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# Nonsmooth and discontinuous speed-gradient algorithms☆

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In this article, nonsmooth extensions of the Speed-Gradient (SG) algorithms in differential and finite forms are proposed. The conditions ensuring achievement of the control goal (convergence of the goal function to zero) are established. Furthermore, conditions under which the control goal is achieved in finite time with the use of nonsmooth or discontinuous SG algorithms are obtained. Theoretical results are illustrated by example of nonsmooth energy-based control for a non-affine in control pendulum-like system.



## 1. Introduction

A number of approaches to stabilization of nonlinear systems are based on the introduction of an appropriate goal/objective functional and the design of a control algorithm providing its asymptotic optimization. Perhaps, most popular ones are based on the design of a controller that minimizes integral performance index. In many cases these approaches lead to a stable closed loop system [1–3]. Alternatively, in [4] it was proposed to minimize the rate of change of a function of the state $V(x)$ along trajectories of the controlled system at each time instant $t$. However, such local minimization is not necessary to achieve stability. If the function $V(x)$ serves as a Lyapunov function for the closed loop system, stability is ensured, provided $V(x(t))$ decreases for all $t$. Such class of algorithms have been studied since late 1970s under the name of Speed-Gradient algorithms [5,6] or (in the affine case) LgV or Jurdjevic–Quinn algorithms [7]. It was shown [8] that a variety of adaptation and control algorithms can be obtained with a proper choice of the controlled system or the goal function. A fundamental result was obtained by E.Sontag [9] who has demonstrated that under special choice of a scalar gain the closed loop becomes globally stable ('universal Sontag's construction').

Stability analysis for an overwhelming majority of the existing algorithms was performed under the assumption of smoothness of the goal function and continuity of the control system right hand sides. In such a case the function $\dot{V}$ is continuous. However, a relaxation of the smoothness assumption provides hopes for better performance of the closed loop system. Therefore it is important to develop a systematic theory allowing one to design nonlinear controllers and to prove stability in nonsmooth situations. Some special cases of relay algorithms were considered in [8]; they provided a new view of the variable structure systems (VSS). However, a comprehensive theory based on the well developed apparatus of nonsmooth analysis [10–14] did not exist until recently.

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The first attempt to study nonsmooth counterparts of SG-controllers was made in [15], where proofs were just outlined. In this paper, we provide a more detailed exposition and some extension of the results of [15]. Besides, a new interesting example is considered. Let us also mention the paper [16], in which a different version of a nonsmooth SG-algorithm was considered and utilized to design an almost global stabilizer of the Brockett integrator that is continuous along solutions of the closed-loop system.

As a basis for formal approach to nonsmooth problems we chose the concept of Hadamard directional differentiability [11]. It allowed us to introduce and analyse the Speed-Subgradient algorithms in differential and finite forms.

An important advantage of nonsmooth optimization and control algorithms is a potential for finite-time convergence: the control goal can be achieved in finite time. It is widely used in variable structure systems (VSS), in switching systems, etc., see [17–21] and references therein. In this paper, we show that under some additional assumptions nonsmooth and discontinuous SG-algorithms in finite form provide finite-time convergence to the goal set.

It is worth noting that in the smooth case the Speed-Gradient algorithms are defined via the gradient of the speed of change (i.e. Lie derivative) of a chosen goal function. In the nonsmooth case, one has to replace the gradient and the derivative with their nonsmooth counterparts, i.e. with a subdifferential and a generalized derivative. Being inspired by the ideas of [11], we utilized Hadamard's directional derivative and the subdifferential (in the sense of convex analysis) of an upper convex approximation of this derivative. Let us note that one can use different tools from nonsmooth and variational analysis, such as the subderivative and the proximal subdifferential [13], in order to construct a different extension of the Speed-Gradient algorithms to a nonsmooth setting.

The structure of the paper is as follows. In Section 2 necessary notions and methods from nonsmooth analysis are outlined. In Section 3 the problem statement is given and two key results concerning nonsmooth SG-algorithms in differential and finite forms are presented. Two examples are described in Section 4.

## 2. Preliminaries

In this section, we recall some notions from nonsmooth analysis [11] and set-valued analysis [22] that are used throughout the article. Denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$, and denote $\mathbb{R}_+ = [0, +\infty)$.

Let a real-valued function $f$ be defined in a neighbourhood of a point $x \in \mathbb{R}^n$. The function $f$ is called *Hadamard directionally differentiable* at the point $x$ if for any $v \in \mathbb{R}^n$ there exists the finite limit

$$f'(x; v) = \lim_{[\alpha, v'] \to [+0, v]} \frac{f(x + \alpha v') - f(x)}{\alpha}$$

(the motivation behind the notation under lim was discussed in [23]). The function $f'(x; \cdot)$ is called the *Hadamard directional derivative* of $f$ at $x$. Note that there exists the elaborate calculus of Hadamard directional derivatives [11]. Observe also that if $n = 1$, then the quantity $f'(x, 1)$ coincides with the right-hand side derivative of $f$ at $x$ that is denoted by $D_+ f(x)$.

It is easy to see that the function $v \to f'(x; v)$ is continuous and positively homogeneous (of degree one), i.e. for any $v \in \mathbb{R}^n$ and $\lambda \geq 0$ one has $f'(x; \lambda v) = \lambda f'(x; v)$. A convex positively homogeneous function $p : \mathbb{R}^n \to \mathbb{R}$ such that $p(v) \geq f'(x; v)$ for all $v \in \mathbb{R}^n$ is called an *upper convex approximation* of the function $f'(x; \cdot)$.

Let $C \subset \mathbb{R}^n$ be an open set. Recall that a function $F$ that maps points from $C$ to possibly empty subset of $\mathbb{R}^m$ is called a *set-valued mapping* (or *multifunction*) from $C$ to $\mathbb{R}^m$. The set-valued mapping $F$ is called *outer semicontinuous* at a point $x_0 \in C$ if for any open set $V \subset \mathbb{R}^m$ with $F(x_0) \subset V$ there exists $\delta > 0$ such that for all $x \in C$ with $|x - x_0| < \delta$ one has $F(x) \subset V$. The set-valued mapping $F$ is called *measurable* if for any open set $V \subset \mathbb{R}^n$ the set $\{x \in C : F(x) \cap V \neq \emptyset\}$ is measurable. One can show that any outer semicontinuous set-valued mapping is measurable.

An important example of an outer semicontinuous (and thus measurable) set-valued mapping is the subdifferential mapping of a convex function. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. It should be noted that throughout this article we consider only finite-valued convex functions.

Recall that the set

$$\partial f(x) = \left\{ v \in \mathbb{R}^n : f(y) - f(x) \geq v^T (y - x) \quad \forall y \in \mathbb{R}^n \right\}$$

is referred to as the *subdifferential* of $f$ at a point $x$. One can verify that the set $\partial f(x)$ is nonempty, convex and compact. As it was mentioned above, the subdifferential mapping $x \to \partial f(x)$ is outer semicontinuous on $\mathbb{R}^n$.

## 3. Nonsmooth speed-gradient: Two algorithms

### 3.1. Problem formulation

Consider the controlled system

$$\dot{x} = F(x, u, t), \quad t \geq 0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of the system state, and $u \in \mathbb{R}^m$ is the control. We assume that the function $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^n$ satisfies the Carathéodory condition, i.e. the mapping $(u, x) \to F(x, u, t)$ is continuous for almost all $t \geq 0$, and the mapping $t \to F(x, u, t)$ is measurable for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Unless otherwise stated, a solution of (1), even in the case of a discontinuous control law, is understood to be a locally absolutely continuous function satisfying (1) for almost all $t$ in its domain.

We pose the general control problem as finding a control law $u(\cdot)$, which ensures the control objective

$$Q_t \le \Delta \quad \text{when} \quad t \ge t_*,$$

where $Q_t = Q(x(t), t)$, $Q(x, t)$ is a nonnegative *goal function* defined on $\mathbb{R}^n \times \mathbb{R}_+$, $\Delta \ge 0$ is some pre-specified threshold, and $t^*$ is the time instant at which the control objective is achieved. The objective can also be formulated as

$$\limsup_{t\to\infty} Q_t \le \Delta,$$

which does not specify the value of $t^*$. In the special case $\Delta = 0$ the control objective takes the form

$$\lim_{t\to\infty} Q(x(t), t) = 0, \tag{2}$$

i.e. the objective is to stabilize the system (1) with respect to the goal function $Q$.

The formulation of the control problem that we use encompasses various control problems, such as partial stabilization, control of system energy, identification and adaptive control (see discussion, as well as various examples and applications, in [24]). In particular, if one takes a control Lyapunov function $V(x)$ of the system (1) as the goal function $Q(x, t)$, then the goal (2) is closely related to asymptotic stability of (1). However, we underline that possible goal functions $Q(x, t)$ are neither exhausted by nor reduced to control Lyapunov functions (see Section 4 below and examples in [24]).

### *3.2. Nonsmooth speed-gradient algorithms*

In order to design a control algorithm suppose that the function $Q$ is locally Lipschitz continuous and Hadamard directionally differentiable. Choose a convex in $u$ function $\omega(x, u, t)$ defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+$, and such that

$$Q'(x, t; F(x, u, t), 1) \le \omega(x, u, t) \quad \forall (x, u, t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+. \tag{3}$$

In particular, if $F$ is affine in $u$, and $p(x, t; \cdot)$ is an upper convex approximation of the function $Q'(x, t; \cdot)$, then one can define

$$\omega(x, u, t) = p(x, t; F(x, u, t), 1).$$

**Remark 1.** Let us explain the motivation behind the definition of $\omega(x, u, t)$. In the smooth case (see [24]), one defines

$$\omega(x, u, t) = \frac{d}{dt} Q(x, t) = \frac{\partial Q}{\partial x}(x, t)^T F(x, u, t) + \frac{\partial Q}{\partial t}(x, t), \tag{4}$$

and utilizes this function in order to design the Speed-Gradient algorithms. It should be noted that in order to prove the convergence of the Speed-Gradient algorithms one must assume that the function (4) is convex in $u$. In order to extend this idea to the nonsmooth case, suppose that $x(t)$ is a solution of (1). Then applying the chain rule for directional derivatives (see, e.g., [11], Theorem I.3.3) one obtains that

$$\frac{d}{dt} Q(x(t), t) = Q'(x(t), t; \dot{x}(t), 1),$$

where $Q'(x(t), t; \dot{x}(t), t)$ is the directional derivative of $Q$ at the point $(x(t), t)$ at the direction $(\dot{x}(t), 1)$. Replacing $\dot{x}(t)$ by the right-hand side of (1) one gets the same expression as in (3). However, in order to retain the convexity of $\omega(x, u, t)$ in $u$ (that is crucial for convergence analysis) in many examples one must replace the equality sign in (4) by the inequality sign.

Take the control algorithm in the form of differential inclusion [25,26]

$$\dot{u} \in -\Gamma \partial_u \omega(x, u, t), \tag{5}$$

where $\Gamma$ is a symmetric positive definite matrix, and $\partial_u \omega(x, u, t)$ is the subdifferential of the function $u \to \omega(x, u, t)$ at the point $(x, u, t)$. The algorithm (5) is a natural generalization of the Speed-Gradient algorithm in differential form [24] to the nonsmooth case. We shall call it the *Speed-Subgradient algorithm*.

Together with the Speed-Subgradient algorithm in differential form (5), let us consider an algorithm in the finite form

$$u \in u_0 - \Gamma \partial_u \omega(x, u, t), \tag{6}$$

where $\Gamma$ is a positive definite gain matrix and $u_0$ is an initial value of the control variable. We will also consider a more general control algorithm

$$u = u_0 + \gamma \psi(x, u, t) \tag{7}$$

where $\gamma > 0$ is a scalar gain and the vector function $\psi$ satisfies the "acute angle" condition: for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $t \ge 0$ there exists $v \in \partial_u \omega(x, u, t)$ such that

$$v^T \psi(x, u, t) \le 0. \tag{8}$$

The algorithm of the form (7) is a generalization of the so-called Speed-Pseudogradient algorithms [24]. Therefore it is natural to call it the *Speed-Pseudosubgradient algorithm*.

It should be noted that (7) is an *equation* with respect to the control variable $u$; in other words, (7) defines the control law $u = u(x, t, \gamma)$ *implicitly*. Therefore, in order to implement the algorithm of the form (7) one should be able to solve this equation, i.e. one should be able either to obtain an explicit expression for $u(x, t, \gamma)$ or to efficiently solve this equation numerically. It should be noted that in many applications either the function $\psi$ does not depend on $u$ (see Section 4) or a solution of (7) can be easily found analytically.

Observe that from the definition of subdifferential it follows that the generalized equation (6) can be rewritten as follows:

$$\omega(x, v, t) - \omega(x, u, t) \geq (-\Gamma^{-1}u + \Gamma^{-1}u_0)^T(v - u) \quad \forall v \in \mathbb{R}^m$$

or, equivalently,

$$(Mu + q)^T(v - u) + \omega(x, v, t) - \omega(x, u, t) \geq 0 \quad \forall v \in \mathbb{R}^m \tag{9}$$

where $M = \Gamma^{-1}$ and $q = -\Gamma^{-1}u_0$. Note that (9) is a linear variational inequality with respect to $u$. Thus, one can apply known results on the existence of solutions of linear variational inequalities in order to prove that controller (6) is correctly defined. In particular, it is easy to check that the variational inequality (9) satisfies all assumptions of part (a) of Corollary 3 in [27], which implies that for any $x \in \mathbb{R}^n$, $u_0 \in \mathbb{R}^m$ and $t \geq 0$ there exists at least one $u$ satisfying (6).

**Remark 2.** Let us underline that the nonsmooth Speed-Gradient algorithms are defined via the classical subdifferential mapping from convex analysis of an upper estimate $\omega(x, u, t)$ of the speed of change of the goal function $Q(x, t)$. The convexity of $\omega(x, u, t)$ in $u$ (or, equivalently, the monotonicity of the subdifferential $\partial_u\omega(x, u, t)$) will play a crucial role in the proofs of stability results presented below.

**Remark 3.** The Speed-Gradient algorithms are intimately related to passivity of the closed loop system [28]. The relation was discovered in [29,30]. Therefore the extension of SG-algorithms to nonsmooth case is close to the framework for multivalued Lurie systems developed in [31–35] for linear, Lipschitz continuous or monotone in some sense nominal systems possessing passivity-like properties. However, the framework adopted in this paper provides stability-like results for a more general case when the system is nonlinear and *non-affine* in control (see Example 3). Furthermore, in [31–33,35] only the regulation and tracking problems are considered, while we study more general class of control problems including partial stabilization, energy control, identification, etc. In particular, in Example 2, we solve the problem of swinging up a pendulum as the energy control problem.

### *3.3. Properties of the nonsmooth speed-gradient algorithms*

Let us discuss the performance of the control system with the proposed control algorithms. At first, we study the Speed-Subgradient algorithm in differential form. The theorem below is a generalization of the corresponding result in the smooth case (see [24]).

**Theorem 1.** *Let the following assumptions hold true:*

1. *the function $F$ satisfies the Carathéodory condition, and for any $r > 0$ there exists a Lebesgue integrable function $m_r : \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality*
$$|F(x, u, t)| \leq m_r(t) \quad \forall t \geq 0$$
*holds true if $|x| \leq r$ and $|u| \leq r$;*
2. *the set-valued mapping $(x, u) \to \partial_u\omega(x, u, t)$ is outer semicontinuous for a.e. $t \geq 0$, the set-valued mapping $t \to \partial_u\omega(x, u, t)$ is measurable for any $x$ and $u$, and for any $r > 0$ there exists a locally integrable function $s_r : \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality*
$$|v| \leq s_r(t) \quad \forall v \in \partial_u\omega(x, u, t) \quad \forall t \geq 0$$
*holds true if $|x| \leq r$ and $|u| \leq r$;*
3. *the function $Q(x, t)$ is nonnegative, uniformly continuous on any set of the form $\{(x, t) : |x| \leq r, t \geq 0\}$ and radially unbounded, i.e.*
$$\inf_{t \geq 0} Q(x, t) \to +\infty \text{ as } |x| \to \infty;$$
4. *there exists $u^* \in \mathbb{R}^m$ and a continuous scalar function $\rho : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho(x, Q(x, t)) = 0$ if and only if $Q(x, t) = 0$, and the inequality*
$$\omega(x, u^*, t) \leq -\rho(x, Q(x, t))$$
*holds for any $t \geq 0$ and $x \in \mathbb{R}^n$.*

*Then for any* $x(0) \in \mathbb{R}^n$ *and* $u(0) \in \mathbb{R}^m$ *all solutions* $(x(t), u(t))$ *of* (1), (5) *are defined and bounded on* $\mathbb{R}_+$, *and*

$$\lim_{t\to\infty} Q(x(t), t) = 0. \tag{10}$$

**Proof.** Note that assumptions 1 and 2 ensure the existence of a solution $(x(t), u(t))$ of the system (1), (5) with arbitrary initial data $(x(0), u(0))$ at least on some finite time interval $[0, t_0)$ (see [25], Theorem 2.7.5).

Introduce the Lyapunov function

$$V(x, u, t) = Q(x, t) + (u - u^*)^T \Gamma^{-1}(u - u^*)/2, \tag{11}$$

and denote $V_0(t) = V(x(t), u(t), t)$. The function $Q$ is Hadamard directionally differentiable, and the functions $x(t), u(t)$ are differentiable for a.e. $t \in [0, t_0)$, as solutions of a differential inclusion. Therefore by the chain rule for directional derivatives ([11], Theorem I.3.3) for a.e. $t \in [0, t_0)$ there exists the right-hand side derivative $D_+V_0(t)$ of the function $V_0$ that has the form

$$D_+V_0(t) = V_0'(t; 1) = Q'(x(t), t; \dot{x}(t), 1) + (u(t) - u^*)^T \Gamma^{-1}\dot{u}(t).$$

Taking into account (1) one gets that for a.e. $t \in [0, t_0)$

$$D_+V_0(t) = Q'(x(t), t; F(x(t), u(t), t), 1) - (u(t) - u^*)^T v(t),$$

where $v(t) = -\Gamma^{-1}\dot{u}(t)$. By the definition of $u(t)$ (see (5)) one has $v(t) \in \partial_u\omega(x(t), u(t), t)$ for a.e. $t \in [0, t_0)$. Hence applying the convexity of $\omega$ in $u$ and assumption 4 one obtains that for a.e. $t \in [0, t_0)$

$$\begin{aligned} D_+V_0(t) &\le \omega(x(t), u(t), t) - (u(t) - u^*)^T v(t) \\ &\le \omega(x(t), u^*, t) \le -\rho(x(t), Q(x(t), t)) \le 0. \end{aligned} \tag{12}$$

Note that $V_0(t)$ is absolutely continuous as the sum of the absolutely continuous function $(u(t) - u^*)^T \Gamma^{-1}(u(t) - u^*)/2$ and the composition of the locally Lipschitz continuous function $Q(\cdot)$ and the absolutely continuous mapping $(x(t), t)$. Hence and from (12) one gets that the function $V_0(t) = V(x(t), u(t), t)$ is nonincreasing. It implies the boundedness of $V(x(t), u(t), t)$, and $Q(x(t), t)$ that, in turn, means the boundedness of $x(t)$ (due to the radial unboundedness of $Q(x, t)$) and $u(t)$ (see (11)). Consequently, all solutions of (1)–(5) exist and bounded on $\mathbb{R}_+$ (see Theorem 2.7.6 in [25]). Thus, it remains to show that (10) holds true.

Let $(x(t), u(t))$ be an arbitrary solution of (1), (5) starting at $(x(0), u(0))$. Recall that $V_0(t)$ is absolutely continuous, while $\rho(x(t), Q(x(t), t))$ is continuous as the composition of continuous functions. Therefore from (12) it follows that

$$\int_0^t \rho(x(\tau), Q(x(\tau), \tau))\, d\tau \le V_0(0) - V_0(t) \quad \forall t \in \mathbb{R}_+.$$

Consequently, applying the fact that $V_0$ is nonincreasing one obtains that

$$\int_0^\infty \rho(x(t), Q(x(t), t))\, dt < +\infty. \tag{13}$$

Note that by definition one has $x(t) - x(s) = \int_s^t F(x(\tau), u(\tau), \tau)\, d\tau$ for any $s, t \in \mathbb{R}_+$. Hence taking into account the boundedness of $x(t)$ and $u(t)$ on $\mathbb{R}_+$, and assumption 1 one gets that there exists a Lebesgue integrable function $m_r$ such that

$$|x(t) - x(s)| \le \int_s^t |m_r(\tau)|\, d\tau \quad \forall t, s \in \mathbb{R}_+.$$

Applying the absolute continuity of the Lebesgue integral one gets that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t, s \in \mathbb{R}_+$ with $|t - s| < \delta$ the following inequalities hold true

$$|x(t) - x(s)| \le \int_s^t |m_r(\tau)|\, d\tau < \varepsilon.$$

Thus, $x(t)$ is uniformly continuous and bounded on $\mathbb{R}_+$. Consequently, taking into account assumptions 3 and 4 one obtains that the function $\rho(x(t), Q(x(t), t))$ is uniformly continuous on $\mathbb{R}_+$ as well. Hence with the use of the Barbalat lemma (see, e.g., [24], Lemma 2.2) and (13) one gets that $\rho(x(t), Q(x(t), t)) \to 0$ as $t \to \infty$.

Let us verify that $Q(x(t), t) \to 0$ as $t \to \infty$. Indeed, arguing by reductio ad absurdum suppose that there exists $\varepsilon > 0$ and an increasing unbounded sequence $\{t_k\} \subset \mathbb{R}_+$ such that $Q(x(t_k), t_k) \ge \varepsilon$ for all $k \in \mathbb{N}$. Since $x(t)$ is bounded on $\mathbb{R}_+$, then applying assumption 3 one gets that there exists $r > 0$ and $c > 0$ such that

$$|x(t)| \le r, \quad Q(x(t), t) \le c \quad \forall t \in \mathbb{R}_+.$$

Define

$$\rho_0 = \inf\big\{\rho(x, s) \mid |x| \le r,\ s \in [\varepsilon, c]\big\}.$$

Assumption 4 implies that $\rho_0 > 0$. Furthermore, by the definition of the sequence $\{t_k\}$ one has

$$\rho\big(x(t_k), Q(x(t_k), t_k)\big) \geq \rho_0 \quad \forall k \in \mathbb{N},$$

which contradicts the fact that $\rho(x(t), Q(x(t), t)) \to 0$ as $t \to \infty$. Thus, $Q(x(t), t) \to 0$ as $t \to \infty$, and the proof is complete.

**Remark 4.** In some problems, the set-valued mapping $(x, u) \to \partial_u\omega(x, u, t)$ might not be outer semicontinuous, which makes the theorem above inapplicable. In this case, one can consider a slightly relaxed version of the Speed-Subgradient algorithm in differential form. Namely, let $\Phi(x, u, t)$ be a compact convex valued multifunction such that $\Phi(x, \cdot, t) = \partial_u\omega(x, \cdot, t)$ for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$ with $Q(x, t) > 0$. Then define a control law as follows

$$\dot{u} \in -\Gamma\Phi(x, u, t). \tag{14}$$

Suppose that all assumptions of Theorem 1 are satisfied with $\partial_u\omega(x, u, t)$ being replaced by $\Phi(x, u, t)$. Then one can verify that for all $x(0) \in \mathbb{R}^n$ and $u(0) \in \mathbb{R}^m$, and for any solution $(x(t), u(t))$ of (1), (14) either $(x(t), u(t))$ is defined and bounded on $\mathbb{R}_+$, and $\lim_{t\to\infty} Q(x(t), t) = 0$ or the control goal is achieved in finite time, i.e. there exists $T > 0$ such that $(x(t), u(t))$ is defined on $[0, T]$ and $Q(x(T), T) = 0$.

Indeed, the validity of assumptions 1 and 2 ensures the existence of a solution $(x(t), u(t))$ of (1), (14) with arbitrary initial data $(x(0), u(0))$ that is defined on a maximal interval of existence $[0, T_{\max})$ (see, e.g., [25], Theorem 2.7.6). If $Q(x(t), t) > 0$ for all $t \in [0, T_{\max})$, then taking into account the fact that $\Phi(x(t), u(t), t) = \partial_u\omega(x(t), u(t), t)$ for all $t \in [0, T_{\max})$ and repeating the proof of the theorem above one can verify that $T_{\max} = +\infty$, the solution $(x(t), u(t))$ is bounded and $Q(x(t), t) \to 0$ as $t \to +\infty$. On the other hand, if there exists $T \in [0, T_{\max})$ such that $Q(x(T), T) = 0$, then it exactly means that the control goal is achieved in finite time.

Note that the achievement of the control goal in finite time does not necessarily mean that $Q(x(t), t) = 0$ for all $t \geq T$ or $\lim_{t\to\infty} Q(x(t), t) = 0$. However, under some additional assumptions on the function $\Phi(x, u, t)$, one can guarantee that in the case of finite time convergence the relaxed control goal $\liminf_{t\to\infty} Q(x(t), t) = 0$ is achieved.

Let us consider now the Speed-Pseudosubgradient algorithm. The theorem below is a simple generalization of Theorem 3.3 from [24].

**Theorem 2.** *Let the following assumptions be valid:*

1. *for any $\gamma > 0$, $u_0 \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $t \geq 0$ there exists a solution $u = \kappa(x, u_0, t, \gamma)$ of* Eq. (7)*, and the function $\kappa$ is locally bounded in $x$ uniformly in $t$;*
2. *a solution of the system (1), (7) exists for all $t \geq 0$, $x(0) \in \mathbb{R}^n$ and $u_0 \in \mathbb{R}^m$;*
3. *the function $Q(x, t)$ is nonnegative and radially unbounded;*
4. *there exist a locally bounded uniformly in $t$ function $u_* : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^m$ and a continuous scalar function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho(s) = 0$ if and only if $s = 0$, and*

$$\omega(x, u_*(x, t), t) \leq -\rho(Q(x, t)) \quad \forall x \in \mathbb{R}^n, t \in \mathbb{R}_+; \tag{15}$$

5. *there exists $\beta > 0$ such that for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $t \geq 0$ the inequality*

$$v^T\psi(x, u, t) \leq -\beta|v| \tag{16}$$

*holds true for some $v \in \partial_u\omega(x, u, t)$.*

*Then for any $x(0) \in \mathbb{R}^n$ and $u_0 \in \mathbb{R}^m$ there exists $\overline{\gamma} > 0$ such that any solution $(x(t), u(t))$ of (1), (7) is bounded and*

$$\lim_{t\to\infty} Q(x(t), t) = 0 \tag{17}$$

*for all $\gamma > \overline{\gamma}$. Moreover, for all $\gamma > \overline{\gamma}$ and $\Delta > 0$ one has*

$$Q(x(t), t) < \Delta \quad \forall t > Q(x(0), 0)/\rho(\Delta). \tag{18}$$

**Proof.** Define the Lyapunov function

$$V(x, t) = Q(x, t)$$

and denote $V_0(t) = V(x(t), t)$, where $(x(\cdot), u(\cdot))$ is a solution of the system (1), (7). Then for a.e. $t \in \mathbb{R}_+$ there exists the right-hand side derivative of the function $V_0$ that has the form

$$D_+V_0(t) = Q'(x(t), t; \dot{x}(t), 1) = Q'(x(t), t; F(x(t), u, t), 1) \leq \omega(x(t), u, t).$$

Applying the convexity of $\omega$ in $u$, and assumptions 4 and 5 one gets that

$$\begin{aligned} D_+V_0(t) &\leq \omega(x(t), u_*(x(t), t), t) + \big[u_0 - u_*(x(t), t) + \gamma\psi(x(t), u, t)\big]^T v \\ &\leq -\rho(Q(x(t), t)) + \big[|u_0 - u_*(x(t), t)| - \gamma\beta\big]|v| \end{aligned} \tag{19}$$

for any $v \in \partial_u \omega(x(t), u, t)$ such that (16) holds true. Note that assumption 5 guarantees that there exists at least one such $v$.

Denote

$$\Omega_0 = \left\{(x, t) \in \mathbb{R}^n \times \mathbb{R}_+ \mid Q(x, t) < V_0(0) + 1\right\}.$$

Taking into account the radial unboundedness of $Q$ and the local boundedness uniformly in $t$ of $u_*(x, t)$ one obtains that

$$d = \sup_{(x,t)\in\Omega_0} |u_0 - u_*(x, t)| < +\infty.$$

Define $\overline{\gamma} = d/\beta$. Let us show that for any $\gamma > \overline{\gamma}$ one has $(x(t), t) \in \Omega_0$ for all $t \in \mathbb{R}_+$. Indeed, let $\gamma > d/\beta$, and suppose that there exists $t_0 > 0$ such that $(x(t_0), t_0) \notin \Omega_0$ or, equivalently, $V_0(t_0) = Q(x(t_0), t_0) \geq V_0(0) + 1$. Define

$$\tau = \inf\left\{t > 0 \mid V_0(t) \geq V_0(0) + 1\right\}.$$

Observe that $\tau > 0$, since $V_0$ is continuous. Therefore $(x(t), t) \in \Omega_0$ for any $t \in [0, \tau)$, and taking into account (19) one gets that

$$D_+V_0(t) \leq -\rho(Q(x(t), t)) \leq 0 \quad \forall t \in [0, \tau).$$

Hence the function $V_0$ is nonincreasing on $[0, \tau)$ and $V_0(\tau) \leq V_0(0)$, which contradicts the definition of $\tau$.

Thus, for any $\gamma > \overline{\gamma}$ one has $(x(t), t) \in \Omega_0$ for all $t \in \mathbb{R}_+$. Therefore from (19) it follows that

$$D_+V_0(t) \leq -\rho(Q(x(t), t)) \leq 0 \quad \forall t \in \mathbb{R}_+ \quad \forall \gamma > \overline{\gamma}. \tag{20}$$

Consequently, the function $V_0(t) = Q(x(t), t)$ is nonincreasing. Hence the solution $(x(\cdot), u(\cdot))$ is bounded due to the radial unboundedness of $Q$ and local boundedness in $x$ uniformly in $t$ of $u = \kappa(x, u_0, t, \gamma)$.

Choose an arbitrary $\Delta > 0$, and define $T_\Delta = \{t \geq 0 : Q(x(t), t) \geq \Delta\}$. Note that since $V_0(t) = Q(x(t), t)$ is nonincreasing, then $T_\Delta$ is a connected set, i.e. it has the form $T_\Delta = [0, t_1]$ for some $t_1 > 0$ (or $t_1 = +\infty$). One has

$$D_+V_0(t) \leq -\rho(\Delta) < 0 \quad \forall t \in T_\Delta.$$

Therefore $\sup T_\Delta \leq V_0(0)/\rho(\Delta) < +\infty$ and

$$V_0(t) = Q(x(t), t) < \Delta \quad \forall t > \sup T_\Delta$$

by the fact that $V_0(t)$ is nonincreasing. Since $\Delta > 0$ is arbitrary, then (17) and (18) hold true.

As a simple application of the theorem above and the standard comparison principle for solutions of ordinary differential equations, we obtain the following result on finite-time convergence of the Speed-Pseudosubgradient algorithm.

**Corollary 1.** *Let all assumptions of Theorem 2 be satisfied, and let the function $\rho$ be locally Lipschitz continuous. Suppose also that for any $z_0 > 0$ there exists $T[z_0] > 0$ such that a solution $z(t)$ of the differential equation*

$$\dot{z} = -\rho(z) \quad z(0) = z_0 \tag{21}$$

*is defined and positive on $[0, T[z_0])$, and $z(t) \to 0$ as $t \to T[z_0]$. Then for any $x(0) \in \mathbb{R}^n$ and $u_0 \in \mathbb{R}^m$ there exists $\overline{\gamma} > 0$ such that for all $\gamma > \overline{\gamma}$ any solution $(x(t), u(t))$ of (1), (7) is bounded and*

$$Q(x(t), t) \to 0 \quad \text{as} \quad t \to T$$

*for some $T < T[z_0]$ with $z_0 = Q(x(0), 0)$.*

Let us discuss assumption 4 of Theorem 2, and the main assumption of the corollary above. Assumption 4 can be roughly interpreted as the assumption on the existence of an "ideal" control law $u_*$ for which the control objective (17) is achieved. Thus, one can say that if the control objective can be achieved with the use of some control law, then it can be achieved with the use of the Speed-Pseudogsubradient algorithm, provided the control law and the corresponding solution of the closed loop system are correctly defined (assumptions 1 and 2 of Theorem 2). It should also be noted that the "ideal" control law $u_*$ can be *unrealizable* since it may depend on unknown parameters.

In the same vein, the main assumption in Corollary 1, roughly speaking, means that the control objective is achieved in finite time, if one uses the "ideal control law" $u_*(x, t)$. Thus, Corollary 1 itself means that the Speed-Pseudosubgradient algorithm converges in finite time, provided there exists some other "ideal control law" for which the control objective is achieved in finite time.

Let us provide a simple example illustrating the above discussion and demonstrating the usage of Theorem 1.

**Example 1** (*Nonlinear Plant Identification*)**.** Consider a pendulum described by the second-order differential equation

$$\ddot{y} = a\sin(y) + bf(t), \tag{22}$$

where $y$ is a generalized coordinate, $f(t)$ is a measurable external force, and $a$, $b$ are unknown parameters. Introduce the model of system (22) of the form

$$\ddot{y}_m = d_1(y - y_m) + d_2(\dot{y} - \dot{y}_m) + a_m \sin(y) + b_m f(t),$$

where $d_1, d_2 > 0$ are introduced to ensure stability. Define $x_1 = y - y_m$, $x_2 = \dot{y} - \dot{y}_m$, $u_1 = a_m$ and $u_2 = b_m$. Then

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -d_1 x_1 - d_2 x_2 + (a - u_1)\sin(y) + (b - u_2)f(t). \end{cases} \tag{23}$$

Introduce the nonsmooth goal function $Q(x) = \sqrt{x^T H x}$, where $H$ is positive definite. Let us apply the Speed-Subgradient algorithm in differential form to the problem under consideration. For any $x \neq 0$ define

$$\omega(x, u) = Q'\big(x; F(x, u)\big) = \frac{1}{2Q(x)}\Big(x^T HAx + x^T A^T Hx \\ + 2(h_{12}x_1 + h_{22}x_2)(a - u_1)\sin(y) + 2(h_{12}x_1 + h_{22}x_2)(b - u_2)f(t)\Big),$$

where $F(x, u)$ is the right-hand side of (23), and $A = \begin{pmatrix} 0 & 1 \\ -d_1 & -d_2 \end{pmatrix}$. If $x = 0$, then set

$$\omega(0, u) = Q'\big(0; F(0, u)\big) = \sqrt{F(0, u)^T HF(0, u)} \\ = \sqrt{h_{22}} \cdot \big|(a - u_1)\sin(y) + (b - u_2)f(t)\big|.$$

One can check that the set-valued mapping $(x, u) \to \partial_u \omega(x, u, t)$ is not outer semicontinuous. That is why we utilize a relaxed version of the Speed-Subgradient algorithm in differential form (see Remark 4). Namely, define the control law as follows

$$\dot{u}_1 \in \begin{cases} \{(h_{12}x_1 + h_{22}x_2)\sin(y)/Q(x)\}, & \text{if } x \neq 0, \\ \text{co}\{-\sqrt{h_{22}}\sin(y), \sqrt{h_{22}}\sin(y)\}, & \text{if } x = 0, \end{cases} \tag{24}$$

and

$$\dot{u}_2 \in \begin{cases} \{(h_{12}x_1 + h_{22}x_2)f(t)/Q(x)\}, & \text{if } x \neq 0, \\ \text{co}\{-\sqrt{h_{22}}f(t), \sqrt{h_{22}}f(t)\}, & \text{if } x = 0. \end{cases} \tag{25}$$

Let us apply Theorem 1 and Remark 4. Clearly, assumptions 1 and 3 of this theorem are valid. Moreover, one can easily verify that the right-hand sides of (24) and (25) are outer semicontinuous and bounded. Therefore it remains to show the existence of an "ideal" control law $u^*$.

Observe that $A$ is stable. Therefore one can choose $H$ as a solution of the Lyapunov equation

$$HA + A^T H = -R,$$

where $R$ is a positive definite matrix. Define $u^* = (a, b)^T$. Then $\omega(0, u^*) = 0$, and for any $x \neq 0$ one has

$$\omega(x, u^*) = -\frac{1}{2Q(x)} x^T Rx \leq -\rho_0 Q(x), \quad \rho_0 = \frac{\lambda_{min}(R)}{2\lambda_{max}(H)}$$

where $\lambda_{min}(R)$ is the minimal eigenvalue of $R$ and $\lambda_{max}(H)$ is the maximal eigenvalue of $H$. Hence assumption 4 of Theorem 1 is valid with $\rho(s) \equiv \rho_0 s$. Thus, all assumptions of Theorem 1 are valid, and either $Q(x(t)) \to 0$ as $t \to \infty$ or there exists $T > 0$ such that $Q(x(T), T) = 0$, which implies that either $|y(t) - y_m(t)| \to 0$ and $|\dot{y}(t) - \dot{y}_m(t)| \to 0$ as $t \to \infty$ or $y(T) = y_m(T)$ and $\dot{y}(T) = \dot{y}_m(T)$ for some $T > 0$. Furthermore, one can verify that under some additional assumptions $u_1(t) \to a$ and $u_2(t) \to b$ as $t \to \infty$.

Note that the ideal control law $u^*$ defined above is unrealizable, since it depends on the unknown parameters $a$ and $b$.

Corollary 1 can be used to obtain new results even in the smooth case. Namely, it provides sufficient conditions for the finite time convergence of *discontinuous* versions of the smooth Speed-Pseudogradient algorithm. In particular, let the functions $F(x, u, t)$ and $Q(x, t)$ be continuously differentiable, and denote

$$\nabla_u \omega(x, u, t) = \left(\frac{\partial f(x, u, t)}{\partial u}\right)^T \frac{\partial Q(x, t)}{\partial x}.$$

Then Corollary 1 gives sufficient conditions for the finite time convergence of the relay algorithm (see [24], Section 3.2.3)

$$u = u_0 - \gamma \,\text{sign}\big(\nabla_u \omega(x, u, t)\big), \tag{26}$$

where the signum function is understood coordinatewise, and the algorithm of the form

$$u = u_0 - \gamma \frac{1}{\|\nabla_u \omega(x, u, t)\|} \nabla_u \omega(x, u, t),$$

where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^m$. However, note that in the case of the relay algorithm (26) assumptions 1 and 2 of Theorem 2 are hard to verify. Nevertheless, let us show that these assumptions are valid in the important case when the function $F(x, u, t)$ is affine in control, provided one understands a solution of a differential equation in the sense of Filippov [25].

**Theorem 3.** *Let $F(x, u, t)$ has the form $F(x, u, t) = f(x, t) + g(x, t)u$, where the functions $f$ and $g$ are defined and continuous on $\mathbb{R}^n \times \mathbb{R}_+$. Let also*

$$u(x, t) = u_0 - \gamma \operatorname{sign}\left( g(x, t)^T \frac{\partial Q(x, t)}{\partial x} \right). \tag{27}$$

*Suppose that the following assumptions are valid:*

1. *the function $Q(x, t)$ is nonnegative, continuously differentiable and radially unbounded;*
2. *there exist a locally bounded uniformly in $t$ function $u_* : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^m$ and a continuous scalar function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho(s) = 0$ if and only if $s = 0$, and*

   $$\omega(x, u_*(x, t), t) \le -\rho(Q(x, t)) \quad \forall x \in \mathbb{R}^n, t \in \mathbb{R}_+,$$

   *where*

   $$\omega(x, u, t) = \frac{\partial Q(x, t)}{\partial x}^T (f(x, t) + g(x, t)u) + \frac{\partial Q(x, t)}{\partial t}. \tag{28}$$

*Then for any $x(0) \in \mathbb{R}^n$ and $u_0 \in \mathbb{R}^m$ there exists $\overline{\gamma} > 0$ such for all $\gamma > \overline{\gamma}$ any Filippov solution $x(t)$ of (1), (27) is defined and bounded on $\mathbb{R}_+$ and*

$$\lim_{t \to \infty} Q(x(t), t) = 0.$$

*Moreover, if the function $\rho$ is locally Lipschitz continuous, and for any $z_0 > 0$ there exists $T[z_0] > 0$ such that a solution $z(t)$ of the differential equation*

$$\dot{z} = -\rho(z), \quad z(0) = z_0 \tag{29}$$

*is defined and positive on $[0, T[z_0])$, and $z(t) \to 0$ as $t \to T[z_0]$, then*

$$Q(x(t), t) \to 0 \quad \text{as} \quad t \to T$$

*for some $T < T[z_0]$ with $z_0 = Q(x(0), 0)$.*

**Proof.** Define the set-valued signum function

$$\operatorname{Sign}(s) = \begin{cases} 1 & \text{if } s > 0, \\ [-1, 1] & \text{if } s = 0, \\ -1 & \text{if } s < 0, \end{cases} \tag{30}$$

and introduce the set-valued control law $U(x, t)$ corresponding to the relay algorithm (27) as follows

$$U(x, t) = u_0 - \gamma \operatorname{Sign}\left( g(x, t)^T \frac{\partial Q(x, t)}{\partial x} \right).$$

Here Sign is the set-valued signum function (30). Applying the facts that $g(x, t)$ is continuous, and $Q(x, t)$ is continuously differentiable, it is easy to verify that the set-valued mapping $U(\cdot, \cdot)$ is outer semicontinuous. Furthermore, note that the mapping $U(\cdot, \cdot)$ is convex-valued, i.e. for any $x$ and $t$ the set $U(x, t)$ is convex. Therefore the right-hand side of the differential inclusion

$$\dot{x}(t) \in f(x, t) + g(x, t)U(x, t) \tag{31}$$

is also outer semicontinuous and convex-valued.

Note that for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$ one has

$$f(x, t) + g(x, t)u(x, t) \in f(x, t) + g(x, t)U(x, t),$$

where $u(x, t)$ is defined by (27). Hence taking into account the fact that the right-hand side of (31) is outer semicontinuous and convex-valued one obtains that any Filippov solution of the system (1), (27) is also a solution of the differential inclusion (31). Therefore it is sufficient to show that the assertion of the theorem holds true for any solution of (31).

By Theorem 2.7.5 from [25] a solution $x(t)$ of (31) is defined at least on some finite time interval $[0, t_0)$. Define the Lyapunov function

$$V(x, t) = Q(x, t)$$

and denote $V_0(t) = V(x(t), t)$. The function $V_0(t)$ is differentiable a.e. on $[0, t_0)$, and its derivative has the form

$$V_0'(t) = \frac{\partial Q(x(t), t)}{\partial x}^T \dot{x}(t) + \frac{\partial Q(x(t), t)}{\partial t} \in \frac{\partial Q(x(t), t)}{\partial x}^T \big(f(x(t), t) + g(x(t), t)U(x(t), t)\big) + \frac{\partial Q(x(t), t)}{\partial t}.$$

Observe that for any $s \in \mathbb{R}$ one has $s \cdot \operatorname{Sign}(s) = |s| = \operatorname{sign}(s) \cdot s$. Therefore

$$\frac{\partial Q(x(t), t)}{\partial x}^T g(x(t), t)U(x(t), t) = \frac{\partial Q(x(t), t)}{\partial x}^T g(x(t), t)u(x(t), t),$$

which, due to (27) and (28), yields

$$\begin{aligned} V_0'(t) &= \omega(x(t), u(x(t), t), t) = \omega(x(t), u_*(x(t), t), t) \\ &\quad + \frac{\partial Q(x(t), t)}{\partial x}^T g(x(t), t)\big[u(x(t), t) - u_*(x(t), t)\big] \\ &\leq -\rho(Q(x(t), t)) + \big[\|u_0 - u_*(x(t), t)\|_\infty - \gamma\big] \left\| g(x(t), t)^T \frac{\partial Q(x(t), t)}{\partial x} \right\|_1, \end{aligned}$$

where

$$\|u\|_\infty = \max\{|u_1|, \ldots, |u_m|\}, \quad \|x\|_1 = \sum_{i=1}^n |x_i|.$$

Denote

$$\Omega_0 = \big\{(x, t) \in \mathbb{R}^n \times \mathbb{R}_+ \mid Q(x, t) < V_0(0) + 1\big\}.$$

Taking into account the radial unboundedness of $Q$ and the local boundedness uniformly in $t$ of $u_*(x, t)$ one obtains that

$$d = \sup_{(x,t)\in\Omega_0} \|u_0 - u_*(x, t)\|_\infty < +\infty.$$

Define $\overline{\gamma} = d$. Then arguing in the same way as in the proof of Theorem 2 one gets that $(x(t), t) \in \Omega_0$ for all $\gamma > \overline{\gamma}$ and for any $t$ from the maximal interval of existence of the solution $x(t)$. Therefore $x(t)$ is defined and bounded on $\mathbb{R}_+$. The rest of the proof almost literally repeats the proofs of Theorem 2 and Corollary 1.

**Remark 5.** Note that Theorems 1 and 2 can be modified to the case when a solution of a differential equation is understood in the sense of Filippov. As it is easily seen from the proof of the theorem above, one simply has to check that a Filippov solution $x(t)$ of (1) satisfies the following inequality

$$V'(x(t), t; \dot{x}(t), 1) \leq \omega(x(t), u, t) \quad \forall t \in \mathbb{R}_+.$$

Then the conclusions of Theorems 1 and 2 are valid for $x(t)$.

## 4. Applications

The theorems above provide a justification of the nonsmooth Speed Gradient algorithms in the general setting. However, some assumptions of these theorems are hard to verify in many particular instances. Moreover, in some important examples, assumption 4 of Theorems 1 and 2 are invalid, since only the inequality of the form

$$\omega(x, u_*, t) \leq 0$$

holds true. Below, we describe two such examples, and demonstrate that even in this situation one can prove that the control goal (17) is achieved.

**Example 2.** Consider a pendulum. Suppose that the suspension point of the pendulum can be moved along the horizontal axis, and that the control variable is the acceleration along this line. The motion of the pendulum is described by the equations

$$\dot{q} = \frac{1}{ml^2} \cdot p, \qquad \dot{p} = -mgl \cdot \sin q + ml \cdot u \cdot \cos q, \tag{32}$$

where $q, p$ are generalized coordinate and momentum, $u$ is the control action, $m, l, g$ are the mass of the pendulum, the length of the pendulum and the gravity acceleration, respectively.

It is convenient to consider motions of the unforced pendulum lying on a cylinder with a circle of radius $l$ in the base, i.e. $-\pi < q \leq \pi$. In other words, it is natural to identify the points $(q_1, p)$ and $(q_2, p)$ with $q_1 - q_2 = 2k\pi$ for some integer $k$. In order to preserve the same phase space for the controlled pendulum only $2\pi$-periodic in $q$ control laws will be considered.

Let the control goal be to make the upright equilibrium of the pendulum globally attractive. Introduce the nonsmooth objective function $Q$ as follows

$$Q(q, p) = |H(q, p) - H_*|,$$

where

$$H(q, p) = \frac{1}{2ml^2} \cdot p^2 + mgl \cdot (1 - \cos q)$$

is the total energy of the unforced pendulum, and $H_* = H(\pi, 0) = 2mgl$ is the total energy of the unforced pendulum at the upright equilibrium.

We use the Speed-Pseudosubgradient algorithm to construct a control law. Clearly, the function $Q$ is locally Lipschitz continuous and Hadamard directionally differentiable. For any $(q, p)$ such that $H(q, p) \neq H_*$ the directional derivative of $Q$ has the form

$$Q'((q, p); v) = \operatorname{sign}(H(q, p) - H_*)\nabla H(q, p)^T v \quad \forall v \in \mathbb{R}^2,$$

where $\nabla H(q, p)$ is the gradient of $H$. Hence for all $(q, p)$ such that $H(q, p) \neq H_*$ one has

$$\begin{aligned} Q'((q, p); F(q, p, u)) &= \operatorname{sign}(H(q, p) - H_*)\nabla H(q, p)^T F(q, p, u) \\ &= \operatorname{sign}(H(q, p) - H_*) \cdot \frac{1}{l} \cdot p \cdot \cos q, \end{aligned}$$

where $F(q, p, u)$ is the right-hand side of (32). Hence according to the Speed-Pseudosubgradient algorithm define the control law as follows

$$u(q, p) = -\gamma \cdot \operatorname{Sign}(H(q, p) - H_*) \cdot p \cdot \cos q \tag{33}$$

(see (30)). Note that the control law (33) is a nonsmooth version of the control law considered in [36]. Observe also that the control law (33) is discontinuous.

**Proposition 1.** *For any $\gamma > 0$ and initial conditions $(q_0, p_0) \neq (0, 0)$ all solutions of the closed-loop system (32), (33) are defined and bounded on $\mathbb{R}_+$, and have a unique $\omega$-limit point $(\pi, 0)$. Thus, the upright equilibrium $(\pi, 0)$ is a unique almost global attractor of the closed-loop system.*

**Proof.** Fix arbitrary $\gamma > 0$ and initial conditions $(q_0, p_0) \neq (0, 0)$. The closed-loop system (32), (33) has the form

$$\dot{q} = \frac{1}{ml^2} \cdot p, \quad \dot{p} \in -mgl \cdot \sin q - \gamma \cdot ml \cdot \operatorname{Sign}(H(q, p) - H_*) \cdot p \cdot \cos^2 q.$$

Observe that the right-hand side of this system is an outer semicontinuous, closed and convex valued multifunction that is bounded on bounded sets. Therefore by [25], Theorem 2.7.6 there exists a solution of (32), (33) starting at $(q_0, p_0)$, and, moreover, all solutions of this system are defined on their maximal interval of existence.

Let $(q(t), p(t))$ be a solution of (32), (33) starting at $(q_0, p_0)$ that is defined on its maximal interval of existence $[0, T_{\max})$. Note that

$$\frac{d}{dt}\Big(H(q(t), p(t)) - H^*\Big)^2 = -\frac{\gamma}{l}\Big|H(q(t), p(t)) - H^*\Big|p^2(t)\cos^2 q(t) \le 0 \tag{34}$$

for a.e. $t \in [0, T_{\max})$. Therefore $H(q(t), p(t)) \le \max\{H^*, H(q_0, p_0)\}$ for all $t \in [0, T_{\max})$, which implies that the solution $(q(t), p(t))$ is bounded and $T_{\max} = +\infty$. Thus, all solutions of (32), (33) are defined and bounded on $\mathbb{R}_+$.

Let us show that the upright equilibrium $(\pi, 0)$ is a unique $\omega$-limit point of $(q(t), p(t))$. Suppose, at first, that $H(q_0, p_0) = H_*$. From (34) it follows that $H(q(t), p(t)) \equiv H^*$, which implies that $dH(q(t), p(t))/dt = 0$ for a.e. $t \ge 0$. On the other hand, by Filippov's Lemma (see, e.g., [22], Theorem 8.2.10), there exists a measurable function $s(t) : \mathbb{R}_+ \to [-1, 1]$ such that for a.e. $t \ge 0$ one has

$$\dot{q}(t) = \frac{1}{ml^2} \cdot p(t), \quad \dot{p}(t) = -mgl \cdot \sin q(t) - \gamma \cdot ml \cdot s(t) \cdot p(t) \cdot \cos^2 q(t).$$

Therefore for a.e. $t \ge 0$ one has

$$\frac{d}{dt}H(q(t), p(t)) = -\frac{\gamma}{l}s(t)p^2(t)\cos^2 q(t),$$

which implies that $s(t)p(t)\cos q(t) = 0$ for a.e. $t \ge 0$, and $(q(t), p(t))$ is a solution of the unforced system

$$\dot{q} = \frac{1}{ml^2} \cdot p, \quad \dot{p} = -mgl \cdot \sin q.$$

Consequently, $(q(t), p(t))$ coincides either with the upright equilibrium or with one of the two homoclinic curves of the unforced pendulum. Hence the solution $(q(t), p(t))$ has a unique $\omega$-limit point, which is the upright equilibrium.

Suppose, now, that $H(q_0, p_0) > H_*$. Then the closed-loop system has the form

$$\dot{q} = \frac{1}{ml^2} \cdot p, \quad \dot{p} = -mgl \cdot \sin q - \gamma \cdot ml \cdot p \cdot \cos^2 q. \tag{35}$$

Observe that the derivative of $H(q, p)$ along solutions of this system has the form

$$\dot{H}(q, p) = -\gamma \cdot \frac{1}{l} \cdot p^2 \cdot \cos^2 q \leq 0.$$

Hence applying Krasovskii–LaSalle's invariance principle one easily obtains that all solutions of the system (35) converge to one of the equilibrium points: $(0, 0)$ or $(\pi, 0)$. Therefore the solution $(q(t), p(t))$ of the closed-loop system (32), (33) starting at $(q_0, p_0)$ either converges to the upright equilibrium (in this case $H(q(t), p(t)) > H_*$ for all $t \in \mathbb{R}_+$) or there exists $T > 0$ such that $H(q(T), p(T)) = H_*$. In the latter case, $H(q(t), p(t)) = H^*$ for all $t \geq T$ due to (34), and arguing in the same way as in the case $H(q_0, p_0) = H^*$ one can check that $(\pi, 0)$ is a unique $\omega$-limit point of $(q(t), p(t))$.

Suppose, finally, that $H(q_0, p_0) < H_*$. Let us prove, at first, that the downward equilibrium $(0, 0)$ is totally unstable. Indeed, since $(q_0, p_0) \neq 0$, then $H(q_0, p_0) > H(0, 0) = 0$. Note that

$$\dot{H}(q(t), p(t)) = \frac{\gamma}{l} p^2(t) \cos^2 q(t) \geq 0 \quad \text{for a.e. } t \geq 0 : H(q(t), p(t)) < H^*.$$

Hence

$$H(q(t), p(t)) \geq H(q_0, p_0) > 0 = H(0, 0) \quad \forall t \geq 0,$$

which implies that the downward equilibrium $(0, 0)$ is totally unstable.

Recall that $H(q_0, p_0) < H_*$. If $H(q(T), p(T)) = H^*$ for some $T > 0$, then $H(q(t), p(t)) = H^*$ for any $t \geq T$ due to (34), and arguing in the same way as in the case $H(q_0, p_0) = H^*$ one can verify that the upright equilibrium $(\pi, 0)$ is a unique $\omega$-limit point of $(q(t), p(t))$. Therefore it remains to consider the case when $H(q(t), p(t)) < H^*$ for all $t \geq 0$. In this case, $(q(t), p(t))$ is a solution of the system

$$\dot{q} = \frac{1}{ml^2} \cdot p, \quad \dot{p} = -mgl \cdot \sin q + \gamma \cdot ml \cdot p \cdot \cos^2 q. \tag{36}$$

Let us show that any solution $(q(t), p(t))$ of this system with $(q(0), p(0)) \neq (0, 0)$ such that $H(q(t), p(t)) < H_*$ for any $t \geq 0$ has a unique $\omega$-limit point $(\pi, 0)$. Then one obtains the desired result.

Denote $C = \{(p, q) : 0 < H(p, q) < H_*\}$. Let $G_0$ be the set of all those $(q, p) \in C$ for which the solution $(q(t), p(t))$ of (36) starting at $(q, p)$ satisfies the inequality $H(q(t), p(t)) < H_*$ for any $t \geq 0$. Denote also $G = G_0 \cup \{(\pi, 0)\}$. The set $G$ is an invariant set of the system (36) by virtue of the fact that this system is autonomous.

Observe that the derivative of the function $V(q, p) = H_* - H(q, p)$ along solutions of (36) has the form

$$\dot{V}(q, p) = -\gamma \cdot \frac{1}{l} \cdot p^2 \cdot \cos^2 q \leq 0.$$

Note also that the function $V$ is nonnegative and continuous on $G$. Hence $V$ is a Lyapunov function of (36) on $G$. Therefore by Krasovskii–LaSalle's invariance principle (see [37], Theorem 6.4) any solution of (36) starting in $G$ converges to the largest invariant set of (36) in the set $E = \{(q, p) \in \operatorname{cl} G : \dot{V}(q, p) = 0\}$, where $\operatorname{cl} G$ is the closure of $G$. It is easy to check that the largest invariant set of (36) in the set $E$ is the union of the downward and the upright equilibriums. Consequently, any trajectory of (36) starting in $G$ converges to the upright equilibrium $(\pi, 0)$, since, as in the case of the closed-loop system (32), (33), the downward equilibrium is totally unstable.

**Remark 6.** Note that the maximum value of the control action can be made arbitrary small by means of a proper choice of the parameter $\gamma > 0$. Indeed, from the proof of the proposition above it follows that $H(q(t), p(t)) < \max\{H_*, H(q(0), p(0))\}$ along any solution $(q(t), p(t))$ of the closed-loop system (32), (33). Hence for any given initial energy level $H(q(0), p(0))$ and any $\gamma > 0$ one has

$$p^2(t) \leq 2ml^2 \big(\max\{H_*, H(q(0), p(0))\} + 2mgl\big),$$

which implies that

$$|u(q, p)| = \gamma \cdot |p| \cdot |\cos q| \leq \gamma \cdot C$$

with some $C > 0$ depending only on the initial energy level $H(q(0), p(0))$ and the parameters of the pendulum. Therefore choosing sufficiently small $\gamma > 0$ one can make the upright equilibrium almost globally attractive with the use of the control with arbitrary small values.

In the next example, we consider a system that is non-affine in control.

**Example 3.** Let the controlled system have the form ([38], Example 1)

$$\dot{q} = p, \qquad \dot{p} = -\frac{a}{l + u} \sin q, \tag{37}$$

where $a > 0$ and $l > 0$ are unknown parameters such that $l \geq l_0$ with $l_0 > 0$. For the sake of convenience, we consider motions of the unforced system lying on a cylinder, i.e. we identify the points $(q_1, p)$ and $(q_2, p)$ with $q_1 - q_2 = 2k\pi$ for some integer $k$. In order to preserve the same phase space for the controlled system only $2\pi$-periodic in $q$ control laws $u = u(q, p)$ will be considered.

Fix $H_* > 0$. Let the control goal be to steer the system to the invariant manifold of the unforced system of the form

$$M = \Big\{(q, p)\Big| H(q, p) = H^*\Big\},$$

where the function

$$H(q, p) = \frac{p^2}{2} + \frac{a}{l}\big(1 - \cos q\big)$$

can be viewed as the total energy of the unforced system.

In order to apply the nonsmooth Speed-Gradient algorithm, introduce the goal function

$$Q(q, p) = \big|H(q, p) - H_*\big|.$$

The function $Q$ is locally Lipschitz continuous, Hadamard directionally differentiable, and

$$Q'((q, p); v) = \operatorname{sign}\big(H(q, p) - H_*\big)\nabla H(q, p)^T v \quad \forall v \in \mathbb{R}^2$$

for any $(q, p)$ such that $H(q, p) \neq H_*$. Hence for any such $(q, p)$ one has

$$\begin{aligned} Q'((q, p); F(q, p, u)) &= \operatorname{sign}\big(H(q, p) - H_*\big)\nabla H(q, p)^T F(q, p, u) \\ &= \operatorname{sign}\big(H(q, p) - H_*\big) \cdot p \cdot \sin q \cdot \left(\frac{a}{l} - \frac{a}{l + u}\right), \end{aligned}$$

where $F(q, p, u)$ is the right-hand side of (37). Note that

$$\frac{\partial}{\partial u} Q'((q, p); F(q, p, u)) = \operatorname{sign}\big(H(q, p) - H_*\big) \cdot p \cdot \sin q \cdot \frac{a}{(l + u)^2}.$$

Therefore according to the Speed-Pseudosubgradient algorithm we define the control law as follows

$$u(q, p) = \gamma\psi(q, p), \quad \psi(q, p) = -\operatorname{Sign}(H(q, p) - H_*) \cdot p \cdot \sin q. \tag{38}$$

Clearly, the function $\psi(q, p)$ satisfies the "acute angle" condition (8), which implies that

$$Q'\Big((q, p); F\big(q, p, u(q, p)\big)\Big) \leq 0$$

for any $(q, p)$ such that $H(q, p) \neq H_*$, i.e. the goal function $Q(q, p)$ is nonincreasing along solutions of the closed-loop system.

**Proposition 2.** *For any initial conditions* $(q(0), p(0)) \neq 0$, *and for all*

$$0 < \gamma < \frac{l_0}{2\sqrt{\max\{H(q(0), p(0)), H_*\}}} \tag{39}$$

*all solutions* $(q(t), p(t))$ *of the closed-loop system* (37), (38) *are defined on* $\mathbb{R}_+$, *and either* $H(q(t), p(t)) \to H_*$ *or* $(q(t), p(t)) \to (\pi, 0)$ *as* $t \to \infty$.

**Proof.** Introduce the set-valued mapping

$$l(q, p) = \begin{cases} \dfrac{a}{l - \gamma \operatorname{sign}(H(q, p) - H_*)p \sin q}, & \text{if } H(q, p) \neq H_*, \\ \operatorname{co}\left\{\dfrac{a}{l - \gamma p \sin q}, \dfrac{a}{l + \gamma p \sin q}\right\}, & \text{if } H(q, p) = H_*, \end{cases}$$

and define an open set $D = \{(q, p) \in \mathbb{R}^2 \mid l_0 - \gamma|p| > 0\}$. Note that from (39) it follows that $(q(0), p(0))$ belongs to the set $D$.

The closed-loop system (37), (38) has the form

$$\dot{q} = p, \qquad \dot{p} = l(q, p) \cdot \sin q. \tag{40}$$

The right-hand side of this system is an outer semicontinuous, closed and convex valued multifunction that is defined and bounded on compact subsets of the set $D$. Consequently, by [25], Theorem 2.7.6 there exists a solution of (40) starting at $(q(0), p(0))$. Moreover, all solutions $(q(t), p(t))$ of this systems starting at $(q(0), p(0))$ are defined on their maximal interval of existence $[0, T_{\max})$, and if $T_{\max} < +\infty$, then $(q(t), p(t))$ reaches the boundary of $D$ as $t \to T_{\max}$.

Let $(q(t), p(t))$ be a solution of (37), (38) starting at $(q(0), p(0)) \neq 0$ that is defined on $[0, T_{\max})$. For a.e. $t \in [0, T_{\max})$ one has

$$\frac{d}{dt}\Big(H(q(t), p(t)) - H_*\Big)^2 = \Big(H(q(t), p(t)) - H_*\Big) \cdot p(t) \cdot \sin q(t) \left( \frac{a}{l} - \frac{a}{l + u_0(q(t), p(t))} \right),$$

where $u_0(q, p) = u(q, p)$ if $H(q, p) \neq H_*$, and $u_0(q, p) = 0$ otherwise. Note that from the definition of the control law (see (38)) it follows that

$$\frac{a}{l} - \frac{a}{l + u_0(q, p)} \begin{cases} > 0, & \text{if } (H(q, p) - H_*)p \sin q < 0, \\ = 0, & \text{if } (H(q, p) - H_*)p \sin q = 0, \\ < 0, & \text{if } (H(q, p) - H_*)p \sin q > 0. \end{cases}$$

Therefore

$$\frac{d}{dt}\Big(H(q(t), p(t)) - H_*\Big)^2 \leq 0 \quad \text{for a.e. } t \in [0, T_{\max}), \tag{41}$$

which yields that $H(q(t), p(t)) \leq \max\{H(q(0), p(0)), H_*\}$ for all $t \in [0, T_{\max})$. Hence

$$|p(t)| \leq \sqrt{2H(q(t), p(t))} \leq \sqrt{2 \max\{H(q(0), p(0)), H_*\}} \quad \forall t \in [0, T_{\max})$$

and $|\dot{q}(t)| \leq \sqrt{2 \max\{H(q(0), p(0)), H_*\}}$ for a.e. $t \in [0, T_{\max})$. Consequently, taking into account (39) one obtains that the trajectory $(q(t), p(t))$ cannot reach the boundary of the set $D$ in finite time, which, in turn, implies that $T_{\max} = +\infty$. Then applying Krasovskii–LaSalle's invariance principle, and arguing in a similar way to the proof of Proposition 1 we arrive at the required result.

**Remark 7.** From the proposition above it follows that any trajectory of the closed-loop system (37), (38) with $\gamma > 0$ being sufficiently small converges either to the invariant manifold $\{(q, p) \mid H(q, p) = H_*\}$ or to the equilibrium point $(\pi, 0)$ of the unforced system. Let us note that this kind of result is typical for Speed-Gradient algorithms (see, e.g., [24], Section 3.6).

## 5. Conclusion

The Speed-Gradient algorithms are used in many nonlinear control and adaptation problems. However their extensions to the nonsmooth case were not available before. In this paper, an important step towards development of nonsmooth versions of Speed-Gradient methods is made. Nonsmooth SG-algorithm in differential and finite form are formulated and conditions for the control goal achievement are obtained. As applications, a nonsmooth energy-based control for swinging up a pendulum is designed, and an energy control problem for a non-affine in control system is solved.

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