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SPEED-GRADIENT CONTROL OF THE BROCKETT INTEGRATOR*

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Abstract. A nonsmooth extension of the speed-gradient algorithms in finite form is proposed. The conditions ensuring control goal (convergence of the goal function to zero) are established. A new algorithm is applied to almost global stabilization of the Brockett integrator that has become a popular benchmark for nonsmooth and discontinuous algorithms. It is proved that the designed control law stabilizes the Brockett integrator for any initial point that does not lie on the x3-axis. Besides, it is shown that the speed-gradient algorithm ensures stabilization with an arbitrarily small control level. An important feature of the proposed control is the fact that it is continuous along trajectories of the closed-loop system.

Key words. nonsmooth systems, nonholonomic integrator, speed-gradient, nonlinear control

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1. Introduction.

The Brockett integrator or nonholonomic integrator

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_1 u_2 - x_2 u_1 \tag{1.1}$$

was introduced in the seminal paper [1]. Since it was proved by Brockett that there exists no smooth time-invariant state feedback stabilizing (1.1) in the origin, it has become a paradigmatic example of systems where smooth feedback fails. This result was further extended to a large class of discontinuous state feedbacks by Ryan [2]. It also was, in a sense, the starting point of application of differential geometry, Lie groups, and Lie algebra methods to nonlinear control that led to the creation of what is now known under the name of geometric control theory. The development of the geometric control theory started in the 1970s [3]. By now, it has grown into a mature area with strong machinery [4, 5, 6, 7]. Since the Brockett integrator is a beautiful and seemingly simple system, it has become a benchmark example for nonlinear control methods.

For several decades many authors have been making efforts to apply their approaches for control of the Brockett integrator. New algorithms for (1.1) were designed via the invariant manifold technique in [8] and via discontinuous transformations in [9]. The sliding mode control was applied in [10]. A family of discontinuous control laws was derived in [11]. A "sample-and-hold" approach based on nonsmooth control Lyapunov functions was proposed in [12]. Logic-based switching was applied in [13]. Methods of optimal control theory were utilized in [14]. A hybrid control law was designed in [15, 16]. An impulsive control was studied in [17]. An interesting gen-

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eral approach, using isospectral flows, to the stabilization of a class of nonholonomic systems that includes the Brockett integrator was developed in [18].

In this paper we make an attempt to control the Brockett integrator by means of the speed-gradient (SG) method. The SG method was proposed at the end of the 1970s as a general framework for the design of control, adaptation, and identification algorithms for nonlinear systems [19]. Since then it was extended in different directions [20, 21] and applied to a variety of problems in physics and mechanics [22, 23, 24]. An intimate relation between the applicability of the SG-method and the passivity of a controlled system was established [25]. In the special case of an affine controlled system the SG-algorithms encompass Jurdjevic–Quinn (LgV) algorithms [26].

The standard procedure of SG-algorithms derivation requires differentiation of the goal function along trajectories of the controlled system. However, in many cases the right-hand sides of the system model are nonsmooth. Sometimes it may be profitable to introduce nonsmooth and even discontinuous terms into control algorithms in order to provide the desired system dynamics, e.g., finite time convergence. Therefore there is a need for a more general framework for the design and analysis of SG-like algorithms in a general nonsmooth case. A first extension of SG-methods to the nonsmooth case has been made in [27]. It should be noted that the assumptions on the controlled system and the goal function that we use in this article (see Theorem 3.1 below) are different from the ones in [27]. Furthermore, in [27] the nonsmooth SG algorithm was applied only to a linear controlled system, while the main goal of the present article is to apply this algorithm to the stabilization problem for the Brockett integrator.

In section 3 of this paper we introduce a further result on the stabilization ability of nonsmooth pseudogradient methods. In section 4 a nonsmooth SG-algorithm for the stabilization of (1.1) is derived, and stability conditions are established. As a further application of nonsmooth SG methods, we consider the energy control problem for a vibrating string in section 5. Necessary preliminary material is presented in section 2.

2. Preliminaries. In this section, we recall some notions from nonsmooth analysis [28] that are used in what follows. Denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$, and denote $\mathbb{R}_+ = [0, +\infty)$.

Let a real-valued function f be defined in a neighborhood of a point $x \in \mathbb{R}^n$. The function f is called *Hadamard directionally differentiable* at the point x if for any $h \in \mathbb{R}^n$ there exists the finite limit

$$f'(x; h) = \lim_{[\alpha,h']\to[+0,h]} \frac{f(x + \alpha h') - f(x)}{\alpha}.$$

The function $f'(x; \cdot)$ is called the *Hadamard directional derivative* of f at x. Note that there exists an elaborate calculus of Hadamard directional derivatives [28]. Observe also that if $n = 1$, then the quantity $f'(x; 1)$ coincides with the right-hand side derivative of f at x that is denoted by $D_+f(x)$.

Recall that the function f is said to be *Hadamard superdifferentiable* at the point x if f is Hadamard directionally differentiable at this point, and there exists a convex compact set $\overline{\partial} f(x) \subset \mathbb{R}^n$ such that

$$f'(x; h) = \min_{v\in\overline{\partial} f(x)} v^T h \quad \forall h \in \mathbb{R}^n.$$

The set $\overline{\partial} f(x)$ is called the (Hadamard) *superdifferential* of f at x. The most common example of a Hadamard superdifferentiable function is the composition of a concave function and a differentiable vector function.

It is easy to see that if functions f_i, $i \in I = \{1, \dots, k\}$, are Hadamard superdifferentiable at a point $x \in \mathbb{R}^n$, then for any $\alpha_i \geq 0$, $i \in I$, the functions $\sum_{i \in I} \alpha_i f_i$ and $\min_{i \in I} f_i$ are Hadamard superdifferentiable at x as well. Note that any semiconcave function [12] is Hadamard superdifferentiable.

3. Nonsmooth speed-gradient algorithm. Consider the controlled system

$$\dot{x} = F(x, u, t), \quad t \geq 0, \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the vector of the system state, and $u \in \mathbb{R}^m$ is the control. We assume that the function $F \colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^n$ is continuous. A solution of (3.1), even in the case of a discontinuous control law, is understood to be an absolutely continuous function satisfying (3.1) for almost all t in its domain.

We pose the general control problem as finding the control law

$$u(t) = U\{x(s), u(s) \colon 0 \leq s \leq t\}$$

which ensures the control objective

$$Q(x(t), t) \leq \Delta \text{ when } t \geq t_*,$$

where $Q(x, t)$ is a nonnegative *goal function* defined on $\mathbb{R}^n \times \mathbb{R}_+$, $\Delta \geq 0$ is some prespecified threshold, and t^* is the time instant at which the control objective is achieved. The objective can be formulated also as

$$\limsup_{t \to \infty} Q(x(t), t) \leq \Delta,$$

which does not specify the value of t^*. In the special case $\Delta = 0$ the control objective takes the form

$$\lim_{t \to \infty} Q(x(t), t)) = 0, \tag{3.2}$$

i.e., the objective is to stabilize the system (3.1) with respect to the goal function Q.

The formulation of the control problem that we use encompasses various control problems, such as partial stabilization, control of system energy, identification, and adaptive control. (See the discussion, as well as various examples and applications, in [21].) In particular, if one takes a control Lyapunov function $V(x)$ of the system (3.1) as the goal function $Q(x, t)$, then the goal (3.2) is closely related to the asymptotic stability of (3.1). However, we underline that possible goal functions $Q(x, t)$ are neither exhausted by nor reduced to control Lyapunov functions. (See section 5 below and examples in [21].)

In order to design a control algorithm suppose that the function Q is locally Lipschitz continuous and Hadamard directionally differentiable. Choose a function $\omega(x, u, t)$ of the form

$$\omega(x, u, t) = g(x, t)^T u,$$

where $g(x, t) \colon \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^m$ is a given function such that

$$Q'(x, t; F(x, u, t), 1) \leq \omega(x, u, t) \quad \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m, t \in \mathbb{R}_+. \tag{3.3}$$

Note that $g(x, t) = \nabla_u \omega(x, u, t)$, where $\nabla_u \omega(x, u, t)$ is the gradient of the function $u \to \omega(x, u, t)$.

Remark 1. The existence of a function $\omega(x,u,t)$ of the form $\omega(x,u,t) = g(x,t)^T u$ that satisfies (3.3) is the basic assumption on the system (3.1) and the goal function $Q(x,t)$ that we implicitly make throughout this article. This assumption is valid, in particular, in the case when the function F is linear in u (i.e., when it has the form $F(x,u,t) = f(x,t)u$), and the function Q is Hadamard superdifferentiable and does not depend on t. Indeed, in this case one can define $\omega(x,u,t) = v(x)^T f(x,t)u$ for any function $v(x)$ such that $v(x) \in \overline{\partial} Q(x)$ for all $x \in \mathbb{R}^n$.

Take the control algorithm in the form

$$u = -\Gamma g(x,t),$$

where Γ is a positive definite gain matrix. We will also consider a more general control algorithm

$$u = \gamma \psi(x,u,t), \tag{3.4}$$

where $\gamma > 0$ is a scalar gain, and the vector function ψ satisfies the "acute angle" condition: $g(x,t)^T \psi(x,u,t) \le 0$ for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $t \in \mathbb{R}_+$. The algorithm of the form (3.4) is a generalization of the so-called speed-pseudogradient algorithms (see [21]). Furthermore, if one takes a control Lyapunov function $V(x)$ of the system (3.1) as the goal function $Q(x,t)$, then the control law (3.4) is a feedback of steepest descent type for V (see [12]). However, since, as was mentioned above, the function $Q(x,t)$ need not be a control Lyapunov function, then the control algorithm (3.4) is not reduced to a feedback of steepest descent type in the general case.

It should be noted that (3.4) is an *equation* with respect to the control variable u; in other words, the equality (3.4) defines the control law $u = u(x,t,\gamma)$ *implicitly.* Therefore, in order to implement the algorithm of the form (3.4) one should be able to efficiently solve this equation, i.e., one should be able either to obtain an explicit expression for $u(x,t,\gamma)$ or to efficiently solve this equation numerically. However, it should be noted that in many applications either the function ψ does not depend on u (see the examples below) or a solution of (3.4) can be found analytically.

Let us discuss the performance of the control systems with the proposed control algorithm (3.4). The following theorem holds true.

THEOREM 3.1. *Let $C \subset \mathbb{R}^n$ be a given set and the following assumptions be valid:*

1. *For any $\gamma > 0$, $x \in \mathbb{R}^n$ and $t \ge 0$ there exists a solution $u = \kappa(x,t,\gamma)$ of* (3.4), *and the function κ is locally bounded in x uniformly in t.*

2. *An absolutely continuous solution of the system* (3.1), (3.4) *exists for all $t \ge 0$ and $x(0) \in \mathbb{R}^n \setminus C$, and $x(t) \notin C$ for any $t \in \mathbb{R}_+$.*

3. *The function $Q(x,t)$ is radially unbounded, i.e.,*

$$\inf_{t \ge 0} Q(x,t) \to \infty \quad \text{as} \quad |x| \to \infty,$$

and nonnegative.

4. *For any $\Delta > 0$ and $r > 0$ there exists $a > 0$ such that $|g(x,t)| \ge a$ for all $x \in \mathbb{R}^n \setminus C$ and $t \in \mathbb{R}_+$ such that $Q(x,t) \ge \Delta$ and $|x| \le r$.*

5. *There exists a continuous function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho(s) = 0$ iff $s = 0$, and $g(x,t)^T \psi(x,u,t) \le -\rho(|g(x,t)|)$ for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $t \in \mathbb{R}_+$.*

Then for any $x(0) \in \mathbb{R}^n \setminus C$ and $\gamma > 0$ a solution of (3.1), (3.4) *is bounded on $\mathbb{R}_+$ and the control goal* (3.2) *is achieved, i.e.,*

$$\lim_{t \to \infty} Q(x(t),t)) = 0.$$

Proof. Let $x(t)$ be a solution of the system (3.1), (3.4). Introduce the Lyapunov function $V(x,t) = Q(x,t)$, and define $V_0(t) = V(x(t),t)$. The function Q is Hadamard directionally differentiable, and the function $x(t)$ is a.e. differentiable, as a solution of a differential equation. Therefore by the chain rule for directional derivatives [28, Theorem I.3.3] for a.e. $t \in \mathbb{R}_+$ there exists the right-hand side derivative $D_+V_0(t)$ of the function V_0 that has the form

$$D_+V_0(t) = Q'(x(t),t; F(x(t),u,t),1) \le \omega(x(t),u,t).$$

Since the function ω has the form $\omega(x,u,t) = g(x,t)^T u$, applying assumption 5 one obtains that

$$D_+V_0(t) \le \gamma g(x,t)^T \psi(x,u,t) \le -\gamma\rho(|g(x,t)|) \le 0. \tag{3.5}$$

Note that the function V_0 is absolutely continuous as the composition of the locally Lipschitz continuous function $Q(x,t)$ and the absolutely continuous function $x(t)$. Hence taking into account (3.5) one gets that the function $V_0(t)$ is nonincreasing, which implies the boundedness of $x(t)$ due to the radial unboundedness of Q and the boundedness of the control u due to the local boundedness in x uniformly in t of the function $u = \kappa(x,t,\gamma)$.

Choose an arbitrary $\Delta > 0$, and denote $T_\Delta = \{t \ge 0 \colon Q(x(t),t) \ge \Delta\}$. Observe that the set T_Δ is connected due to the fact that the function $V_0(t) = Q(x(t),t)$ is nonincreasing. From assumption 4 it follows that there exists $a > 0$ such that $|g(x,t)| > a$ for all $t \in T_\Delta$. Hence with the use of (3.5) one gets that $D_+V_0(t) \le -\gamma\rho(a) < 0$ for any $t \in T_\Delta$. Consequently, $\sup T_\Delta \le V_0(0)/\gamma\rho(a)$, and

$$V_0(t) = Q(x(t),t) < \Delta \quad \forall t > \sup T_\Delta.$$

Since $\Delta > 0$ is arbitrary, (3.2) holds true. ∎

Remark 2. Let all assumptions of the theorem above be valid, and suppose that the function $\psi(x,u,t)$ is bounded for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $t \in \mathbb{R}_+$. Then the control goal (3.2) can be achieved for an arbitrarily small control input. Indeed, by choosing sufficiently small $\gamma > 0$, one can obtain that the inequality $|u| = \gamma|\psi(x,u,t)| < \varepsilon$ holds true for an arbitrary small prespecified $\varepsilon > 0$.

The following lemma allows one to slightly improve Theorem 3.1.

LEMMA 3.2. *Let assumptions* 1–3 *of Theorem* 3.1 *hold true, and let*

$$g(x,t)^T \psi(x,u,t) \le 0 \quad \forall u \in \mathbb{R}^m$$

for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$ such that $Q(x,t) > 0$. Suppose that $x(t)$ is a solution of the system (3.1), (3.4) *with $x(0) \in \mathbb{R}^n \setminus C$ such that $Q(x(T),T) = 0$ for some $T \ge 0$. Then $Q(x(t),t) = 0$ for all $t \ge T$.*

Proof. Arguing by reductio ad absurdum, suppose that there exists $t_0 > T$ such that $Q(x(t_0),t_0) > 0$. Denote

$$\tau = \sup\big\{t \in [T,t_0] \colon Q(x(t),t) = 0\big\}.$$

Then $T \le \tau < t_0$, $Q(x(\tau),\tau) = 0$, and for any $t \in (\tau,t_0]$ one has $Q(x(t),t) > 0$. Hence for a.e. $t \in (\tau, t_0]$ one has

$$D_+V_0(t) = Q'(x(t), t; F(x(t), u, t), 1) \leq \omega(x(t), u, t) := \gamma g(x, t)^T \psi(x, u, t) \leq 0,$$

where $V_0(t) = Q(x(t), t)$. Consequently, the function V_0 is nonincreasing on $[\tau, t_0]$. Therefore $Q(x(t_0), t_0) \leq Q(x(\tau), \tau) = 0$, which contradicts the definition of t_0. ∎

Remark 3. From the lemma above it follows that Theorem 3.1 holds true in the case when the inequality

$$Q'(x, t; F(x, u, t), 1) \leq \omega(x, u, t) \quad \forall u \in \mathbb{R}^m$$

(see (3.3)) is satisfied only for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$ such that $Q(x, t) > 0$. Indeed, if $Q(x(t), t) > 0$ for all $t \in \mathbb{R}_+$, then arguing in the same way as in the proof of Theorem 3.1 one obtains that $Q(x(t), t) \to 0$ as $t \to \infty$. On the other hand, if $Q(x(T), T) = 0$ for some $T \geq 0$, then arguing in the same way as in the proof of the previous lemma one gets that $Q(x(t), t) = 0$ for all $t \geq T$, which implies the desired result.

4. Stabilization of the Brockett integrator.

4.1. Problem formulation. Let us apply the theory discussed above to the construction of an arbitrarily small stabilizing feedback control for the Brockett integrator (1.1). Since there is no continuous feedback control that stabilizes this system [1], the standard SG algorithms cannot be applied in this case. That is why we utilize the nonsmooth version of SG-algorithm developed in this paper.

As was mentioned above, the Brockett integrator is expressed as (1.1). We impose the additional constraint on control $u_1^2 + u_2^2 \leq \varepsilon$, where $\varepsilon > 0$ is arbitrary. Inspired by the ideas of Clarke [12], we introduce the goal function $Q(x)$ as follows:

$$Q(x) = \left(\sqrt{x_1^2 + x_2^2} - |x_3|\right)^2 + x_3^2 = x_1^2 + x_2^2 + 2x_3^2 - 2|x_3|\sqrt{x_1^2 + x_2^2}.$$

Note that the function Q is radially unbounded, and $Q(x) = 0$ iff $x = 0$. It was shown in [12] that Q is a control Lyapunov function for the Brockett integrator.

4.2. Feedback construction. Let us apply the algorithm (3.4) to the construction of a control law. For the sake of convenience, denote $\sigma(x) = \sqrt{x_1^2 + x_2^2}$. The function Q is locally Lipschitz continuous and Hadamard directionally differentiable. Its directional derivative has the form

$$Q'(x; h) = 2x_1h_1 + 2x_2h_2 + 4x_3h_3 - 2|x_3|\frac{(x_1h_1 + x_2h_2)}{\sigma(x)} - 2\,\mathrm{sign}(x_3)\sigma(x)h_3$$

in the case $x_3 \neq 0$ and $\sigma(x) \neq 0$, and

$$Q'(x, h) = \begin{cases} 2x_2h_1 + 2x_2h_2 - 2|h_3|\sigma(x) \text{ if } x_3 = 0, \\ 4x_3h_3 - 2|x_3|\sqrt{h_1^2 + h_2^2} \text{ if } \sigma(x) = 0. \end{cases}$$

Let $x_3 \neq 0$ and $\sigma(x) \neq 0$. Then the function $Q'(x; \cdot)$ is linear, i.e., Q is differentiable. Therefore in this case define

$$\begin{aligned} \omega(x, u) = Q'(x)^T F(x, u) &= 2x_1u_1 + 2x_2u_2 + 4x_3(x_1u_2 - x_2u_1) \\ &\quad - 2|x_3|(x_1u_1 + x_2u_2)/\sigma(x) - 2\,\mathrm{sign}(x_3)\sigma(x)(x_1u_2 - x_2u_1), \end{aligned}$$

where $F(x, u)$ is the right-hand side of (1.1).

Let now $x_3 = 0$. Then Q is Hadamard superdifferentiable, and its superdifferential has the form

$$\overline{\partial} Q(x) = \operatorname{co}\left\{(2x_1, 2x_2, 2\sigma(x))^T, (2x_1, 2x_2, -2\sigma(x))^T\right\},$$

where "co" stands for the convex hull. Note that $(2x_1, 2x_2, 0) \in \overline{\partial} Q(x)$, and define

$$\omega(x, u) = (2x_1, 2x_2, 0)^T F(x, u) = 2x_1 u_1 + 2x_2 u_2 \geq Q'(x; F(x, u)). \tag{4.1}$$

Finally, let $\sigma(x) = 0$. Then Q is also Hadamard superdifferentiable and

$$\overline{\partial} Q(x) = \left\{(-2|x_3|v_1, -2|x_3|v_2, 4x_3)^T \colon |(v_1, v_2)| \leq 1\right\}.$$

Therefore choose an arbitrary $v = (v_1, v_2) \in \mathbb{R}^2$ such that $|v| = 1$, and define

$$\begin{aligned} \omega(x, u) &= (-2|x_3|v_1, -2|x_3|v_2, 4x_3)^T F(x, u) \\ &= -2|x_3|(v_1 u_1 + v_2 u_2) \geq Q'(x; F(x, u)). \end{aligned} \tag{4.2}$$

(Recall that $\sigma(x) = 0$, i.e., $x_1 = x_2 = 0$.) Note that the choice of v can depend on x_3, i.e., one can choose $v = v(x_3)$.

Define the control law as follows:

$$u(x) = \gamma\psi(x), \quad \psi(x) = -|\nabla_u \omega(x, u)|^{-1} \nabla_u \omega(x, u). \tag{4.3}$$

Thus, the control has the form

$$u(x) = \begin{cases} 0 \text{ if } x = 0, \\ -\gamma\sigma(x)^{-1}(x_1, x_2)^T \text{ if } x_3 = 0, \sigma(x) \neq 0, \\ \gamma v(x_3) \text{ if } \sigma(x) = 0, x_3 \neq 0, \\ -\gamma|\nabla_u \omega(x, u)|^{-1} \nabla_u \omega(x, u) \text{ if } \sigma(x) \neq 0, x_3 \neq 0, \end{cases} \tag{4.4}$$

where $\nabla_u \omega(x, u) = (\partial\omega/\partial u_1, \partial\omega/\partial u_2)$ and

$$\frac{\partial\omega}{\partial u_1}(x, u) = 2x_1 - 4x_2 x_3 - 2\frac{|x_3|x_1}{\sigma(x)} + 2\operatorname{sign}(x_3) x_2 \sigma(x), \tag{4.5}$$

$$\frac{\partial\omega}{\partial u_2}(x, u) = 2x_2 + 4x_1 x_3 - 2\frac{|x_3|x_2}{\sigma(x)} - 2\operatorname{sign}(x_3) x_1 \sigma(x) \tag{4.6}$$

in the case $\sigma(x) \neq 0$ and $x_3 \neq 0$.

Let us show that $|\nabla_u \omega(x, u)| \neq 0$ for any x such that $\sigma(x) \neq 0$ and $x_3 \neq 0$. Indeed, multiplying by $\sigma(x)$ in (4.5) one gets that for any x such that $x_1 \neq 0$ and $x_3 \neq 0$ the following holds true:

$$\sigma(x)\frac{\partial\omega}{\partial u_1}(x, u) = 2\operatorname{sign}(x_3) x_2 \sigma^2(x) + (2x_1 - 4x_2 x_3)\sigma(x) - 2|x_3|x_1 \neq 0,$$

since the discriminant of the quadratic equation

$$l(s) = 2\operatorname{sign}(x_3) x_2 s^2 + (2x_1 - 4x_2 x_3)s - 2|x_3|x_1 = 0$$

has the form $D = 4x_1^2 + 16x_2^2 x_3^2 > 0$. Analogously, $\partial\omega/\partial u_2 \neq 0$ for all x such that $x_2 \neq 0$ and $x_3 \neq 0$. Thus, $|\nabla_u \omega(x, u)| \neq 0$ for any $x \in \mathbb{R}^3$ such that $x_3 \neq 0$ and $\sigma(x) \neq 0$. Hence the control law (4.4) is corretly defined.

Remark 4. (i) Observe that in the case $\sigma(x) \neq 0$ and $x_3 \neq 0$, the set of limit points of $u(x)$ (see (4.4)–(4.6)) as $\sigma(x) \to 0$ is the circle of radius γ centered at the origin. Interestingly, according to the algorithm one defines $u(x)$, when $\sigma(x) = 0$, as an arbitrary element of this circle.

(ii) From the definition it follows that the control law (4.4) is a feedback of steepest descent type for the control Lyapunov function $Q(x)$. However, we want to point out that we understand a solution of a differential equation in the classical sense as opposed to the sample-and-hold sense in [12], where a different discontinuous stabilizing feedback of steepest descent type for the Brockett integrator was constructed. Moreover, we will show that unlike the feedback controller proposed in [12], the control law (4.4) is continuous along solutions of the closed-loop system for almost all initial points.

(iii) Note that the state feedback (4.4) is not upper semicontinuous as a set-valued mapping. Hence it does not fall into the class of discontinuous state feedbacks [2] that do not stabilize Brockett integrator.

4.3. Properties of the designed control law. Let us verify that all assumption of Theorem 3.1 hold true with

$$C = \{x \in \mathbb{R}^3 \colon \sigma(x) = 0, x_3 \neq 0\}.$$

Then we can conclude that the control law (4.4) stabilizes the Brockett integrator for any $\gamma > 0$ and any initial point $x(0)$ that does not belong to the x_3-axis. Moreover, by choosing $\gamma = \sqrt{\varepsilon}$ and taking into account the fact that $|u(x)| = \gamma$ (see (4.3)) one obtains that the proposed control satisfies the constraint $u_1(x)^2 + u_2(x)^2 \leq \varepsilon$, i.e., it can be made arbitrary small.

Clearly, assumptions 1 and 3 of Theorem 3.1 are satisfied. Since

$$\psi(x) = -|\nabla_u \omega(x, u)|^{-1} \nabla_u \omega(x, u),$$

assumption 5 is also satisfied with $\rho(s) \equiv s$.

PROPOSITION 4.1. *Assumption* 4 *of Theorem* 3.1 *is satisfied in the example under consideration.*

Proof. Introduce a set-valued mapping $G \colon \mathbb{R}^3 \rightrightarrows \mathbb{R}^2$. Define $G(x) = \nabla_u \omega(x, u)$ if $\sigma(x) \neq 0$ and $x_3 \neq 0$,

$$G(x) = \big\{(2x_1, 2x_2)^T, (2x_1 + 2x_2\sigma(x), 2x_2 - 2x_1\sigma(x))^T, \\ (2x_1 - 2x_2\sigma(x), 2x_2 + 2x_1\sigma(x))^T\big\}$$

if $x_3 = 0$, and $G(x) = \{-2|x_3|w \colon |w| = 1\}$, if $\sigma(x) = 0$. By definition (see (4.1) and (4.2)), one has $\nabla_u \omega(x, u) \in G(x)$ for any x. Furthermore, it is easy check that $0 \in G(x)$ iff $x = 0$.

As mentioned above, in the case $\sigma(x) \neq 0$ and $x_3 \neq 0$, the set of limit points of $\nabla_u \omega(x, u)$ (see (4.5) and (4.6)) as $\sigma(x) \to 0$ is the circle of radius $2|x_3|$ centered at the origin. Note also that in the same case the set of limit points of $\nabla_u \omega(x, u)$ as $x_3 \to 0$ consists of two points:

$$(2x_1 + 2x_2\sigma(x), 2x_2 - 2x_1\sigma(x))^T, \quad (2x_1 - 2x_2\sigma(x), 2x_2 + 2x_1\sigma(x))^T.$$

Thus, in the case when $\sigma(x) = 0$ or $x_3 = 0$ the set $G(x)$ consists of $\nabla_u \omega(x, u)$ and all limit points of $\nabla_u \omega(y, u)$ as $y \to x$. Therefore it is easy to verify that the set-valued mapping G is upper semicontinuous, i.e., for any x and any open set V such that $G(x) \subset V$ there exists $\delta > 0$ such that for any y with $|y - x| < \delta$ one has $G(y) \subset V$.

Arguing by reductio ad absurdum, suppose that assumption 4 does not hold true. Then there exists $\Delta > 0$, $r > 0$, and a sequence $\{x^{(n)}\}$ such that

$$Q(x^{(n)}) \geq \Delta, \quad |x^{(n)}| \leq r, \quad |\nabla_u \omega(x^{(n)}, u)| \leq \frac{1}{n}.$$

Consequently, there exists a subsequence, which we denote again by $\{x^{(n)}\}$, converging to some x^*. Since the function Q is continuous and $Q(x) = 0$ iff $x = 0$, then $Q(x^*) \geq \Delta$ and $x^* \neq 0$, which implies $0 \notin G(x^*)$. Hence applying the upper semicontinuity of the set-valued mapping G one gets that there exists $a > 0$ and $\delta > 0$ such that

$$\inf_{y \in G(x)} |y| > a \quad \forall x \in \mathbb{R}^3 \colon |x - x^*| < \delta.$$

Therefore for all sufficiently large n one has $\inf\{|y| \colon y \in G(x^{(n)})\} > a$, which contradicts the fact that $|\nabla_u \omega(x^{(n)}, u)| \to 0$ as $n \to \infty$, since $\nabla_u \omega(x^{(n)}, u) \in G(x^{(n)})$ for all n. ∎

It remains to check that assumption 2 holds true, i.e., to verify that the closed-loop system (1.1), (4.4) has a solution for any initial data $x(0) \in \mathbb{R}^n \setminus C$ and $x(t) \notin C$ for all $t \in \mathbb{R}_+$. We prove a stronger assertion that, in particular, implies that for any $x(0) \notin C$ there exists a classical (i.e., continuously differentiable) solution of the closed-loop system.

PROPOSITION 4.2. *Let $\sigma(x(0)) \neq 0$. Then a solution $x(t)$ of the closed-loop system* (1.1), (4.4) *exists on $\mathbb{R}_+$, $x(t) \notin C$ for all $t \geq 0$, and the control does not switch for any $t \in \mathbb{R}_+$ such that $x(t) \neq 0$. Thus, $x(t)$ is a classical solution of the system* (1.1), (4.4) *either on $\mathbb{R}_+$ or on some finite time interval $[0, t_0)$, $t_0 > 0$. In the latter case, $x(t) \to 0$ as $t \to t_0$, and $x(t)$ is an absolutely continuous solution of* (1.1), (4.4) *that is continuously differentiable on $\mathbb{R}_+ \setminus \{t_0\}$.*

Proof. Let $x_3(0) = 0$. Then the closed-loop system takes the form

$$\dot{x}_1 = -\gamma x_1 / \sigma(x), \quad \dot{x}_2 = -\gamma x_2 / \sigma(x), \quad \dot{x}_3 = 0.$$

Hence, obviously, a solution of this system exists on $\mathbb{R}_+$, $x_3(t) = 0$ for any $t \in \mathbb{R}_+$, and the control does not switch.

Let, now, $x_3(0) \neq 0$. Then the closed-loop system takes the form

$$\dot{x}_1 = -\gamma |\nabla_u \omega(x, u)|^{-1} \frac{\partial \omega}{\partial u_1}(x, u), \quad \dot{x}_2 = -\gamma |\nabla_u \omega(x, u)|^{-1} \frac{\partial \omega}{\partial u_2}(x, u), \tag{4.7}$$

$$\dot{x}_3 = -\gamma |\nabla_u \omega(x, u)|^{-1} \left(x_1 \frac{\partial \omega}{\partial u_2}(x, u) - x_2 \frac{\partial \omega}{\partial u_1}(x, u) \right), \tag{4.8}$$

where $\nabla_u \omega(x, u)$ has the form (4.5), (4.6). Clearly, a continuously differentiable solution $x(t)$ of the system (4.7), (4.8) exists at least on some finite time interval. Denote by $[0, t_0)$ the maximal interval of existence of this solution. Since the function Q is radially unbounded and by the definition of the control law (4.3) one has

$$\frac{d}{dt} Q(x(t), t) \leq -\gamma |\nabla_u \omega(x, u)| < 0 \quad \forall t \in [0, t_0),$$

and then $x(t)$ is bounded on $[0, t_0)$. Therefore, either $t_0 = +\infty$ and, thus, $x(t)$ is a continuously differentiable solution of the closed-loop system that is defined and bounded on $\mathbb{R}_+$, and the control does not switch, or at least one of the functions $x_3(t)$ and $\sigma(x(t))$ tends to zero as $t \to t_0$.

Let us show that $x_3(t) \to 0$ as $t \to t_0$ iff $\sigma(x(t)) \to 0$ as $t \to t_0$. In other words, $t_0 < +\infty$ iff the trajectory $x(t)$ reaches the origin at $t = t_0$, which yields the required result. Note also that if t_0 is finite, and $x(t) \to 0$ as $t \to t_0$, then the control switches to zero at time $t = t_0$ and $x(t) \equiv 0$ for any $t \geq t_0$. Thus, the solution $x(t)$ of the closed-loop system (1.1), (4.4) is absolutely continuous on $\mathbb{R}_+$ and continuously differentiable on $\mathbb{R}_+ \setminus \{t_0\}$.

Let $x_3(0) > 0$, and suppose that $t_0 < +\infty$ and $x_3(t) \to 0$ as $t \to t_0$. The cases when $x_3(0) < 0$ or $\sigma(x(t)) \to 0$ as $t \to t_0$ can be considered in the same way.

It is clear that $x_3(t) > 0$ for all $t \in [0, t_0)$. Choose $\varepsilon > 0$. Then there exists $\delta > 0$ such that $0 < x_3(t) < \varepsilon/2$ for any $t \in [t_0 - \delta, t_0)$. Observe that for the closed-loop system one has

$$\dot{x}_3 = -\frac{2\gamma\sigma^2(x)}{|\nabla_u \omega(x,u)|}(2x_3 - \sigma(x)), \quad \frac{d}{dt}\sigma(x) = -\frac{2\gamma}{|\nabla_u \omega(x,u)|}(\sigma(x) - x_3). \tag{4.9}$$

Therefore there exists $s \in [t_0 - \delta, t_0)$ such that $\sigma(s) < \varepsilon$, because otherwise from (4.9) it follows that $\dot{x}_3(t) > 0$ on $[t_0 - \delta, t_0)$ or, equivalently, the function x_3 is strictly increasing on $[t_0 - \delta, t_0)$, which contradicts the fact that $x_3(t) \to 0$ as $t \to t_0$.

Let us check that $\sigma(x(t)) < \varepsilon$ for any $t \in [s, t_0)$. Arguing by reductio ad absurdum, suppose that there exists $\bar{t} \in (s, t_0)$ such that $\sigma(x(\bar{t})) \geq \varepsilon$. Denote

$$\tau = \inf\big\{t \in (s, t_0) \colon \sigma(x(t)) = \varepsilon\big\}.$$

Clearly, $\tau > s$, $\sigma(x(\tau)) = \varepsilon$, and for any $t \in [s, \tau)$ one has $\sigma(x(t)) < \varepsilon$. Hence due to the continuity of $\sigma(x(t))$ there exists $\xi \in [s, \tau)$ such that $\varepsilon/2 < \sigma(x(t)) < \varepsilon$ for all $t \in (\xi, \tau)$. Therefore with the use of (4.9) one gets that $\dot{\sigma}(x(t)) < 0$ on (ξ, τ), which implies that $\sigma(x(t))$ is strictly decreasing on (ξ, τ) and, thus, $\sigma(x(\tau)) < \varepsilon$, which contradicts the definition of τ. Hence $\sigma(x(t)) < \varepsilon$ for any $t \in [s, t_0)$. Since $\varepsilon > 0$ is arbitrary, $\sigma(x(t)) \to 0$ as $t \to t_0$. ∎

Thus, the designed control law (4.4) stabilizes the Brockett integrator for any $\gamma > 0$ and any initial point $x(0)$ that does not lie on the x_3-axis. Moreover, for any such initial point the control does not switch and, thus, is continuous along the solutions of the closed-loop system. Therefore for any $x(0) \notin C$ there exists a unique classical (i.e., continuously differentiable) solution of the closed-loop system. Finally, according to Remark 2 the maximum value of the control can be made arbitrary small.

Remark 5. Let us discuss the case when when the initial point lies on the x_3-axis. According to the control algorithm(4.4) one chooses $u(x(0)) = \gamma v$ for an arbitrary $v \in \mathbb{R}^2$ such that $|v| = 1$. The closed-loop system at this point takes the form

$$\dot{x}_1 = \gamma v_1 \quad \dot{x}_2 = \gamma v_2 \quad \dot{x}_3 = 0.$$

Thus, according to the algorithm, the control input "pushes the point off the x_3-axis," and then the control switches. However, it is not obvious whether an absolutely continuous solution of the closed-loop system (1.1), (4.4) with the initial point $x(0)$ lying on the x_3-axis exists. If such a solution exists, then the proposed control law (4.4) stabilizes the Brockett integrator for an arbitrary initial data.

Since the set of limit points of the control (4.4) as $\sigma(x) \to 0$ is the circle with radius γ centered at the origin, and $u(x) \to \gamma v$ as $\sigma(x) \to 0$ if the limit is taken along

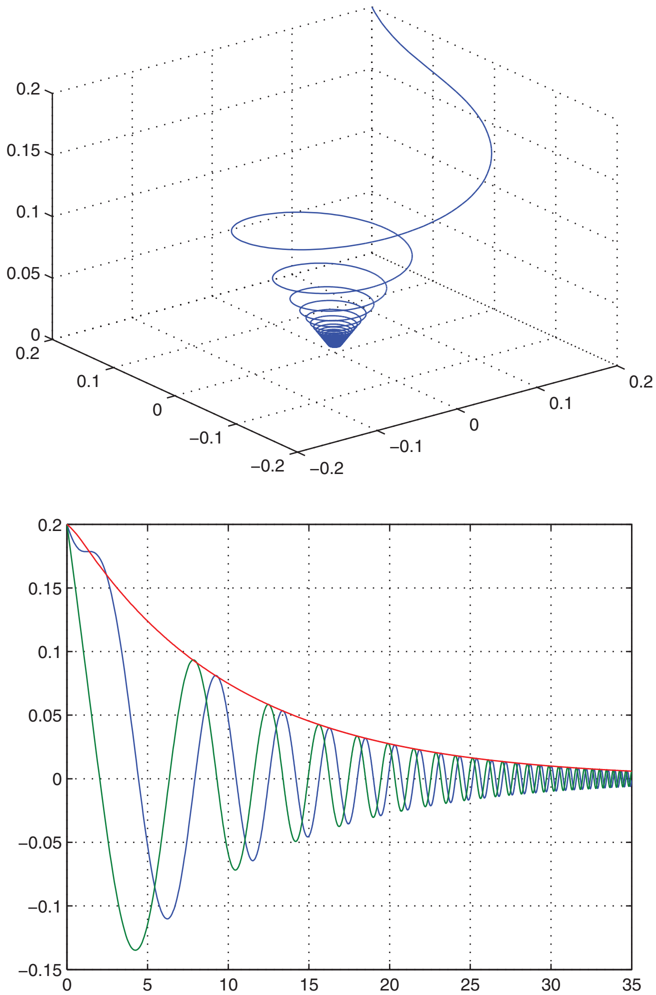

FIG. 1. *Simulation results of the closed-loop system.*

the ray $(\alpha v_1, \alpha v_2, x_3)$, $\alpha \geq 0$, it is natural to expect that a solution of the closed-loop system (1.1), (4.4) exists. However, the proof of the existence of a solution is outside the scope of this article and is left for future research.

4.4. Simulation. Simulation of the closed-loop system with the following parameters was performed: $\gamma = 0.1$ and $x(0) = (0.2, 0.2, 0.2)$. Simulation results demonstrate convergence of the trajectory to the origin; see Figure 1.

5. Energy control of a vibrating string. Theorem 3.1 furnishes sufficient conditions for the convergence of the nonsmooth SG algorithm (3.4). However, in some important examples assumption 4 of this theorem is invalid. In this section, we present such an example and demonstrate that even in this case one can prove that the control goal

$$\lim_{t \to \infty} Q(x(t), t) = 0$$

is achieved with the use the Krasovskii–LaSalle invariance principle.

Consider an undamped vibrating string (see, e.g., [29, 30]). The equation of motion of this string can be written in the form

$$\ddot{r} + \omega_0^2(1 + K|r|^2)r = u, \tag{5.1}$$

where $r = (x, y)$ is the displacement of an element of the string in the xy-plane that is perpendicular to the string, $u = (u_1, u_2) \in \mathbb{R}^2$ represents the forcing term, $K > 0$ is a nonlinear coefficient that takes into account the finite stretching of the string, and $\omega_0 = k\sqrt{T_0/\mu}$ with T_0 being the average tension of the string, μ its linear mass density and $k = 2\pi/\lambda$, where λ is the wavelength.

The system (5.1) can be written in the form

$$\dot{q} = p, \quad \dot{p} = -\omega_0^2\big(1 + K(q_1^2 + q_2^2)\big)q + u, \tag{5.2}$$

where $q = (q_1, q_2) = r$ and $p = (p_1, p_2) = \dot{r}$. The Hamiltonian for the system (5.2) has the form

$$H(q, p) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\omega_0^2}{2}(q_1^2 + q_2^2) + \frac{\omega_0^2}{4}K(q_1^2 + q_2^2)^2.$$

We pose the control problem as finding the control law $u = u(q, p)$, which ensures the objective

$$H(q, p) \to H^* \quad \text{as} \quad t \to +\infty, \tag{5.3}$$

where $H^* \geq 0$ is prespecified. Thus, the control objective is to reach a required energy level H^*.

Introduce the following nonsmooth goal function:

$$Q(q, p) = |H(q, p) - H^*|. \tag{5.4}$$

It is easy to see that the function Q is locally Lipschitz continuous and Hadamard directionally differentiable. For any $h \in \mathbb{R}^4$ its directional derivative has the form

$$Q'(q, p; h) = \operatorname{sign}(H(q, p) - H^*)\Big(\omega_0^2\big(1 + K(q_1^2 + q_2^2)\big)q_1h_1 + \omega_0^2\big(1 + K(q_1^2 + q_2^2)\big)q_2h_2 + p_1h_3 + p_2h_4\Big)$$

in the case $H(q, p) \neq H^*$, and

$$Q'(q, p; h) = \Big|\omega_0^2\big(1 + K(q_1^2 + q_2^2)\big)q_1h_1 + \omega_0^2\big(1 + K(q_1^2 + q_2^2)\big)q_2h_2 + p_1h_3 + p_2h_4\Big|$$

in the case $H(q, p) = H^*$. If $H(q, p) \neq H^*$, then the function Q is differentiable, and we define

$$\omega(q, p, u) = Q'(q, p)^T F(q, p, u) = \operatorname{sign}(H(q, p) - H^*)\big(p_1u_1 + p_2u_2\big),$$

where $F(q, p, u)$ is the right-hand side of (5.2). In the case $H(q, p) = H^*$, there is no linear in u function $\omega(q, p, u)$ such that

$$Q'(q, p; F(q, p, u)) \leq \omega(q, p, u) \quad \forall q, p, u \in \mathbb{R}^2.$$

However, taking into account Remark 3 we define $\omega(q, p, u) \equiv 0$ for any q and p such that $H(q, p) = H^*$. Then according to the nonsmooth SG algorithm we define the control law as follows:

$$u = -\gamma \nabla_u \omega(q, p, u).$$

Thus, the control law has the form

$$u(q, p) = -\gamma \operatorname{sign}(H(q, p) - H^*)p, \tag{5.5}$$

where $\operatorname{sign}(0) = 0$.

It is easy to see that assumption 4 of Theorem 3.1 is not satisfied in the example under consideration. Therefore, one cannot apply Theorem 3.1 in order to prove the achievement of the control goal (5.3). However, the following result holds true.

Proposition 5.1. *For any $\gamma > 0$ and $q(0), p(0) \in \mathbb{R}^2$ such that $|q(0)| + |p(0)| \neq 0$ and $H(q(0), p(0)) \neq H^*$ there exists an absolutely continuous solution $(q(t), p(t))$ of the system* (5.2), (5.5) *that is defined and bounded on $\mathbb{R}_+$, and the control goal* (5.3) *is achieved. Moreover, if $H^* \neq 0$, then there exists $T > 0$ such that*

$$H(q(t), p(t)) \to H^* \quad \text{as} \quad t \to T, \tag{5.6}$$

i.e., the control goal is achieved in finite time.

Proof. Note that a solution $(q(t), p(t))$ of the system (5.2), (5.5) exists at least on some finite time interval. Furthermore, since

$$\frac{d}{dt} H(q(t), p(t)) = -\gamma \operatorname{sign}(H(q(t), p(t)) - H^*)|p|^2,$$

$H(q(t), p(t)) \leq \max\{H(q(0), p(0)), H^*\}$, which implies that $(q(t), p(t))$ is bounded. Hence either $H(q(t), p(t)) \neq H^*$ for all t, and $(q(t), p(t))$ is a continuously differentiable solution of the system (5.2), (5.5) that is defined and bounded on $\mathbb{R}_+$ or there exists some $T \geq 0$ such that $H(q(t), p(t)) \to H^*$ as $t \to T$. In the latter case, the control switches to zero at time $t = T$, and $(q(t), p(t))$ coincides with a solution of the system (5.2) with $u = 0$, since the total energy $H(q, p)$ is conserved along solutions of the unforced system (5.2). Thus, $(q(t), p(t))$ is an absolutely continuous solution $(q(t), p(t))$ of the system (5.2), (5.5) that is defined and bounded on $\mathbb{R}_+$. It remains to show that the control goal (5.3) is achieved, and in the case $H^* \neq 0$ there exists $T > 0$ such that $H(q(t), p(t)) \to H^*$ as $t \to T$.

Suppose, at first, that $H(q(0), p(0)) > H^*$. Then the closed-loop system takes the form

$$\dot{q} = p, \quad \dot{p} = -\omega_0^2\big(1 + K(q_1^2 + q_2^2)\big)q - \gamma p. \tag{5.7}$$

Clearly, there exists a unique solution $(q_0(t), p_0(t))$ of the system (5.7) satisfying $q_0(0) = q(0)$ and $p_0(0) = p(0)$ that is defined and bounded on $\mathbb{R}_+$ by virtue of the fact that

$$\frac{d}{dt} H(q_0(t), p_0(t)) = -\gamma |p_0(t)|^2 \leq 0 \quad \forall t \in \mathbb{R}_+.$$

Applying the Krasovskii–LaSalle invariance principle to the system (5.7) with $H(q, p)$ as a Lyapunov function one obtains that $(q_0(t), p_0(t)) \to (0, 0)$ as $t \to \infty$. Consequently, $H(q_0(t), p_0(t)) \to 0$ as $t \to \infty$. Therefore $H(q(t), p(t)) \to 0$ as $t \to \infty$ in the case $H^* = 0$, and $H(q(t), p(t)) \to H^*$ as $t \to T$ for some $T > 0$ in the case $H^* > 0$, where $(q(t), p(t))$ is a solution of the closed-loop system (5.2), (5.5).

Suppose, now, that $H(q(0), p(0)) < H^*$. Then the closed-loop system takes the form

$$\dot{q} = p, \quad \dot{p} = -\omega_0^2\bigl(1 + K(q_1^2 + q_2^2)\bigr)q + \gamma p. \tag{5.8}$$

Denote by $(q_0(t), p_0(t))$ a unique solution of the system (5.8) satisfying $q_0(0) = q(0)$ and $p_0(0) = p(0)$ that exists at least on some finite time interval. Also, denote by $[0, t_0)$ the maximal interval of existence of this solution. Observe that

$$\frac{d}{dt} H(q_0(t), p_0(t)) = \gamma |p_0(t)|^2 \geq 0 \quad \forall t \in [0, t_0). \tag{5.9}$$

Hence the function $H(q_0(\cdot), p_0(\cdot))$ is nondecreasing.

Note that if $t_0 < +\infty$, then $H(q_0(t), p_0(t)) \to \infty$ as $t \to t_0$. Therefore, either there exists $T \in (0, t_0)$ such that $H(q_0(t), p_0(t)) < H^*$ for all $t \in [0, T)$ and $H(q_0(T), p_0(T)) = H^*$ or

$$\lim_{t \to t_0} H(q_0(t), p_0(t)) < H^*.$$

In the former case one has $H(q(t), p(t)) \to H^*$ as $t \to T$, where $(q(t), p(t))$ is a solution of (5.2), (5.5), while in the latter case one has $t_0 = +\infty$ and $H(q_0(t), p_0(t)) < H^*$ for all $t \geq 0$. Let us show that the latter case is impossible.

Indeed, denote

$$D = \{(q_0(t), p_0(t)) \colon t \geq 0\}.$$

Clearly, D is a bounded invariant set of the system (5.8). Observe that the function $V(q, p) = H^* - H(q, p)$ is nonnegative and continuous on the set D, and its derivative along solutions of the system (5.8) has the form

$$\frac{d}{dt} V(q, p) = -\gamma |p|^2 \leq 0.$$

Hence V is a Lyapunov function of the system (5.8) on the invariant set D. Therefore, by the Krasovskii–LaSalle invariance principle (see [31, Theorem 6.4]) any solution of (5.8) starting in D converges to the largest invariant set of (5.8) in the set

$$E = \left\{ (q, p) \in \operatorname{cl} D \colon \frac{d}{dt} V(q, p) = 0 \right\},$$

where $\operatorname{cl} D$ is the closure of the set D. Note that $dV(q, p)/dt = 0$ iff $p = 0$, and the only invariant set of the system (5.8) contained in the set $\{(q, p) \in \mathbb{R}^4 \colon p = 0\}$ is the equilibrium point $(0, 0)$. Therefore any solution of (5.8) starting in D must converge to the origin. In particular, one has $(q_0(t), p_0(t)) \to (0, 0)$ as $t \to \infty$. However, from the estimate (5.9) and the fact that $|q(0)| + |p(0)| \neq 0$ it follows that

$$H(q_0(t), p_0(t)) \geq H(q(0), p(0)) > 0,$$

which contradicts the fact that $(q_0(t), p_0(t)) \to (0, 0)$ as $t \to \infty$. Thus, the case when $H(q_0(t), p_0(t)) < H^*$ for all $t \in \mathbb{R}_+$ is impossible, which completes the proof. ∎

Remark 6. Note that one can use the smooth goal function

$$Q(q, p) = \frac{1}{2}\bigl(H(q, p) - H^*\bigr)^2$$

instead of the nonsmooth goal function (5.4) and apply the standard SG algorithm in order to design a control law for the problem under consideration. However, there are no results on the finite-time convergence of the smooth SG algorithm. In contrast, the use of the nonsmooth SG algorithm allows one to guarantee finite-time convergence to any nonzero energy level.

6. Conclusion. It is well known that relaxation of the smoothness condition may greatly improve performance of the control systems (recall variable structure systems). However, many specific problems are still to be examined. The contribution of this paper is twofold. On the one hand, we propose a nonsmooth extension of the SG algorithms that, in turn, extend the classical $L_g V$ control. On the other hand, we present yet another almost global stabilizer for the Brockett integrator. Its additional features are the possibility of stabilization with an arbitrarily small control level and continuity of the control along trajectories of the closed-loop system.

An avenue for further research is testing nonsmooth SG algorithms for various nonlinear control problems.

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