  $H$ -quasidifferentiable functions and  $H$ -codifferentiable functions. It is clear that a function  $F: \Omega \rightarrow E$  is weakly  $H$ -codifferentiable at a point  $x \in \Omega$  if and only if  $F$  is Dini  $H$ -quasidifferentiable at this point. Also, it is easy to see that if  $F$  is Dini  $H$ -quasidifferentiable at a point  $x \in \Omega$ , and there exists  $(p, q) \in \mathcal{D}_H F(x)$  such that  $p$  and  $q$  are Lipschitz continuous in a neighbourhood of zero, then  $F$  is Hadamard  $H$ -quasidifferentiable at  $x$ . The following proposition, which is, partly, a generalization of Theorem 2.1 from [26], reveals a connection between strongly  $H$ -codifferentiable functions and Hadamard  $H$ -quasidifferentiable functions.

**PROPOSITION 3.18** *Let  $X$  be a normed space,  $E$  be an order complete normed lattice, and  $F: \Omega \rightarrow E$  be an arbitrary function. For the function  $F$  to be Hadamard  $H$ -quasidifferentiable at a point  $x \in \Omega$  it is sufficient and, in the case when  $X$  is finite dimensional,*

necessary that  $F$  is strongly  $H$ -codifferentiable at this point and for any  $(\Phi, \Psi) \in F'_H[x]$  the sum  $\Phi + \Psi$  is finite and continuous on  $X$ .

*Proof Sufficiency* Let  $(\Phi, \Psi) \in F'_H[x]$ ,  $g \in X$  and sequences  $\{g_n\} \subset X$ ,  $\{\alpha_n\} \subset (0, +\infty)$  such that  $g_n \rightarrow g$  and  $\alpha_n \rightarrow 0$  as  $n \rightarrow \infty$  be arbitrary. Since  $(\Phi, \Psi) \in F'_H[x]$  and the sum  $\Phi + \Psi$  is continuous, then

$$\frac{1}{\alpha_n \|g_n\|} \|F(x + \alpha_n g_n) - F(x) - \Phi(\alpha_n g_n) - \Psi(\alpha_n g_n)\| \rightarrow 0$$

and  $\|\Phi(g_n) + \Psi(g_n) - \Phi(g) - \Psi(g)\| \rightarrow 0$  as  $n \rightarrow \infty$ . Consequently

$$\begin{aligned} \left\| \frac{F(x + \alpha_n g_n) - F(x)}{\alpha_n} - \Phi(g) - \Psi(g) \right\| &\leq \|\Phi(g_n) + \Psi(g_n) - \Phi(g) - \Psi(g)\| \\ &+ \|g_n\| \frac{1}{\alpha_n \|g_n\|} \|F(x + \alpha_n g_n) - F(x) - \Phi(\alpha_n g_n) - \Psi(\alpha_n g_n)\| \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . Therefore, the function  $F$  is Hadamard  $H$ -quasidifferentiable at the point  $x$  and  $(p, q) \in \mathcal{D}_H F(x)$  if and only if  $(p, q) \in F'_H[x]$ .

*Necessity* Ab absurdo, suppose that  $F$  is not strongly  $H$ -codifferentiable at the point  $x$ . Fix an arbitrary  $(p, q) \in \mathcal{D}_H F(x)$ . It is clear that there exist  $\varepsilon > 0$  and a sequence of admissible argument increments  $\{\Delta x_n\} \subset X$  such that  $\|\Delta x_n\| \rightarrow 0$  and for any  $n \in \mathbb{N}$

$$\frac{1}{\|\Delta x_n\|} \|F(x + \Delta x_n) - F(x) - p(\Delta x_n) - q(\Delta x_n)\| > \varepsilon. \quad (8)$$

Denote  $\alpha_n = \|\Delta x_n\|$ ,  $g_n = \Delta x_n/\alpha_n$ . Applying the fact that  $X$  is finite dimensional one gets that there exists a subsequence  $\{g_{n_k}\}$  converging to some  $g^* \in X$ ,  $\|g^*\| = 1$ .

Since  $F$  is Hadamard  $H$ -quasidifferentiable, then there exists  $k_1 \in \mathbb{N}$  such for all  $k > k_1$  one has

$$\left\| \frac{F(x + \alpha_{n_k} g_{n_k}) - F(x)}{\alpha_{n_k}} - p(g^*) - q(g^*) \right\| < \frac{\varepsilon}{4}.$$

It is well-known and easy to check, that the directional derivative  $F'(x, g)$  of the Hadamard directionally differentiable function  $F$  is continuous with respect to  $g$ . Therefore, the sum  $p + q$  is continuous on  $X$ . Hence, there exists  $k_2 \in \mathbb{N}$  such that for any  $k > k_2$  one has  $\|p(g_{n_k}) + q(g_{n_k}) - p(g^*) - q(g^*)\| < \varepsilon/4$ . Taking into account the fact that  $(p, q) \in \mathcal{D}_H F(x)$  one gets that for any  $k > \max\{k_1, k_2\}$

$$\begin{aligned} &\frac{1}{\|\Delta x_{n_k}\|} \|F(x + \Delta x_{n_k}) - F(x) - p(\Delta x_{n_k}) - q(\Delta x_{n_k})\| \\ &\leq \left\| \frac{F(x + \alpha_{n_k} g_{n_k}) - F(x)}{\alpha_{n_k}} - p(g^*) - q(g^*) \right\| \\ &\quad + \|p(g_{n_k}) + q(g_{n_k}) - p(g^*) - q(g^*)\| \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}, \end{aligned}$$

which contradicts (8). Thus, the function  $F$  is strongly  $H$ -codifferentiable at the point  $x$  and taking into account the fact that the equivalence relations  $\sigma$  and  $\sigma_s$  coincide in the case when any  $h \in H$  is p.h., one gets that  $(p, q) \in \mathcal{D}_H F(x)$  if and only if  $(p, q) \in F'_H[x]$ .



Furthermore, for any  $(\Phi, \Psi) \in F'_H[x]$  the sum  $\Phi + \Psi$  is finite and continuous on  $X$ , since for any  $(p, q) \in \mathcal{D}_H F(x)$  the sum  $p(\cdot) + q(\cdot) = F'(x, \cdot)$  is finite and continuous on  $X$ .  $\square$

*Remark 6* It is to be mentioned that the notion of strong  $H$ -codifferentiability in the case when any function  $h \in H$  is positively homogeneous of degree one is closely related to the notion of semidifferentiability introduced in [25].

Let us briefly discuss two well-known examples of  $H$ -quasidifferentiable functions. Let  $X$  be a locally convex Hausdorff topological vector space over the real field and  $E = \mathbb{R}$ . It is easy to verify that if  $H = X^*$ , then a function  $F: \Omega \rightarrow \mathbb{R}$  is Dini  $H$ -quasidifferentiable at a point  $x \in \Omega$  if and only if  $F$  is quasidifferentiable at this point (cf. [7,26,27]).

Suppose now that  $H$  consists of all finite l.s.c. positively homogeneous convex functions  $h: X \rightarrow \mathbb{R}$  (or u.s.c. positively homogeneous concave functions  $h: X \rightarrow \mathbb{R}$ ). Then one can show that a function  $F: \Omega \rightarrow \mathbb{R}$  is Dini  $H$ -superdifferentiable ( $H$ -subdifferentiable) at a point  $x \in \Omega$  if and only if there exists an upper exhaustor (lower exhaustor) [23] of a functions  $F$  at this point.

*Remark 7* Note that the notion of quasidifferentiable functions in order complete vector lattices (cf. [7]) coincide with the notion of  $H$ -quasidifferentiable functions for the set  $H = \mathcal{B}(X, Y)$ . Also, the notion of quasidifferentiable in the generalized sense functions introduced in [28] is the particular case of the notion of  $H$ -quasidifferentiable functions, when the set  $H$  consists of all finite l.s.c. positively homogeneous convex and finite u.s.c. positively homogeneous concave functions.

### 3.4. Abstract convex approximations of nonsmooth functions

In this section we consider the concept of abstract convex approximations of nonsmooth functions, that is closely related to the notion of  $H$ -codifferentiability. These approximations are a very convenient tool for studying various kinds of optimization problems. We will use them to derive necessary conditions for an extremum of an  $H$ -codifferentiable function. The notion of abstract convex approximation is a natural generalization of the notion of convex approximation (see [29] and references therein).

Let, as earlier,  $H$  be a nonempty set of functions mapping  $X$  to  $\bar{E}$ , and let  $F: \Omega \rightarrow E$  be an arbitrary function.

*Definition 3.19* An  $H$ -convex function  $\varphi: X \rightarrow \bar{E}$  is called a weak upper  $H$ -convex approximation (or weak upper abstract convex approximation with respect to  $H$ ) of the function  $F$  at a point  $x \in \Omega$  if

- (1)  $\varphi(0) \geq 0$  and  $0 \in \text{int dom } \varphi$ ;
- (2) for any  $\Delta x \in X$  there exist  $\alpha_0 > 0$  and a function  $\beta: (0, \alpha_0) \rightarrow E$  such that  $\text{co}\{x, x + \alpha_0 \Delta x\} \subset \Omega \cap \text{dom } \varphi$ ,  $\beta(\alpha) \rightarrow 0$  as  $\alpha \downarrow 0$  and

$$F(x + \alpha \Delta x) - F(x) \leq \varphi(\alpha \Delta x) + \alpha \beta(\alpha) \quad \forall \alpha \in [0, \alpha_0).$$

*Definition 3.20* An  $H$ -concave function  $\psi: X \rightarrow \bar{E}$  is referred to as a weak lower  $H$ -concave approximation (or weak lower abstract concave approximation with respect to  $H$ ) of the function  $F$  at a point  $x \in \Omega$  if

- (1)  $\psi(0) \leq 0$  and  $0 \in \text{int dom } \psi$ ;
- (2) for any  $\Delta x \in X$  there exist  $\alpha_0 > 0$  and a function  $\beta: (0, \alpha_0) \rightarrow E$  such that  $\text{co}\{x, x + \alpha_0 \Delta x\} \subset \Omega \cap \text{dom } \psi$ ,  $\beta(\alpha) \rightarrow 0$  as  $\alpha \downarrow 0$  and

$$F(x + \alpha \Delta x) - F(x) \geq \psi(\alpha \Delta x) - \alpha \beta(\alpha) \quad \forall \alpha \in [0, \alpha_0).$$

*Definition 3.21* Let  $X$  be a normed space. An  $H$ -convex function  $\varphi: X \rightarrow \bar{E}$  is called a strong upper  $H$ -convex approximation of the function  $F$  at  $x \in \Omega$  if

- (1)  $\varphi(0) \geq 0$  and  $0 \in \text{int dom } \varphi$ ;
- (2) there exists  $r > 0$  and a function  $\beta: B(0, r) \rightarrow E$  such that  $\beta(\Delta x) \rightarrow 0$  as  $\Delta x \rightarrow 0$  and

$$F(x + \Delta x) - F(x) \leq \varphi(\Delta x) + \|\Delta x\| \beta(\Delta x) \quad \forall \Delta x \in B(0, r).$$

One can also define a strong lower  $H$ -concave approximation of the function  $F$  at a point  $x \in \Omega$ .

It is natural to expect that an upper  $H$ -convex approximation (lower  $H$ -concave approximation) does not usually provide enough information about the behaviour of the function  $F$  in a neighbourhood of a point  $x$ . Therefore, we have to use various families of upper  $H$ -convex (lower  $H$ -concave) approximations. Families of these approximations that are of the most importance for the study of optimization problems is called an exhaustive families.

*Definition 3.22* A family  $\{\varphi_\lambda\}$ ,  $\lambda \in \Lambda$  of weak upper  $H$ -convex approximations of the function  $F$  at a point  $x \in \Omega$  is said to be exhaustive if  $\inf_{\lambda \in \Lambda} \varphi_\lambda(0) = 0$  and for any admissible  $\Delta x \in X$

$$F(x + \Delta x) - F(x) = \inf_{\lambda \in \Lambda} \varphi_\lambda(\Delta x) + o(\Delta x, x),$$

where  $o(\alpha \Delta x, x)/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ .

*Definition 3.23* A family  $\{\psi_\lambda\}$ ,  $\lambda \in \Lambda$  of weak lower  $H$ -concave approximations of the function  $F$  at a point  $x \in \Omega$  is referred to as exhaustive if  $\sup_{\lambda \in \Lambda} \psi_\lambda(0) = 0$  and for any  $\Delta x \in X$

$$F(x + \Delta x) - F(x) = \sup_{\lambda \in \Lambda} \psi_\lambda(\Delta x) + o(\Delta x, x),$$

where  $o(\alpha \Delta x, x)/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ .

*Definition 3.24* Let  $X$  be a normed space. A family  $\{\varphi_\lambda\}$ ,  $\lambda \in \Lambda$  of strong upper  $H$ -convex approximations of the function  $F$  at a point  $x \in \Omega$  is said to be exhaustive if  $\inf_{\lambda \in \Lambda} \varphi_\lambda(0) = 0$  and for any  $\Delta x \in X$

$$F(x + \Delta x) - F(x) = \inf_{\lambda \in \Lambda} \varphi_\lambda(\Delta x) + o(\Delta x, x),$$

where  $o(\Delta x, x)/\|\Delta x\| \rightarrow 0$  as  $\Delta x \rightarrow 0$ .

The exhaustive family of strong lower  $H$ -concave approximations is defined in a similar way.

The following proposition reveals an obvious connection between upper  $H$ -convex (lower  $H$ -concave) approximations and  $H$ -codifferentials.

**PROPOSITION 3.25** *Let the set  $H$  be closed under addition, and let for any  $h \in H$  one has  $0 \in \text{int dom } h$ . Suppose that a function  $F : \Omega \rightarrow E$  is weakly (strongly)  $H$ -codifferentiable at a point  $x \in \Omega$ . Then for any  $(\Phi, \Psi) \in \delta F_H[x]$  and for all  $h \in \text{supp}^-(\Phi, H)$  and  $p \in \text{supp}^+(\Psi, H)$  the function  $\Phi + p$  is a weak (strong) upper  $H$ -convex approximation of  $F$  at  $x$  and the function  $h + \Psi$  is a weak (strong) lower  $H$ -concave approximation of  $F$  at  $x$ . Moreover, for any  $(\Phi, \Psi) \in \delta F_H[x]$  and for any  $U, V \subset H$  such that  $\Phi$  is generated by  $U$  and  $\Psi$  is generated by  $V$  the family  $\{\Phi + p\}$ ,  $p \in V$  is an exhaustive family of weak (strong) upper  $H$ -convex approximations of  $F$  at  $x$ , and the family  $\{h + \Psi\}$ ,  $h \in U$  is an exhaustive family of weak (strong) lower  $H$ -concave approximations of  $F$  at  $x$ .*

#### 4. Calculus of abstract codifferentiable functions

In this section we discuss the problem of computing  $H$ -codifferentials and construct the  $H$ -codifferential calculus. We also consider the problem of continuity of  $H$ -codifferentials, which is very important for practical applications. We study only the Hausdorff continuity of  $H$ -codifferentials; however, one could reformulate all results of this sections to the case of other types of continuity.

In the following propositions and theorems we mostly study weakly  $H$ -codifferentiable functions but all results of this section are also valid for strongly  $H$ -codifferentiable functions.

*Remark 1* (i) We do not discuss any formulae for computing  $H$ -quasidifferentials, upper  $H$ -convex (lower  $H$ -concave) approximations, and exhaustive families of these approximations. One can easily derive them arguing in a similar way to the cases of exhaustive families of nonhomogeneous convex approximations [29] and quasidifferentiable functions.[7]

(ii) One can consider the main results of this section as sufficient conditions for the set of all  $H$ -codifferentiable (or all continuously  $H$ -codifferentiable) at a given point functions to be a cone, a group under addition (or multiplication), a linear space, an algebra, a lattice or a vector lattice.

As earlier mentioned, we suppose that  $X$  is a Hausdorff topological vector space over the field of real or complex numbers,  $E$  is an order complete Hausdorff topological vector lattice,  $H$  is an arbitrary nonempty set of functions  $h : X \rightarrow \overline{E}$ , and  $\Omega \subset X$  is an open set.

It is obvious that if a function  $F : \Omega \rightarrow E$  is weakly  $H$ -codifferentiable at  $x \in \Omega$ , then for any  $c \in E$  the function  $F+c$  is also weakly  $H$ -codifferentiable at  $x$ ,  $\delta(F+c)_H[x] = \delta F_H[x]$  and  $D_H^w(F+c)(x) = D_H^w F(x)$ . It is easy to check that the following propositions hold true.

**PROPOSITION 4.1** *Let a function  $F : \Omega \rightarrow E$  be weakly  $H$ -codifferentiable at a point  $x \in \Omega$ , and let  $\alpha \in \mathbb{R}$  be arbitrary. Suppose also that  $H$  is a cone in the case  $\alpha \neq 0$ , and  $0 \in H$  in the case  $\alpha = 0$ . Then the function  $\alpha F$  is weakly  $H$ -codifferentiable at the*

point  $x$ ,  $\delta(\alpha F)_H[x] = \alpha\delta F_H[x]$  and  $D_H^w(\alpha F)(x) = \alpha D_H^w F(x)$  in the case  $\alpha \geq 0$ , and the function  $\alpha F$  is weakly  $(-H)$ -codifferentiable at  $x$ ,  $\delta(\alpha F)_{(-H)}[x] = \alpha\delta F_H[x]$  and  $D_{(-H)}^w(\alpha F)(x) = \alpha D_H^w F(x)$  in the case  $\alpha < 0$ .

**COROLLARY 4.2** *Let all assumptions of the previous proposition be satisfied, and let  $(H, d)$  be a metric space. Suppose that the mapping  $h \rightarrow \alpha h$  is uniformly continuous on  $H$  (if  $\alpha < 0$  and  $(-H) \not\subseteq H$ , then we suppose that the set  $(-H)$  is equipped with a metric; in particular, one can suppose that  $d(-h, -p) = d(h, p)$  for all  $h, p \in H$ ). Suppose also that the function  $F$  is Hausdorff continuously weakly  $H$ -codifferentiable at the point  $x$ . Then the function  $\alpha F$  is Hausdorff continuously weakly  $H$ -codifferentiable at the point  $x$ .*

**PROPOSITION 4.3** *Let functions  $F_1, F_2: \Omega \rightarrow E$  be weakly  $H$ -codifferentiable at a point  $x \in \Omega$ , and let the set  $H$  be closed under addition. Then the function  $F_1 + F_2$  is weakly  $H$ -codifferentiable at the point  $x$ ,  $\delta(F_1 + F_2)_H[x] = \delta(F_1)_H[x] + \delta(F_2)_H[x]$  and  $D_H^w(F_1 + F_2)(x) = D_H^w F_1(x) + D_H^w F_2(x)$ .*

**COROLLARY 4.4** *Let all assumption of the previous proposition be satisfied, and let  $(H, d)$  be a metric space. Suppose that the mapping  $(h, p) \rightarrow h + p$  is uniformly continuous on  $H \times H$ ; in particular, one can suppose that there exists  $C > 0$  such that*

$$d(h_1 + h_2, p_1 + p_2) \leq C(d(h_1, p_1) + d(h_2, p_2)) \quad \forall h_1, h_2, p_1, p_2 \in H.$$

*Suppose also that the functions  $F_1$  and  $F_2$  are Hausdorff continuously weakly  $H$ -codifferentiable at the point  $x$ . Then the function  $F_1 + F_2$  is Hausdorff continuously weakly  $H$ -codifferentiable at  $x$ .*

Let us study the problem of finding the  $H$ -codifferential of the superposition of functions.

**THEOREM 4.5** *Let  $X, Y$  be arbitrary normed spaces,  $E$  be an order complete normed lattice. Suppose that the following conditions are satisfied:*

- (1) *a function  $F: \Omega \rightarrow E$  is strongly  $H$ -codifferentiable at a point  $x \in \Omega$ ;*
- (2)  *$F'_H[x]$  is Lipschitz continuous in a neighbourhood of zero;*
- (3)  *$S \subset Y$  is an open set,  $y \in S$  is arbitrary;*
- (4) *a function  $G: S \rightarrow X$  is continuous and Gâteaux differentiable at the point  $y$ , and  $G(y) = x$ .*

*Then there exists an open set  $\mathcal{O} \subset S$  such that  $y \in \mathcal{O}$ , the function  $T = F \circ G$  is defined on  $\mathcal{O}$  and weakly  $\widehat{H}$ -codifferentiable at the point  $y$ , where*

$$\widehat{H} = \{\hat{h}: Y \rightarrow \mathbb{R} \mid \hat{h} = h \circ \delta G[y], h \in H\}$$

*and  $\delta G[y]$  is the Gâteaux derivative of the function  $G$  at the point  $y$ . Moreover, for any  $(\Phi, \Psi) \in F'_H[x]$  and  $(U, V) \in D_H^s F(x)$  one has*

$$\delta T_{\widehat{H}}[y] = [\Phi \circ \delta G[y], \Psi \circ \delta G[y]]_w, \quad (9)$$

$$D_{\widehat{H}}^w T(y) = \left[ \left\{ \hat{h} = h \circ \delta G[y] \in \widehat{H} \mid h \in U \right\}, \left\{ \hat{p} = p \circ \delta G[y] \in \widehat{H} \mid p \in V \right\} \right]_w. \quad (10)$$

*Proof* Note that the right-hand sides of equalities (9) and (10) do not depend on the choice of  $(\Phi, \Psi) \in F'_H[x]$  and  $(U, V) \in D^s_H F(x)$ . Hence these formulae are correct.

Since the function  $G$  is continuous at the point  $y$  and the set  $\Omega$  is open, then there exists  $\mu > 0$  such that  $\mathcal{O}(y, \mu) \subset S$  and  $G(\mathcal{O}(y, \mu)) \subset \Omega$ . Denote  $\mathcal{O} = \mathcal{O}(y, \mu)$ . It is clear that the composition  $F \circ G$  is defined at least on  $\mathcal{O}$ .

Fix an arbitrary  $\Delta y \in \mathcal{O}(0, \mu)$ . The function  $G$  is Gâteaux differentiable at the point  $y$  hence

$$G(y + \Delta y) = G(y) + \delta G[y](\Delta y) + o_G(\Delta y),$$

where  $o_G(\alpha \Delta y)/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ . Denote  $\omega(\Delta y) = \delta G[y](\Delta y) + o_G(\Delta y)$ . It is obvious that there exists  $\alpha_0 \in (0, 1)$  such that for all  $\alpha \in (0, \alpha_0)$

$$\|\omega(\alpha \Delta y)\| \leq \alpha(\|\delta G[y]\| + 1)\|\Delta y\|.$$

In particular, one has that  $\|\omega(\alpha \Delta y)\| \rightarrow 0$  as  $\alpha \downarrow 0$ .

Fix an arbitrary  $(\Phi, \Psi) \in F'_H[x]$  such that the functions  $\Phi(\cdot)$  and  $\Psi(\cdot)$  are Lipschitz continuous in a neighbourhood of zero (such  $\Phi$  and  $\Psi$  exist since  $F'_H[x]$  is Lipschitz continuous in a neighbourhood of zero). The function  $F$  is strongly  $H$ -codifferentiable at the point  $x = G(y)$ , therefore for any admissible  $\Delta x \in X$  one has

$$F(x + \Delta x) - F(x) = \Phi(\Delta x) + \Psi(\Delta x) + o_F(\Delta x),$$

where  $\|o_F(\Delta x)\|/\|\Delta x\| \rightarrow 0$  as  $\Delta x \rightarrow 0$  or, equivalently,  $\|o_F(\Delta x)\| = \beta(\Delta x)\|\Delta x\|$ , where  $\beta(\Delta x) \rightarrow 0$  as  $\Delta x \rightarrow 0$ . Thus, one gets

$$\begin{aligned} T(y + \Delta y) - T(y) &= F(G(y + \Delta y)) - F(G(y)) \\ &= F(G(y) + \omega(\Delta y)) - F(G(y)) \\ &= \Phi(\omega(\Delta y)) + \Psi(\omega(\Delta y)) + o_F(\omega(\Delta y)). \end{aligned}$$

For any  $\alpha \in (0, \alpha_0)$  one has

$$\|o_F(\omega(\alpha \Delta y))\| \leq \alpha\beta(\omega(\alpha \Delta y))(\|G'[y]\| + 1)\|\Delta y\|.$$

Since  $\beta(\Delta x) \rightarrow 0$  as  $\Delta x \rightarrow 0$  and  $\|\omega(\alpha \Delta y)\| \rightarrow 0$  as  $\alpha \downarrow 0$ , then  $\beta(\omega(\alpha \Delta y)) \rightarrow 0$  as  $\alpha \downarrow 0$ . Therefore,  $\|o_F(\omega(\alpha \Delta y))\|/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ . It remains to note that since the functions  $\Phi(\cdot)$  and  $\Psi(\cdot)$  are Lipschitz continuous in a neighbourhood of zero, then

$$\Phi(\omega(\Delta y)) + \Psi(\omega(\Delta y)) = (\Phi \circ \delta G[y])(\Delta y) + (\Psi \circ \delta G[y])(\Delta y) + o(\Delta y),$$

where  $o(\alpha \Delta y, y)/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ . □

Let us recall some definitions from lattice theory (see [11,12,30]). Let  $Y$  be an order complete vector lattice. Denote by  $L(E, Y)_+$  the set of all positive linear operators mapping  $E$  to  $Y$ . A linear operator  $T: E \rightarrow Y$  is said to be a complete lattice homomorphism, if for any bounded from above set  $A \subset E$  one has  $T \sup_{x \in A} x = \sup_{x \in A} Tx$ , and for any bounded from below set  $B \subset E$  one has  $T \inf_{x \in B} x = \inf_{x \in B} Tx$ . It is clear that any complete lattice homomorphism  $T$  is a positive operator.

A linear operator  $T: E \rightarrow Y$  is said to be completely regular, if there exist complete lattice homomorphisms  $S, R: E \rightarrow Y$  such that  $T = S - R$ . One can verify that the representation  $T = S - R$  of the completely regular operator  $T$  as the difference of two

complete lattice homomorphisms is not unique. It is easy to see that any linear mapping  $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ , where  $\mathbb{R}^m$  and  $\mathbb{R}^n$  are endowed with the canonical order relations, is completely regular.

**THEOREM 4.6** *Let  $X$  be a normed space,  $E$  and  $Y$  be order complete normed lattices. Suppose that the following conditions are satisfied:*

- (1) *the set  $H$  is closed under addition and for any  $h \in H$  one has  $-h \in H$ ;*
- (2) *a function  $F: \Omega \rightarrow E$  is weakly  $H$ -codifferentiable at a point  $x$ ;*
- (3)  *$\delta F_H[x]$  is Lipschitz continuous in a neighbourhood of zero;*
- (4)  *$\Sigma \subset E$  is an open set such that  $F(x) \in \Sigma$ ;*
- (5) *a function  $G: \Sigma \rightarrow Y$  is Fréchet differentiable at a point  $F(x)$ ;*
- (6) *the Fréchet derivative  $G'[F(x)]$  of  $G$  at  $F(x)$  is a completely regular linear mapping;*
- (7) *the function  $T = G \circ F$  is defined on an open set  $\mathcal{O} \subset \Omega$  (in particular, one can suppose that  $\Sigma = E$  or that  $F$  is continuous at  $x$ ).*

Then  $T$  is weakly  $\widehat{H}$ -codifferentiable at a point  $x$ , where  $\widehat{H} = \{S \circ h + R \circ p' \mid h, p \in H, S, R \in L(E, Y)_+\}$ . Furthermore, for all complete lattice homomorphisms  $S, R: E \rightarrow Y$  such that  $G'[F(x)] = S - R$ , and for any  $(\Phi, \Psi) \in \delta F_H[x]$  and  $(U, V) \in D_H^w F(x)$  one has

$$\delta T_{\widehat{H}}[x] = [S \circ \Phi - R \circ \Psi, S \circ \Psi - R \circ \Phi]_w, \quad (11)$$

$$D_{\widehat{H}}^w T(x) = [\{S \circ h - R \circ p \mid h \in U, p \in V\}, \{S \circ p - R \circ h \mid h \in U, p \in V\}]^w. \quad (12)$$

*Proof* Fix arbitrary complete lattice homomorphisms  $S, R: E \rightarrow Y$  such that  $G'[F(x)] = S - R$ , and an arbitrary  $(\Phi, \Psi) \in \delta F_H[x]$ . Arguing in a similar way to the proof of Theorem 4.5 one can show that for any admissible argument increment  $\Delta x \in X$

$$T(x + \Delta x) - T(x) = G'[F(x)](\Phi(\Delta x) + \Psi(\Delta x)) + o(\Delta x, x),$$

where  $o(\alpha \Delta x, x)/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ . Let us show that the function  $K(\cdot) = G'[F(x)](\Phi(\cdot) + \Psi(\cdot))$  can be represented as the sum of  $\widehat{H}$ -convex and  $\widehat{H}$ -concave functions. Indeed, let  $\Phi$  be generated by  $U \subset H$ , and  $\Psi$  be generated by  $V \subset H$ . Hence for any  $x \in X$

$$\begin{aligned} K(x) &= (S - R)(\sup_{h \in U} h(x) + \inf_{p \in V} p(x)) \\ &= \sup_{h \in U, p \in V} (S(h(x)) - R(p(x))) + \inf_{h \in U, p \in V} (S(p(x)) - R(h(x))). \end{aligned}$$

Taking into account the assumptions about the set  $H$  one gets that  $K$  is the sum of  $\widehat{H}$ -convex and  $\widehat{H}$ -concave functions.

It remains to note that formulae (11)–(12) do not depend on the choice of complete lattice homomorphisms  $S, R: E \rightarrow Y$ , such that  $G'[F(x)] = S - R$ ,  $(\Phi, \Psi) \in \delta F_H[x]$  and  $(U, V) \in D_H^w F(x)$ , since

$$(S \circ \Phi - R \circ \Psi) + (S \circ \Psi - R \circ \Phi) = G'[F(x)](\Phi + \Psi),$$

and for any function  $w: X \rightarrow E$  and  $x \in X$  one has  $G'[F(x)](w(\alpha x))/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ , whenever  $w(\alpha x)/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ .  $\square$

As a simple, yet useful corollary to the previous proposition one gets the following result.

**THEOREM 4.7** *Let  $X$  be an arbitrary normed space. Suppose that the following conditions are satisfied:*

- (1) *the set  $H$  is a linear subspace of  $\mathbb{R}^X$  (where  $\mathbb{R}^X$  is the set of all functions mapping  $X$  to  $\mathbb{R}$ );*
- (2) *functions  $F_i: \Omega \rightarrow \mathbb{R}$  are weakly  $H$ -codifferentiable at a point  $x \in \Omega$ ,  $i \in I = \{1, \dots, d\}$ ;*
- (3)  *$\delta(F_i)_H[x]$  are Lipschitz continuous in a neighbourhood of zero,  $i \in I$ ;*
- (4)  *$S \subset \mathbb{R}^d$  is an open set such that  $y = (F_1(x), \dots, F_d(x)) \in S$ ;*
- (5) *a function  $g: S \rightarrow \mathbb{R}$  is differentiable at the point  $y$ ;*
- (6) *the function  $T(\cdot) = g(F_1(\cdot), \dots, F_d(\cdot))$  is defined on an open set  $\mathcal{O} \subset \Omega$ .*

*Then the function  $T$  is weakly  $H$ -codifferentiable at the point  $x$  and*

$$\delta T_H[x] = \sum_{i \in I} \frac{\partial g}{\partial y_i}(y) \delta(F_i)_H[x], \quad D_H^w T(x) = \sum_{i \in I} \frac{\partial g}{\partial y_i}(y) D_H^w F_i(x).$$

**COROLLARY 4.8** *Let all assumptions of the previous theorem be satisfied, and let  $(H, d)$  be a metric space. Suppose that the mapping  $(\alpha, h) \rightarrow \alpha h$ ,  $\alpha \in \mathbb{R}$  is uniformly continuous on  $\mathbb{R} \times H$ , and the mapping  $(h, p) \rightarrow h + p$  is uniformly continuous on  $H \times H$  (in particular, one can suppose that  $d$  is a norm). Suppose also that all functions  $F_i$  are continuous and Hausdorff continuously weakly  $H$ -codifferentiable at the point  $x$ , and the function  $g$  is continuously differentiable at the point  $y$ . Then the function  $T$  is Hausdorff continuously weakly  $H$ -codifferentiable at  $x$ .*

**Remark 2** Let functions  $F, F_1, F_2: \Omega \rightarrow \mathbb{R}$  be (continuously)  $H$ -codifferentiable at a point  $x \in \Omega$ , and let  $F \neq 0$  in a neighbourhood of  $x$ . As simple corollaries to Theorem 4.7 one gets the  $H$ -codifferentiability (continuous  $H$ -codifferentiability) of the functions  $F_1 \cdot F_2$  and  $1/F$  at the point  $x$  under suitable assumptions on the set  $H$ .

Let us consider the supremum and the infimum of  $H$ -codifferentiable functions.

**THEOREM 4.9** *Let functions  $F_i: \Omega \rightarrow E$  be weakly  $H$ -codifferentiable at a point  $x \in \Omega$ ,  $i \in I = \{1, \dots, n\}$ . Suppose that the set  $H$  satisfies the following assumptions:*

- (1)  *$H$  is closed under addition;*
- (2) *for any  $h \in H$  one has  $-h \in H$ ;*
- (3)  *$H$  is closed under vertical shifts, i.e. for any  $c \in E$ ,  $h \in H$  one has  $h + c \in H$ .*

*Then the functions  $F = \sup_{i \in I} F_i$  and  $G = \inf_{i \in I} F_i$  are weakly  $H$ -codifferentiable at the point  $x$ . Moreover, for any  $(\Phi_i, \Psi_i) \in \delta(F_i)_H[x]$  and  $(U_i, V_i) \in D_H^w F_i(x)$ ,  $i \in I$ , one has*



$$\delta F_H[x] = \left[ \sup_{i \in I} \left( F_i(x) - F(x) + \Phi_i - \sum_{j \in I \setminus \{i\}} \Psi_j \right), \sum_{k \in I} \Psi_k \right]_w, \quad (13)$$

$$\delta G_H[x] = \left[ \sum_{k \in I} \Phi_k, \inf_{i \in I} \left( F_i(x) - G(x) + \Psi_i - \sum_{j \in I \setminus \{i\}} \Phi_j \right) \right]_w, \quad (14)$$

and

$$D_H^w F(x) = \left[ \bigcup_{i \in I} \left\{ \{F_i(x) - F(x)\} + U_i - \sum_{j \in I \setminus \{i\}} V_j \right\}, \sum_{k \in I} V_k \right]_w, \quad (15)$$

$$D_H^w G(x) = \left[ \sum_{k \in I} U_k, \bigcup_{i \in I} \left\{ \{F_i(x) - G(x)\} + V_i - \sum_{j \in I \setminus \{i\}} U_j \right\} \right]_w. \quad (16)$$

*Proof* Note that the right-hand sides of formulae (13)–(16) do not depend on the choice of  $(\Phi_i, \Psi_i) \in \delta(F_i)_H[x]$  and  $(U_i, V_i) \in D_H^w F_i(x)$ . Therefore, these formulae are correct.

We only consider the function  $F$ , since the assertion about the function  $G$  is proved in a similar way. Fix arbitrary  $(\Phi_i, \Psi_i) \in \delta F_H[x]$  and  $(U_i, V_i) \in D_H^w F(x)$ ,  $i \in I$ . For any admissible argument increment  $\Delta x \in X$  one has

$$F_i(x + \Delta x) = F_i(x) + \Phi_i(\Delta x) + \Psi_i(\Delta x) + o_i(\Delta x, x),$$

where  $o_i(\alpha \Delta x, x)/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ . Hence

$$F(x + \Delta x) - F(x) = \sup_{i \in I} (F_i(x) - F(x) + \Phi_i(\Delta x) + \Psi_i(\Delta x) + o_i(\Delta x, x)).$$

Applying the simple inequality

$$\begin{aligned} & \left| \sup_{i \in I} (F_i(x) - F(x) + \Phi_i(\Delta x) + \Psi_i(\Delta x) + o_i(\Delta x, x)) \right. \\ & \left. - \sup_{i \in I} (F_i(x) - F(x) + \Phi_i(\Delta x) + \Psi_i(\Delta x)) \right| \\ & \leq \sum_{i \in I} |o_i(\Delta x, x)| \end{aligned}$$

one gets

$$F(x + \Delta x) - F(x) = \sup_{i \in I} (F_i(x) - F(x) + \Phi_i(\Delta x) + \Psi_i(\Delta x)) + o(\Delta x, x),$$

where  $o(\alpha \Delta x, x)/\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ . It remains to note that

$$\begin{aligned} & \sup_{i \in I} (F_i(x) - F(x) + \Phi_i(\Delta x) + \Psi_i(\Delta x)) \\ & = \sup_{i \in I} \left( F_i(x) - F(x) + \Phi_i(\Delta x) - \sum_{j \in I \setminus \{i\}} \Psi_j(\Delta x) \right) + \sum_{k=1}^n \Psi_k(\Delta x), \end{aligned}$$

and the fact that the right-hand side of the last equality is the sum of  $H$ -convex and  $H$ -concave functions.  $\square$

**COROLLARY 4.10** *Suppose that all assumptions of the previous theorem are satisfied and  $0 \in H$  (or, equivalently, for any  $c \in E$  the function  $h \equiv c$  belongs to  $H$ ). Denote by  $\ell: E \rightarrow H$  the natural embedding of  $E$  in  $H$ , i.e.  $(\ell(y))(\cdot) \equiv y$  for all  $y \in E$ . Let  $(H, d)$  be a metric space such that the following assumptions are satisfied:*

- (1) *the mapping  $(h, p) \rightarrow h + p$  is uniformly continuous on  $H \times H$ ;*
- (2) *the mapping  $h \rightarrow -h$  is uniformly continuous on  $H$ ;*
- (3) *the quotient topology on  $\ell(E)$  induced by  $\ell$  is finer than the topology induced by the metric  $d$*

*(therefore, if a function  $T: \Omega \rightarrow E$  is continuous, then the function  $\ell \circ T: \Omega \rightarrow H$  is also continuous). Suppose also that all functions  $F_i$  are continuous and Hausdorff continuously weakly  $H$ -codifferentiable at the point  $x$ ,  $i \in I$ . Then the functions  $F$  and  $G$  are Hausdorff continuously weakly  $H$ -codifferentiable at  $x$ .*

**Remark 3** One can easily prove that under suitable assumptions the supremum of an infinite family of weakly  $H$ -hypodifferentiable functions is also weakly  $H$ -hypodifferentiable (and that the infimum of an infinite family of weakly  $H$ -hyperdifferentiable functions is weakly  $H$ -hypodifferentiable).

### 5. Necessary optimality conditions

In this section we derive necessary optimality conditions for  $H$ -quasidifferentiable and  $H$ -codifferentiable functions with the use of upper abstract convex and lower abstract concave approximations. Then we show how they can be transformed into more constructive necessary optimality conditions in some particular cases.

#### 5.1. General necessary conditions for an extremum

In this section we only consider the case  $E = \mathbb{R}$ ; however, one can modify main results of this section to the case of general order complete topological vector lattices.

We need an auxiliary definition (see [18,19]).

**Definition 5.1** Let  $f: X \rightarrow \overline{\mathbb{R}}$  be an arbitrary function such that  $f(0) = 0$ . The function  $f$  is said to be subhomogeneous (superhomogeneous) if for any  $\Delta x \in X$  and  $\alpha \in (0, 1)$  one has

$$f(\alpha \Delta x) \leq \alpha f(\Delta x) \quad (f(\alpha \Delta x) \geq \alpha f(\Delta x)).$$

The class of all subhomogeneous (or superhomogeneous) functions is very broad. In particular, any convex (concave) function  $f: X \rightarrow \mathbb{R}$  such that  $f(0) = 0$  is subhomogeneous (superhomogeneous). Also, any positively homogeneous of degree  $\lambda \geq 1$  ( $\lambda \in (0, 1]$ ) function is subhomogeneous (superhomogeneous).

Let  $A \subset X$  be a convex set, and let the set  $H$  be closed under vertical shifts. If  $x \in X$  then denote  $A - x = \{y \in X \mid y = a - x, a \in A\}$ . Consider the following optimization problem

$$f_0(x) \rightarrow \inf, \quad x \in A, \quad f_i(x) \leq 0, \quad i \in I, \quad (17)$$

where  $f_i: X \rightarrow \mathbb{R}, i \in I_0 = \{0\} \cup I, I = \{1, \dots, n\}$ .

**THEOREM 5.2** *Let functions  $\varphi_i: X \rightarrow \overline{\mathbb{R}}$  be weak upper  $H$ -convex approximations of the functions  $f_i$  at a point  $x^* \in A$  such that  $\varphi_i(0) = 0, i \in I_0$ . Suppose that  $x^*$  is a point of local minimum in problem (17), and the  $H$ -convex function*

$$g(\cdot) = \sup\{\varphi_0(\cdot), \varphi_1(\cdot) + f_1(x^*), \dots, \varphi_n(\cdot) + f_n(x^*)\} \quad (18)$$

*is subhomogeneous. Then 0 is a point of global minimum of the function  $g$  on the set  $A - x^*$ . Moreover, if  $A = X$  and  $0 \in H$ , then  $0 \in \underline{\partial}_H g(0)$ .*

*Proof* Since  $x^*$  is a point of local minimum in problem (17), then it is obvious that  $x^*$  is a point of local minimum of the function

$$F(\cdot) = \max\{f_0(\cdot) - f_0(x^*), f_1(\cdot), \dots, f_n(\cdot)\}$$

on the set  $A$ . It is easy to check that the function  $g$  (cf. (18)) is a weak upper  $H$ -convex approximation of the function  $F$  at  $x^*$  and  $g(0) = 0$ .

Suppose that 0 is not a point of global minimum of the function  $g$  on the set  $A - x^*$ . Then there exists  $y \in A$  such that  $g(y - x^*) = -m < 0 = g(0)$ . Denote  $\Delta x = y - x^*$ . Since  $g$  is a weak upper  $H$ -convex approximation of the function  $F$  at the point  $x^*$  and  $g$  is subhomogeneous, then there exists  $\delta \in (0, 1)$  such that

$$F(x^* + \alpha \Delta x) - F(x^*) \leq g(\alpha \Delta x) + \frac{m}{2} \alpha \leq \alpha g(\Delta x) + \frac{m}{2} \alpha = -\frac{m}{2} \alpha \quad \forall \alpha \in (0, \delta),$$

which contradicts the fact that  $x^*$  is a point of local minimum of  $F$  on  $A$ . □

Arguing in a similar way one can prove the following theorem, which is the ‘mirror version’ of the previous one.

**THEOREM 5.3** *Let functions  $\psi_i: X \rightarrow \overline{\mathbb{R}}$  be weak lower  $H$ -concave approximations of the functions  $f_i$  at a point  $x^* \in A$  such that  $\psi_i(0) = 0, i \in I_0$ . Suppose that  $x^*$  is a point of local maximum in the problem*

$$f_0(x) \rightarrow \sup, \quad x \in A, \quad f_i(x) \geq 0, \quad i \in I, \quad (19)$$

*and the  $H$ -concave function*

$$g(\cdot) = \inf\{\psi_0(\cdot), \psi_1(\cdot) + f_1(x^*), \dots, \psi_n(\cdot) + f_n(x^*)\}$$

*is superhomogeneous. Then 0 is a point of global maximum of the function  $g$  on the set  $A - x^*$ . Moreover, if  $A = X$  and  $0 \in H$ , then  $0 \in \overline{\partial}_H g(0)$ .*

One can obtain necessary optimality conditions in terms of abstract convex approximations for more general optimization problems, although it requires more restrictive assumptions. Namely, let  $X$  be a normed space and  $M \subset \Omega$  be a nonempty set. For any  $x \in \text{cl } M$  denote by  $T_M(x)$  the contingent cone to the set  $M$  at the point  $x$  (see [31], Chapter 4). The following theorem holds true.

**THEOREM 5.4** *Let  $x^* \in X$  be a point of local minimum in the problem*

$$f_0(x) \rightarrow \inf, \quad x \in M, \quad f_i(x) \leq 0, \quad i \in I, \quad (20)$$

*Suppose that a function  $\varphi_i: X \rightarrow \overline{\mathbb{R}}$  is a strong upper  $H$ -convex approximation of the function  $f_i$  at the point  $x^*$  such that  $\varphi_i(0) = 0$ , and  $\varphi_i$  is Lipschitz continuous in a neighbourhood of zero,  $i \in I_0$ . Suppose also that the  $H$ -convex function*

$$g(x) = \sup\{\varphi_0(x), \varphi_1(x) + f_1(x^*), \dots, \varphi_n(x) + f_n(x^*)\} \quad x \in X \quad (21)$$

*is subhomogeneous. Then 0 is a point of global minimum of  $g$  on  $T_M(x^*)$ .*

*Proof* It is clear that  $x^*$  is a point of local minimum of the function

$$F(\cdot) = \max\{f_0(\cdot) - f(x^*), f_1(\cdot), \dots, f_n(\cdot)\}$$

on the set  $M$ . Also, it is easy to verify that the function  $g$  (cf. (21)) is a strong upper  $H$ -convex approximation of  $F$  at  $x^*$ . Moreover,  $g$  is Lipschitz continuous in a neighbourhood of zero and  $g(0) = 0$ .

Suppose that there exists  $v \in T_M(x^*)$  such that  $g(v) = -m < g(0)$ . Since  $v \in T_M(x^*)$ , then there exist sequences  $\{h_n\} \subset (0, +\infty)$  and  $\{v_n\} \subset X$  such that  $x^* + h_n v_n \in M$ ,  $h_n \downarrow 0$  and  $v_n \rightarrow v$  as  $n \rightarrow \infty$ .

Applying the fact that  $g$  is a strong upper  $H$ -convex approximation of  $F$  at  $x^*$  one has that there exist  $r > 0$  and a function  $\beta: B(0, r) \rightarrow \mathbb{R}$  such that  $\beta(\Delta x) \rightarrow 0$  as  $\Delta x \rightarrow 0$  and

$$F(x^* + \Delta x) - F(x^*) \leq g(\Delta x) + \beta(\Delta x)\|\Delta x\| \quad \forall \Delta x \in B(0, r).$$

Hence, there exists  $n_1 \in \mathbb{N}$  such that for any  $n > n_1$  one has  $|\beta(h_n v_n)|\|v_n\| \leq m/3$ . Since  $g$  is Lipschitz continuous in a neighbourhood of zero and  $v_n \rightarrow v$ , then there exist  $L > 0$  and  $n_2 \in \mathbb{N}$  such that for all  $n > n_2$

$$|g(h_n v_n) - g(h_n v)| \leq L h_n \|v_n - v\| \leq \frac{m}{3} h_n.$$

Therefore, taking into account the subhomogeneity of  $g$ , one gets that for any  $n > \max\{n_1, n_2\}$

$$\begin{aligned} F(x^* + h_n v_n) - F(x^*) &\leq g(h_n v_n) + \beta(h_n v_n)h_n \|v_n\| \\ &\leq g(h_n v) + \frac{2m}{3} h_n \leq -m h_n + \frac{2m}{3} h_n < 0, \end{aligned}$$

which contradicts the fact that  $x^*$  is a point of local minimum of  $F$  on  $M$ . □

*Remark 1* One can easily prove an analogous theorem about necessary condition for a local maximum in the problem

$$f_0(x) \rightarrow \sup, \quad x \in M, \quad f_i(x) \geq 0, \quad i \in I$$

in terms of strong lower  $H$ -concave approximations.

As obvious corollaries to the previous theorems one gets the following necessary optimality conditions for  $H$ -codifferentiable functions.

**THEOREM 5.5** *Let the functions  $f_i, i \in I_0$  be weakly  $H$ -codifferentiable at a point  $x^* \in A$ , and let  $x^*$  be a point of local minimum in problem (17). Suppose that the set  $H$  is closed under addition and for any  $h \in H$  one has  $0 \in \text{int dom } h$ . Then for any  $(\Phi_i, \Psi_i) \in \delta(f_i)_H(x^*)$  and  $p_i \in \bar{\partial}_H \Psi_i(0), i \in I_0$  such that the  $H$ -convex function*

$$g(\cdot) = \sup\{\Phi_0(\cdot) + p_0(\cdot), \Phi_1(\cdot) + p_1(\cdot) + f_1(x^*), \dots, \Phi_n(\cdot) + p_n(\cdot) + f_n(x^*)\}$$

*is subhomogeneous the function  $g$  attains a global minimum on the set  $A - x^*$  at the origin.*

**THEOREM 5.6** *Let the functions  $f_i, i \in I_0$  and the set  $H$  be as in the previous theorem. Suppose that  $x^*$  is a point of local maximum in problem (19). Then for any  $(\Phi_i, \Psi_i) \in \delta(f_i)_H(x)$  and  $h_i \in \underline{\partial}_H \Phi_i(0), i \in I_0$  such that the  $H$ -concave function*

$$g(\cdot) = \inf\{h_0(\cdot) + \Psi_0(\cdot), h_1(\cdot) + \Psi_1(\cdot) + f_1(x^*), \dots, h_n(\cdot) + \Psi_n(\cdot) + f_n(x^*)\}$$

*is superhomogeneous the function  $g$  has a global maximum value on the set  $A - x^*$  at the origin.*

In general, upper abstract convex approximations are more convenient for the study of minimization problems, whereas lower abstract concave approximations are more convenient for the study of maximization problems. Necessary conditions for a maximum can be expressed in terms of upper abstract convex approximations, although these conditions are much more cumbersome than the ones stated in Theorem 5.3.

We need additional notation. Denote

$$\gamma(x, A) = \{g \in X \mid \exists \alpha > 0: x + \alpha g \in A\}$$

and  $\Gamma(x, A) = \text{cl } \gamma(x, A)$ . It is easy to see that both  $\gamma(x, A)$  and  $\Gamma(x, A)$  are nonempty convex cones.

**THEOREM 5.7** *Let  $\{\varphi_\lambda\}, \lambda \in \Lambda$  be an exhaustive family of weak upper  $H$ -convex approximations of the function  $f_0$  at a point  $x^* \in A$ , and let  $x^*$  be a point of local maximum of the function  $f_0$  on the set  $A$ . Suppose that for any  $g \in \Gamma(x^*, A)$  there exists  $\alpha_g > 0$  such that for any  $\lambda \in \Lambda$  the function  $\alpha \rightarrow \varphi_\lambda(\alpha g), \alpha \in [0, \alpha_g)$  is convex. Then for any  $\varepsilon > 0$  and  $g \in \gamma(x^*, A)$  there exists  $\lambda \in \Lambda$  such that*

$$\varphi'_\lambda(0, g) \leq \varepsilon, \tag{22}$$

*where  $\varphi'_\lambda(0, g)$  is the directional derivative of the function  $\varphi_\lambda$  at the origin in the direction  $g$ . Moreover, if  $\Lambda$  is finite, then for any  $g \in \gamma(x^*, A)$  there exists  $\lambda \in \Lambda$  such that*

$$\varphi'_\lambda(0, g) \leq 0.$$

*If, in addition, for any  $\lambda \in \Lambda$  the function  $\varphi'_\lambda(0, \cdot)$  is continuous on  $\Gamma(x^*, A)$ , then for any  $g \in \Gamma(x^*, A)$  there exists  $\lambda \in \Lambda$  such that the last inequality holds true.*

*Proof* Note that since for any  $\lambda \in \Lambda$  the function  $z_{\lambda, g}(\alpha) = \varphi_\lambda(\alpha g), \alpha \in [0, \alpha_g), g \in \Gamma(x^*, A)$  is convex then, as it is well-known, there exists the right-hand derivative  $(z_{\lambda, g})'_+(0)$  and the following equalities hold true

$$(z_{\lambda, g})'_+(0) = \lim_{\alpha \downarrow 0} \frac{\varphi(\alpha g) - \varphi(0)}{\alpha} = \varphi'_\lambda(0, g) = \inf_{\alpha \in (0, \alpha_g)} \frac{\varphi_\lambda(\alpha g) - \varphi_\lambda(0)}{\alpha} \quad (23)$$

(cf. [32], Proposition 4.1.3).

Suppose that there exists  $\varepsilon > 0$  and  $g \in \gamma(x^*, A)$  such that inequality (22) does not hold true for any  $\lambda \in \Lambda$ . Hence, applying (23) one gets

$$\varphi_\lambda(\alpha g) \geq \varphi_\lambda(0) + \varepsilon \alpha \quad \forall \alpha \in [0, \alpha_g] \quad \forall \lambda \in \Lambda. \quad (24)$$

Taking into account the facts that  $g \in \gamma(x^*, A)$  and the set  $A$  is convex, one can suppose that  $\text{co}\{x^*, x^* + \alpha_g g\} \subset A$ .

Since  $\{\varphi_\lambda\}, \lambda \in \Lambda$ , is an exhaustive family of weak upper  $H$ -convex approximations of  $f_0$  at  $x^*$ , then  $\inf_{\lambda \in \Lambda} \varphi_\lambda(0) = 0$  and there exists  $\delta > 0$  such that

$$f_0(x^* + \alpha g) - f_0(x^*) \geq \inf_{\lambda \in \Lambda} \varphi_\lambda(\alpha g) - \frac{\varepsilon}{2} \alpha \quad \forall \alpha \in (0, \delta).$$

Thus, taking into account (24) one has

$$f_0(x^* + \alpha g) - f_0(x^*) \geq \inf_{\lambda \in \Lambda} \varphi_\lambda(0) + \frac{\varepsilon}{2} \alpha = \frac{\varepsilon}{2} \alpha \quad \forall \alpha \in (0, \min\{\delta, \alpha_g\}),$$

which contradicts the fact that  $x^*$  is a point of local maximum of  $f_0$  on  $A$ . □

*Remark 2* (i) An analogous theorem about necessary conditions for a minimum in terms of weak lower abstract concave approximations also holds true.

(ii) One can construct a numerical method for finding stationary points of an  $H$ -codifferentiable function (as well as a numerical method for finding a solution of the equation  $F(x) = 0$ , where  $F$  is  $H$ -codifferentiable) based on the method for the search of a local minimizer of a nonsmooth function having a continuous approximation (see [18,19]).

### 5.2. Necessary conditions for an extremum of abstract quasidifferentiable function

Let us consider necessary optimality conditions for  $H$ -quasidifferentiable functions. We only discuss necessary conditions for a minimum, since necessary conditions for a maximum are symmetrical to them. All necessary optimality conditions stated below immediately follows from the necessary conditions for an extremum of a directionally differentiable function. Therefore, we omit the proofs.

Let all functions  $h \in H$  be p.h., and, as earlier, suppose that  $f_0: \Omega \rightarrow \mathbb{R}$  is an arbitrary function,  $A \subset \Omega$  is a nonempty convex set.

**THEOREM 5.8** *Let the function  $f_0$  be Dini (Hadamard)  $H$ -quasidifferentiable at a point  $x^* \in A$ . Suppose that  $x^*$  is a point of local minimum of the function  $f_0$  on the set  $A$ . Then for any  $(\Phi, \Psi) \in \mathcal{D}_H f_0(x^*)$  and for all  $p \in \text{supp}^+(\Psi, H)$  the function  $\Phi + p$  attains a global minimum value on the set  $\gamma(x^*, A)$  ( $\Gamma(x^*, A)$ ) at the origin. Also, for any  $U \subset H$  such that  $\Phi = \sup_{h \in U} h$ , for any  $\varepsilon > 0$  and for all  $g \in \gamma(x^*, A)$  ( $g \in \Gamma(x^*, A)$ ) there exists  $h \in U$  such that  $h(g) + \Psi(g) \geq -\varepsilon$ . Moreover, if there exists  $U \subset H$  such that*

- (1)  $\Phi$  is generated by  $U$ ,
- (2) for any  $x \in X$  there exists  $h \in U$  such that  $\Phi(x) = h(x)$  (in particular, if  $U$  is finite)

then for any  $g \in \gamma(x^*, A)$  ( $g \in \Gamma(x^*, A)$ ) there exists  $h \in U$  such that  $h(g) + \Psi(g) \geq 0$ .

**COROLLARY 5.9** *Suppose that all assumption of the previous theorem are satisfied,  $x^* \in \text{int } A$ , and let  $0 \in H$ . Then for any  $(\Phi, \Psi) \in \mathcal{D}_H f_0(x^*)$  and for all  $p \in \text{supp}^+(\Psi, H)$  one has  $0 \in \underline{\partial}_H(\Phi + p)(0)$ .*

*Remark 3* Applying Theorems 5.5 and 5.8 for different particular sets  $H$  one can easily obtain well-known necessary optimality conditions for codifferentiable and quasidifferentiable functions, and for functions having upper (lower) exhauster or upper (lower) coexhauster.[7,23,24,33]

### 5.3. Some particular cases

Let us consider how general necessary optimality conditions for  $H$ -codifferentiable functions can be easily transformed into more convenient conditions in some particular cases. In this subsection,  $X$  is a real Banach space,  $E = \mathbb{R}$ ,  $A \subset \Omega$  is a nonvoid closed convex set. Note, that if the set  $H$  is closed under vertical shifts then, without loss of generality, we may assume that for any weakly  $H$ -codifferentiable function  $f$  and for all  $(\Phi, \Psi) \in \delta f_H$  one has  $\Phi(0) = \Psi(0) = 0$ .

Let  $f, f_i: X \rightarrow \mathbb{R}$  be arbitrary functions,  $i \in I_0 = \{0\} \cup I$ , where  $I = \{1, \dots, n\}$ . For any  $x \in X$  denote  $R(x) = \{0\} \cup \{i \in I \mid f_i(x) = 0\}$ .

*Example 5.10* Let  $H$  coincide with the set of all continuous affine function  $h: X \rightarrow \mathbb{R}$ . Then, as it was shown in Example 3.10, the function  $f$  is weakly  $H$ -codifferentiable at a point  $x \in \Omega$  iff  $f$  is codifferentiable at this point.

Let us derive necessary optimality conditions for a codifferentiable function in the problem with smooth equality and codifferentiable inequality constraints.

**PROPOSITION 5.11** *Let  $Y$  be a Banach space, a mapping  $F: X \rightarrow Y$  be continuously Fréchet differentiable at a point  $x^* \in X$ , the functions  $f_i$  be Fréchet codifferentiable at a point  $x^*$ ,  $i \in I_0$ . Suppose that the Fréchet derivative  $F'[x^*]$  of the map  $F$  at  $x^*$  is surjective, and  $x^*$  is a point of local minimum in the problem*

$$f_0(x) \rightarrow \inf, \quad F(x) = 0, \quad f_i(x) \leq 0, \quad i \in I.$$

Then for any  $(0, q_i) \in \bar{d}f_i(x^*)$ ,  $i \in R(x^*)$  there exists  $y^* \in Y^*$  such that

$$(0, y^* \circ F'[x^*]) \in \left( \text{co} \bigcup_{i \in R(x^*)} (\underline{d}f_i(x^*) + \{(0, q_i)\}) \right).$$

*Proof* Fix an arbitrary  $(0, q_i) \in \bar{d}f_i(x^*)$ ,  $i \in I_0$  and define

$$\varphi_i(x) = \max_{(a,p) \in \underline{d}f_i(x^*) + (0,q_i)} (a + p(x)) \quad \forall x \in X, \forall i \in I_0.$$

It is easy to check that the function  $\varphi_i$  is a strong upper  $H$ -convex approximation of  $f_i$  at  $x^*$ ,  $\varphi_i(0) = 0$  and  $\varphi_i$  is Lipschitz continuous in a neighbourhood of zero ([34], Corollary 2.2.12),  $i \in I_0$ .



Denote  $M = \{x \in X \mid F(x) = 0\}$ . By virtue of Theorem 5.4 one has that 0 is a point of global minimum of the convex function

$$g(\cdot) = \max\{\varphi_0(\cdot), \varphi_1(\cdot) + f_1(x^*), \dots, \varphi_n(\cdot) + f_n(x^*)\}$$

on the set  $T_M(x^*)$ . Taking into account the Lusternik theorem (see [32], Section 0.2) one has that  $\text{Ker } F'[x^*] \subset T_M(x^*)$ , where  $\text{Ker } F'[x^*]$  is the kernel of the linear operator  $F'[x^*]$ . Therefore, applying the necessary and sufficient condition for a minimum of a convex function on a closed convex set ([32], Theorem 1.1.2') and the theorem about the subdifferential of the maximum of a finite family of convex functions ([34], Corollary 2.8.11), one gets

$$\partial g(0) \cap (-N(0, \text{Ker } F'[x^*])) \neq \emptyset, \quad \partial g(x^*) = \text{co} \bigcup_{i \in R(x^*)} \partial g_i(0),$$

where  $\partial g(0)$  is the subdifferential of the convex function  $g$  at 0 and  $N(0, \text{Ker } F'[x^*]) = \{p \in X^* \mid p(x) \leq 0 \forall x \in \text{Ker } F'[x^*]\}$  is the normal cone to the set  $\text{Ker } F'[x^*]$  at the point 0. By virtue of the theorem about the subdifferential of the supremum ([32], Theorem 4.2.3) one has  $\{0\} \times \partial g_i(0) \subset \underline{d}f_i(x^*) + \{(0, q_i)\}$ . Hence

$$\left( \text{co} \bigcup_{i \in R(x^*)} (\underline{d}f_i(x^*) + \{(0, q_i)\}) \right) \cap (\{0\} \times (-N(0, \text{Ker } F'[x^*]))) \neq \emptyset$$

It remains to note that  $N(0, \text{Ker } F'[x^*])$ , as the annihilator of the subspace  $\text{Ker } F'(x^*)$ , coincides with the image of the adjoint operator of  $F'[x^*]$  ([35], Theorem 6.5.10), i.e. for any  $p \in N(0, \text{Ker } F'[x^*])$  there exists  $y^* \in Y^*$  such that  $p = y^* \circ F'[x^*]$ .  $\square$

*Example 5.12* Let  $H$  consist of all proper l.s.c. convex functions  $h: X \rightarrow \overline{\mathbb{R}}$  such that  $0 \in \text{int dom } h$ . Let us recall that in this case if there exists an upper coexhauster of the function  $f$  at the point  $x \in \Omega$ , then the function  $f$  is weakly  $H$ -hyperdifferentiable at this point (cf. Example 3.13).

Arguing in a similar way to the proof of Proposition 5.11 one can get the following result.

**PROPOSITION 5.13** *Let  $Y$  be a Banach space, a mapping  $F: X \rightarrow Y$  be continuously Fréchet differentiable at a point  $x^* \in X$ . Suppose that there exist Fréchet upper coexhausters  $\overline{E}_i(x^*)$  of the functions  $f_i$  at a point  $x^*$ ,  $i \in I_0$ . Suppose also that the operator  $F'[x^*]$  is surjective, and  $x^*$  is a point of local minimum in the problem*

$$f_0(x) \rightarrow \inf, \quad F(x) = 0, \quad f_i(x) \leq 0, \quad i \in I.$$

*Then for any  $C_i \in \overline{E}_i(x^*)$ ,  $i \in R(x^*)$  there exists  $y^* \in Y^*$  such that*

$$(0, y^* \circ F'[x^*]) \in \left( \text{co} \bigcup_{i \in R(x^*)} C_i \right).$$

*where  $\overline{e}_i(x^*) = \{C \in \overline{E}_i(x^*) \mid \max_{(a,p) \in C} a = 0\}$ .*

Let us obtain necessary conditions for a maximum in terms of a lower coexhauster.

**PROPOSITION 5.14** *Suppose that there exists an upper coexhauster  $\overline{E}(x^*)$  of the function  $f$  at a point  $x^* \in A$ , and let  $x^*$  be a point of local maximum of the function  $f$  on the set  $A$ . Then for any  $\varepsilon > 0$  and  $g \in \gamma(x^*, A)$  there exists  $C \in \overline{E}(x^*)$  such that  $p(g) \leq \varepsilon$  for all  $(a_C, p) \in C$ , where  $a_C = \max_{(a,p) \in C} a$ . Moreover, if the family  $\overline{E}(x^*)$  is finite, then for any  $g \in \Gamma(x^*, A)$  there exists  $C \in \overline{E}(x^*)$  such that  $p(g) \leq 0$  for all  $(a_C, p) \in C$ .*

*Proof* By virtue of Theorem 5.7 one has that for any  $\varepsilon > 0$  and  $g \in \gamma(x^*, A)$  there exists  $C \in \overline{E}(x^*)$  such that  $h'(0, g) \leq \varepsilon$ , where  $h(\cdot) = \max_{(a,p) \in C} (a + p(\cdot))$ . It remains to note that  $h'(0, g) = \max_{p \in \underline{\partial}h(0)} p(g)$ , where  $\underline{\partial}h(0) = \{p \in X^* \mid (a_C, p) \in C\}$  (see, e.g. [32], Chapter 4).  $\square$

*Remark 4* One could also consider the case when the set  $H$  consists of all proper u.s.c. concave functions  $h: X \rightarrow \overline{\mathbb{R}}$  such that  $0 \in \text{int dom } h$ .

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