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THE SMALL BALL ASYMPTOTICS IN HILBERT NORM FOR THE KAC-KIEFER-WOLFOWITZ PROCESSES*

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(*Translated by the authors*)

Аннотация. We consider the problem of small ball behavior in L_2-norm for some Gaussian processes of statistical interest. The problem is reduced to the spectral asymptotics for some integral-differential operators. To find these asymptotics we construct the complete asymptotic expansions of some oscillation integrals with slowly varying amplitudes.

Key words. Small ball asymptotics, Gaussian processes, spectral asymptotics, slowly varying functions.

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1. Introduction. The theory of small ball behavior for norms of Gaussian processes is intensively developed in last decades (see, for example, reviews [1], [2]; the latest references can be found at the site [3]). The most explored case is that of L_2-norm. Let $X(t)$, $0 \leqq t \leqq 1$, be a Gaussian process with zero mean and covariance function $\mathcal{G}(t,s) = \mathbf{E}\,X(t)X(s)$, $t,s \in [0,1]$. We put

$$\|X\| = \left(\int_0^1 X^2(t)\,dt\right)^{1/2}.$$

We are interested in sharp asymptotics of $\mathbf{P}\{\|X\| < \varepsilon\}$ as $\varepsilon \to 0$. Theoretically the problem of small deviation asymptotics was solved by Sytaya in [4], but in an implicit way. Then the efforts of many authors starting from [5], [6], [7] were aimed at the simplification of the expression for $\mathbf{P}\{\|X\| < \varepsilon\}$ under different assumptions.

By the well-known Karhunen-Loève expansion, the following distributional equiality holds:

$$\|X\|^2 \stackrel{d}{=} \sum_{k=1}^{\infty} \lambda_k \xi_k^2.$$

Here ξ_k, $k \in \mathbf{N}$, are independent standard Gaussian random variables (r.v.), while $\lambda_k > 0$ ($k \in \mathbf{N}$, $\sum \lambda_k < \infty$) are eigenvalues of the integral operator with the kernel $\mathcal{G}(s,t)$. Thus, the original problem is reduced to the description of the asymptotic behavior of $\mathbf{P}\{\sum_{k=1}^{\infty} \lambda_k \xi_k^2 \leqq \varepsilon^2\}$ as $\varepsilon \to 0$. The main difficulty is that the explicit formulas for eigenvalues are known only for a limited number of processes (see [8], [9]). If one knows sufficiently precise asymptotics for λ_k then it is possible to obtain the small ball asymptotics up to a constant using well-known comparison principle of Wenbo Li.

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Proposition 1 (see [8], [10]). *Let ξ_k be a sequence of independent standard Gaussian r.v. Suppose that λ_k and $\widetilde{\lambda}_k$ are two positive non-increasing summable sequences such that $\prod \widetilde{\lambda}_k/\lambda_k < \infty$. Then*

$$\mathbf{P}\Big\{\sum_{k=1}^{\infty}\lambda_k\xi_k^2<\varepsilon^2\Big\}\sim\mathbf{P}\Big\{\sum_{k=1}^{\infty}\widetilde{\lambda}_k\xi_k^2<\varepsilon^2\Big\}\cdot\Big(\prod_{k=1}^{\infty}\frac{\widetilde{\lambda}_k}{\lambda_k}\Big)^{1/2},\qquad\varepsilon\to 0. \tag{1}$$

In papers [11], [12] there was selected the concept of the *Green Gaussian process.* For such process the covariance function $\mathcal{G}(s,t)$ is the Green function for an ordinary differential operator. This allows to study asymptotics for λ_k using the methods of spectral theory of ODEs, originated from the classical works of G. Birkhoff [13], [14] and J. D. Tamarkin [15], [16] (further development of this theory can be found in [17]).

The sharp asymptotics of small ball probabilities in L_2-norm with various weights for a large class of particular processes were calculated in papers [18], [19] and [20] with the approach of [11], [12]. See in this connection also [21], [22], [23].

In the paper [24] the spectrum perturbation of the covariance operator was considered under a finite-dimensional perturbation of the Gaussian process. It was shown that for "non-critical" perturbations the eigenvalues $\widetilde{\lambda}_k$ of the perturbed operator are asymptotically close to the original ones such that $\prod \widetilde{\lambda}_k/\lambda_k < \infty$. A similar result was proved in [25] for a special class of operators.

We consider the small deviation problem in $L_2[0,1]$ for Gaussian processes $X^{(1)}$, $X^{(2)}, X^{(3)}$ with zero mean and covariance functions of the form

$$G_1(s,t)=G(s,t)-h_1(s)h_1(t), \tag{2}$$
$$G_2(s,t)=G(s,t)-h_2(s)h_2(t), \tag{3}$$
$$G_3(s,t)=G(s,t)-h_1(s)h_1(t)-h_2(s)h_2(t). \tag{4}$$

Here $G(s,t)=\min(s,t)-st$ is the Green function of the boundary value problem

$$\mathcal{L}u:=-u''=\lambda u,\quad u(0)=u(1)=0,$$

and

$$h_1(t)=\varphi(\Phi^{-1}(t)),\quad h_2(t)=\varphi(\Phi^{-1}(t))\,\frac{\Phi^{-1}(t)}{\sqrt{2}},$$

where

$$\varphi(s)=\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{s^2}{2}\Big),\quad \Phi(t)=\int_{-\infty}^{t}\varphi(s)\,ds$$

are the standard normal density and distribution function, respectively.

The processes $X^{(1)}, X^{(2)}, X^{(3)}$ appear as limiting ones when building goodness-of-fit tests of ω^2-type for testing normality with estimated mean and/or variance. In the paper [26] M. Kac, J. Kiefer and J. Wolfowitz proved the convergence of the empirical processes with estimated parameters to the limit process in the sense of finite-dimensional distributions. Analogous results were obtained independently by I. I. Gikhman [27], [28]. Afterwards J. Durbin [29] gave the rigorous proof of weak convergence to the limit, and described the limit processes for a much wider class of empirical processes.

It is well known that eigenfunctions and eigenvalues of the integral operator with the kernel $G(s,t)$ are

$$y_k(t) = \sin(\pi k t), \quad \lambda_k = (\pi k)^{-2}.$$

Let $\lambda_k^{(i)}$ be eigenvalues of integral operators with kernels $G_i(s,t)$, $i = 1,2,3$; $k \in \mathbf{N}$. In the cases (2) and (3) we deal with one-dimensional perturbations of the operator with kernel $G(s,t)$. According to the minimax principle [30, § 9.2], eigenvalues of the perturbed operator and the original ones interlace. Note that both perturbations h_1 and h_2 are "critical" (in terms of our problem this means that $\int h_i(t)(\mathcal{L}h_i)(t)\,dt = 1$, $i = 1,2$). Thus the relation $\prod \lambda_k^{(i)}/\lambda_k < \infty$ does not hold. Moreover, since $\mathcal{L}h_i \notin L_2[0,1]$, the result of [24, Theorem 2] is also not applicable. We construct the eigenvalues asymptotics for the operators (2)–(4) using explicit formulas for Fredholm determinants from [26] and [24]. In this way one can construct the complete asymptotic expansion of $\lambda_k^{(i)}$, but we restrict ourselves to the approximation $\widetilde{\lambda}_k^{(i)}$ such that $\prod \widetilde{\lambda}_k^{(i)}/\lambda_k^{(i)} < \infty$.

Note that the function $h_1(t)$ is even with respect to the point $t = 1/2$. Thus the eigenfunctions odd with respect to the point $t = 1/2$ and corresponding eigenvalues do not change under perturbation (2). For simplicity we denote them $\lambda_{2k}^{(1)} = \lambda_{2k}$, $k \in \mathbf{N}$, despite of the fact that it can break the eigenvalues enumeration in the decreasing order. Similarly, since the function $h_2(t)$ is odd with respect to the point $t = 1/2$, eigenfunctions even with respect to $t = 1/2$ and corresponding eigenvalues do not change under the perturbation (3). We denote them $\lambda_{2k-1}^{(2)} = \lambda_{2k-1}$, $k \in \mathbf{N}$. Besides, one can easily see that $\lambda_{2k}^{(3)} = \lambda_{2k}^{(1)}$ and $\lambda_{2k-1}^{(3)} = \lambda_{2k-1}^{(2)}$. Notice also that the quadratic forms of the perturbed operators (2)–(4) do not exceed the quadratic form of the original operator. Therefore the minimax principle provides $\lambda_{2k-1}^{(1)} \leqq \lambda_{2k-1}$ and $\lambda_{2k}^{(2)} \leqq \lambda_{2k}$.

The equations for the eigenvalues of the integral operator with the kernel $G_3(s,t)$ were derived in [26], see also [24, example 5]. After some transformations of these equations we get that the quantities

$$\omega_{2k-1}^{(1)} := \left(\lambda_{2k-1}^{(1)}\right)^{-1/2}, \quad \omega_{2k}^{(2)} := \left(\lambda_{2k}^{(2)}\right)^{-1/2}$$

are roots of the equations

$$D_1(\omega) := \frac{2\sin(\omega/2)}{\omega}\,\mathcal{C}_1^2(\omega) + \frac{\cos(\omega/2)}{\omega^2} - \frac{4\cos(\omega/2)}{\omega}\,\mathcal{I}_1(\omega) = 0, \tag{5}$$

$$D_2(\omega) := -\frac{\cos(\omega/2)}{\omega}\,\mathcal{C}_2^2(\omega) + \frac{3\sin(\omega/2)}{2\omega^2} - \frac{2\sin(\omega/2)}{\omega}\,\mathcal{I}_2(\omega) = 0, \tag{6}$$

respectively, where

$$\mathcal{C}_1(\omega) = \int_0^{1/2} \Phi^{-1}(t)\cos(\omega t)\,dt, \quad \mathcal{C}_2(\omega) = \int_0^{1/2} (\Phi^{-1}(t))^2\cos(\omega t)\,dt,$$

$$\mathcal{I}_1(\omega) = \int_0^{1/2}\int_0^t \Phi^{-1}(t)\Phi^{-1}(s)\sin(\omega t)\cos(\omega s)\,ds\,dt,$$

$$\mathcal{I}_2(\omega) = \int_0^{1/2}\int_0^t (\Phi^{-1}(t))^2(\Phi^{-1}(s))^2\sin(\omega t)\cos(\omega s)\,ds\,dt.$$

The paper is organized as follows. In Section 2 we calculate the asymptotics for a class of integrals with slowly varying amplitudes (see the definition in Appendix).

Integrals $\mathcal{C}_1(\omega)$, $\mathcal{C}_2(\omega)$, $\mathcal{I}_1(\omega)$ and $\mathcal{I}_2(\omega)$ are particular examples of this class. For these integrals we construct the complete asymptotic expansion with an error estimate. In Section 3 we derive the asymptotic equations for the roots $\omega_{2k-1}^{(1)}$ and $\omega_{2k}^{(2)}$ and calculate their asymptotics (formulas (24) and (25)). In Sections 4 and 5 we apply these results to the problem of small ball asymptotics for the processes $X^{(1)}, X^{(2)}, X^{(3)}$ (see formulas (29), (35) and (36)). In Appendix we prove an auxiliary lemma on the Fredholm determinants and some properties of the function $\Phi^{-1}(t)$.

We use letter C to denote various positive constants which exact values are not important. To indicate that C depends on some parameters, we list them in the parentheses: $C(\dots)$.

2. Asymptotics of integrals with slowly varying amplitudes. Let a function $F(t)$ be defined on the interval $(0, 1/2]$, $F(1/2) = 0$, and let the functions $F_0(t) = F(t)$, $F_{n+1}(t) = tF_n'(t)$, $n \geqq 0$, be slowly varying at zero. We introduce the following notation:

$$\int_{S_N(x_1)} F_M \, d\mu_N := \int_1^{x_1} \cdots \int_1^{x_N} F_M\left(\frac{x_{N+1}}{\omega}\right) \frac{dx_{N+1}}{x_{N+1}} \cdots \frac{dx_2}{x_2}, \qquad N \geqq 1,\ M \geqq 0.$$

THEOREM 1. *We have*

$$\mathcal{C}(\omega) := \int_0^{1/2} F(t) \cos(\omega t)\, dt = \sum_{k=1}^N c_k^{\cos} \frac{F_k(1/\omega)}{\omega} + R_N^{\cos}, \qquad \omega \to \infty, \tag{7}$$

where

$$c_k^{\cos} = -\int_0^\infty \frac{\sin x}{x} \frac{\ln^{k-1} x}{(k-1)!}\, dx, \qquad k \geqq 1,$$

$$R_N^{\cos} = -\int_0^{\omega/2} \frac{\sin x_1}{x_1} \int_{S_N(x_1)} F_{N+1}\, d\mu_N \frac{dx_1}{\omega} + O\left(\frac{L_N(\omega)}{\omega^2}\right). \tag{8}$$

Here $L_N(\omega)$ is a slowly varying function at infinity. Moreover, we have an error estimate:

$$|R_N^{\cos}| \leqq C(F, N) \frac{|F_{N+1}(1/\omega)|}{\omega}. \tag{9}$$

Proof. Integrating by parts we obtain

$$\int_0^{1/2} F(t)\cos(\omega t)\, dt = -\int_0^{\omega/2} F_1\left(\frac{x}{\omega}\right) \frac{\sin x}{x} \frac{dx}{\omega}.$$

In what follows we will need the representation for the function F_M:

$$F_M\left(\frac{x}{\omega}\right) = F_M\left(\frac{1}{\omega}\right) + \int_1^x F_{M+1}\left(\frac{y}{\omega}\right) \frac{dy}{y} \quad \forall M \geqq 0. \tag{10}$$

Using formula (10) with $M = 1$, we arrive at

$$\mathcal{C}(\omega) = -\int_0^\infty \frac{\sin x}{x}\, dx \cdot \frac{F_1(1/\omega)}{\omega} - \frac{1}{\omega} \underbrace{\int_0^{\omega/2} \int_1^x F_2\left(\frac{y}{\omega}\right) \frac{dy}{y} \frac{\sin x}{x}\, dx}_{=R_1} + O\left(\frac{1}{\omega^2}\right).$$

This gives (7) for $N=1$. Then we integrate R_1 by parts and get

$$R_1 = \frac{1-\cos x}{x}\int_1^x F_2\left(\frac{y}{\omega}\right)\frac{dy}{y}\Bigg|_{x=0}^{x=\omega/2} - \int_0^{\omega/2}\frac{1-\cos x}{x^2}F_2\left(\frac{x}{\omega}\right)dx + \int_0^{\omega/2}\frac{1-\cos(x)}{x^2}\int_1^x F_2\left(\frac{y}{\omega}\right)\frac{dy}{y}\,dx. \tag{11}$$

To estimate these integrals we need the following statement.

PROPOSITION 2 (see [31, Sect. 1]). *Let $\mathbf{F}(t) > 0$ be a slowly varying function at zero. Then for arbitrary $\alpha > 0$ there exists $\varepsilon > 0$ such that the function $\mathbf{F}(t)t^\alpha$ increases, and the function $\mathbf{F}(t)t^{-\alpha}$ decreases in ε-neighbourhood of zero.*

Let $\alpha > 0$. We take ε such that Proposition 2 holds. Then for ω large $(1/\omega < \varepsilon)$ we have the estimates

$$\left|\mathbf{F}\left(\frac{y}{\omega}\right)\right|\left(\frac{y}{\omega}\right)^\alpha \leqq \left|\mathbf{F}\left(\frac{1}{\omega}\right)\right|\omega^{-\alpha} \quad \text{if } y \in (0,1], \tag{12}$$

$$\left|\mathbf{F}\left(\frac{y}{\omega}\right)\right|\left(\frac{\omega}{y}\right)^\alpha \leqq \left|\mathbf{F}\left(\frac{1}{\omega}\right)\right|\omega^{\alpha} \quad \text{if } y \in (1,\varepsilon\omega]. \tag{13}$$

Moreover, by the continuity of $\mathbf{F}$ and (13), we obtain

$$\begin{aligned}\left|\mathbf{F}\left(\frac{y}{\omega}\right)\right|\left(\frac{\omega}{y}\right)^\alpha &\leqq C(\alpha,F)\left|\mathbf{F}\left(\frac{\varepsilon\omega}{\omega}\right)\right|\left(\frac{\omega}{\varepsilon\omega}\right)^\alpha \\ &\leqq C(\alpha,F)\left|\mathbf{F}\left(\frac{1}{\omega}\right)\right|\omega^{\alpha} \quad \text{if } y \in \left[\varepsilon\omega, \frac{\omega}{2}\right].\end{aligned} \tag{14}$$

Let us estimate $\int_1^x F_2(y/\omega)y^{-1}\,dy$. Using the estimate (12) for $\mathbf{F} = F_2$, we obtain for $x \in (0,1]$

$$\left|\int_1^x F_2\left(\frac{y}{\omega}\right)\frac{dy}{y}\right| \leqq \omega^\alpha \int_1^x \left|F_2\left(\frac{y}{\omega}\right)\right|\left(\frac{y}{\omega}\right)^\alpha \frac{dy}{y^{1+\alpha}} \leqq \left|F_2\left(\frac{1}{\omega}\right)\right|\frac{|x^{-\alpha}-1|}{\alpha}. \tag{15}$$

Using the estimates (13) and (14), we obtain for $x \in [1,\omega/2]$

$$\begin{aligned}\left|\int_1^x F_2\left(\frac{y}{\omega}\right)\frac{dy}{y}\right| &\leqq \omega^{-\alpha}\int_1^x \left|F_2\left(\frac{y}{\omega}\right)\right|\left(\frac{\omega}{y}\right)^\alpha \frac{dy}{y^{1-\alpha}} \\ &\leqq C(\alpha,F)\left|F_2\left(\frac{1}{\omega}\right)\right|\frac{|x^{\alpha}-1|}{\alpha}.\end{aligned} \tag{16}$$

We substitute the estimates (15) and (16) with $\alpha = 1/2$ into the expression (11) and obtain the estimate (9) for $N=1$. For $N>1$ we proceed by induction. Namely, we substitute formula (10) with $M=N$ into (8) and estimate the remainder term using Proposition 2. This completes the proof of Theorem 1.

THEOREM 2. *We have*

$$\mathcal{S}(\omega) := \int_0^{1/2} F(t)\sin(\omega t)\,dt = \frac{F(1/\omega)}{\omega} + \sum_{k=1}^N c_k^{\sin}\frac{F_k(1/\omega)}{\omega} + R_N^{\sin}, \quad \textit{as } \omega \to \infty,$$

where

$$
\begin{aligned}
c_k^{\sin} &= -\int_0^1 \frac{1-\cos x}{x}\frac{\ln^{k-1}x}{(k-1)!}\,dx + \int_1^\infty \frac{\cos x}{x}\frac{\ln^{k-1}x}{(k-1)!}\,dx, \qquad k \geqq 1,\\
R_N^{\sin} &= -\int_0^1 \frac{1-\cos(x_1)}{x_1}\int_{S_N(x_1)} F_{N+1}\,d\mu_N\,\frac{dx_1}{\omega}\\
&\quad + \int_1^{\omega/2}\frac{\cos x_1}{x_1}\int_{S_N(x_1)} F_{N+1}\,d\mu_N\,\frac{dx_1}{\omega} + O\Big(\frac{L_N(\omega)}{\omega^2}\Big).
\end{aligned}
$$

Here $L_N(\omega)$ is a slowly varying function at infinity.

Moreover, we have an error estimate:

$$
|R_N^{\sin}| \leqq C(F,N)\frac{|F_{N+1}(1/\omega)|}{\omega}.
$$

The proof is similar to the proof of Theorem 1.

Remark 1. The following integrals can be reduced to $\mathcal{C}(\omega)$ and $\mathcal{S}(\omega)$:

$$
\begin{aligned}
\int_0^{1/2}\int_0^{\tau} F(t)F(\tau)\sin(\omega t)\sin(\omega\tau)\,dt\,d\tau &= \frac{\mathcal{S}^2(\omega)}{2},\\
\int_0^{1/2}\int_0^{\tau} F(t)F(\tau)\cos(\omega t)\cos(\omega\tau)\,dt\,d\tau &= \frac{\mathcal{C}^2(\omega)}{2}.
\end{aligned}
$$

Theorem 3. *We have*:

$$
\begin{aligned}
\mathcal{I}(\omega) := \int_0^{1/2}\int_0^{\tau} F(t)F(\tau)\sin(\omega\tau)\cos(\omega t)\,dt\,d\tau &= \frac{1}{2\omega}\int_0^{1/2} F^2(t)\,dt\\
+ \sum_{n=2}^{N}\sum_{\substack{k+m=n,\\ k,m\geqq 1}} a_{k,m}\frac{F_k(1/\omega)F_m(1/\omega)}{\omega^2} &+ R_N^{sc}, \qquad \text{as } \omega\to\infty
\end{aligned}
\tag{17}
$$

where

$$
a_{k,m} = -\int_0^\infty \frac{\sin x}{x}\frac{\ln^{k-1}x}{(k-1)!}\int_x^\infty \frac{\cos y}{y}\frac{\ln^{m-1}y}{(m-1)!}\,dy\,dx.
$$

Moreover, we have an error estimate

$$
|R_N^{sc}| \leqq C(F,N)\sum_{\substack{i+j=N+1,\\ i,j\geqq 1}} \frac{|F_i(1/\omega)F_j(1/\omega)|}{\omega^2}.
\tag{18}
$$

Proof. We change the order of integration and integrate by parts. This gives

$$
\mathcal{I}(\omega) = \frac{1}{2\omega}\int_0^{1/2} F^2(t)\,dt - \frac{1}{\omega^2}\int_0^{\omega/2} F_1\Big(\frac{x}{\omega}\Big)\frac{\sin x}{x}\int_x^{\omega/2} F_1\Big(\frac{y}{\omega}\Big)\frac{\cos y}{y}\,dy\,dx.
$$

Using formula (10) with $M=1$, we get

$$\begin{aligned}
&\int_0^{\omega/2} F_1\Big(\frac{x}{\omega}\Big)\,\frac{\sin x}{x}\int_x^{\omega/2} F_1\Big(\frac{y}{\omega}\Big)\,\frac{\cos y}{y}\,dy\,dx\\
&\quad= F_1^2\Big(\frac{1}{\omega}\Big)\int_0^{\omega/2}\frac{\sin x}{x}\int_x^{\omega/2}\frac{\cos y}{y}\,dy\,dx\\
&\qquad+ F_1\Big(\frac{1}{\omega}\Big)\underbrace{\int_0^{\omega/2}\frac{\sin x}{x}\int_x^{\omega/2}\frac{\cos y}{y}\int_1^y F_2\Big(\frac{z}{\omega}\Big)\,\frac{dz}{z}\,dy\,dx}_{=R_{11}}\\
&\qquad+ \underbrace{\int_0^{\omega/2}\frac{\sin x}{x}\int_1^x F_2\Big(\frac{z}{\omega}\Big)\,\frac{dz}{z}\int_x^{\omega/2} F_1\Big(\frac{y}{\omega}\Big)\frac{\cos y}{y}\,dy\,dx}_{=R_{12}}\,.
\end{aligned}\tag{19}$$

It is easy to see that

$$\int_0^{\omega/2}\frac{\sin x}{x}\int_x^{\omega/2}\frac{\cos y}{y}\,dy\,dx=\int_0^{\infty}\frac{\sin x}{x}\int_x^{\infty}\frac{\cos y}{y}\,dy\,dx+O\Big(\frac{1}{\omega}\Big).$$

Hence, we obtain formula (17) for $N=2$.

To estimate the integral R_{11} we split it into three terms:

$$\begin{aligned}
R_{11}&=\int_0^1\frac{\sin x}{x}\int_x^1\frac{\cos y}{y}\int_1^y F_2\Big(\frac{z}{\omega}\Big)\,\frac{dz}{z}\,dy\,dx\\
&\quad+\int_0^1\frac{\sin x}{x}\int_1^{\omega/2}\frac{\cos y}{y}\int_1^y F_2\Big(\frac{z}{\omega}\Big)\,\frac{dz}{z}\,dy\,dx\\
&\quad+\int_1^{\omega/2}\frac{\sin x}{x}\int_x^{\omega/2}\frac{\cos y}{y}\int_1^y F_2\Big(\frac{z}{\omega}\Big)\,\frac{dz}{z}\,dy\,dx=:R_{111}+R_{112}+R_{113}.
\end{aligned}$$

Note that the integrand in R_{111} does not change the sign. By (15) and the inequalities

$$\frac{\sin x}{x}\leqq 1,\qquad \cos y\leqq 1,$$

we obtain

$$|R_{111}|\leqq\Big|F_2\Big(\frac{1}{\omega}\Big)\Big|\int_0^1\int_x^1\frac{y^{-\alpha}-1}{\alpha y}\,dy\,dx=C(\alpha)\Big|F_2\Big(\frac{1}{\omega}\Big)\Big|.$$

Next,

$$|R_{112}|\leqq\Big|\int_1^{\omega/2}\frac{\cos y}{y}\int_1^y F_2\Big(\frac{z}{\omega}\Big)\frac{dz}{z}\,dy\Big|.$$

Integrating by parts, we have

$$\begin{aligned}
|R_{112}|\leqq\Big|&\frac{1+\sin y}{y}\int_1^y F_2\Big(\frac{z}{\omega}\Big)\,\frac{dz}{z}\Big|_{y=1}^{y=\omega/2}\\
&-\int_1^{\omega/2}\frac{1+\sin y}{y^2}\Big(F_2\Big(\frac{y}{\omega}\Big)-\int_1^y F_2\Big(\frac{z}{\omega}\Big)\,\frac{dz}{z}\Big)\,dy\Big|.
\end{aligned}$$

Substitution $y=1$ vanishes; by (16), substitution $y=\omega/2$ is $O(|F_2(\frac{1}{\omega})|\cdot\omega^{\alpha-1})$. Using inequalities (16), (13) and (14) for $\mathbf{F}=F_2$, we estimate the last integral as follows:

$$\left|\int_1^{\omega/2}\frac{1+\sin y}{y^2}\left(F_2\left(\frac{y}{\omega}\right)-\int_1^y F_2\left(\frac{z}{\omega}\right)\frac{dz}{z}\right)dy\right|$$
$$\leqq C(\alpha,F)\left|F_2\left(\frac{1}{\omega}\right)\right|\int_1^{\omega/2}\frac{1+\sin y}{y^{2-\alpha}}\,dy\leqq C(\alpha,F)\left|F_2\left(\frac{1}{\omega}\right)\right|.$$

Consider now R_{113}. Integrating by parts we obtain

$$\begin{aligned}R_{113}=&\frac{1-\cos x}{x}\int_x^{\omega/2}\frac{\cos y}{y}\int_1^y F_2\left(\frac{z}{\omega}\right)\frac{dz}{z}\,dy\Big|_{x=1}^{x=\omega/2}\\&+\int_1^{\omega/2}\frac{1-\cos x}{x^2}\int_x^{\omega/2}\frac{\cos y}{y}\int_1^y F_2\left(\frac{z}{\omega}\right)\frac{dz}{z}\,dy\,dx\\&+\int_1^{\omega/2}\frac{(1-\cos x)\cos x}{x^2}\int_1^x F_2\left(\frac{z}{\omega}\right)\frac{dz}{z}\,dx.\end{aligned}$$

Substitution $x=\omega/2$ vanishes; substitution $x=1$ is estimated in terms of R_{112}. Using (16) we estimate the last integral via $C(\alpha,F)|F_2(1/\omega)|$. As for the second term we integrate by parts with respect to y and arrive at

$$\begin{aligned}&\int_1^{\omega/2}\frac{1-\cos x}{x^2}\int_x^{\omega/2}\frac{\cos y}{y}\int_1^y F_2\left(\frac{z}{\omega}\right)\frac{dz}{z}\,dy\,dx\\&=\int_1^{\omega/2}\frac{1-\cos x}{x^2}\left[\frac{1+\sin y}{y}\int_1^y F_2\left(\frac{z}{\omega}\right)\frac{dz}{z}\Big|_{y=x}^{y=\omega/2}\right.\\&\qquad\left.+\int_x^{\omega/2}\frac{1+\sin y}{y^2}\left(\int_1^y F_2\left(\frac{z}{\omega}\right)\frac{dz}{z}-F_2\left(\frac{y}{\omega}\right)\right)dy\right]dx.\end{aligned}$$

This expression is estimated in the same way as R_{112}. As a result, we obtain

$$|R_{11}|\leqq C(\alpha,F)\left|F_2\left(\frac{1}{\omega}\right)\right|.$$

In a similar way we estimate R_{12}. Setting $\alpha=1/2$ we obtain (18) for $N=2$. For $N>2$ we proceed by induction. Namely, we substitute formula (10) into (19) and estimate the remainder term using Proposition 2. This completes the proof of Theorem 3.

3. Asymptotics of eigenvalues. For convenience we denote $\mathcal{F}_0(x):=\Phi^{-1}(x)$, $\mathcal{F}_{n+1}(x):=x\mathcal{F}_n'(x)$, $n\geqq 0$. Note that by Theorem 5 (see Appendix) the functions $F=\mathcal{F}_0$ and $F=\mathcal{F}_0^2$ satisfy the assumptions from the beginning of Section 2. We write down the asymptotics of the integrals $\mathcal{C}_1(\omega)$, $\mathcal{C}_2(\omega)$, $\mathcal{I}_1(\omega)$, $\mathcal{I}_2(\omega)$ using the formulas (7) and (17) for $N=2$ (this precision is sufficient for our goals as one can see from the next Section):

$$
\begin{aligned}
&\mathcal{C}_1(\omega) = -\frac{\pi}{2}\frac{\mathcal{F}_1}{\omega} + \frac{\gamma\pi}{2}\frac{\mathcal{F}_2}{\omega} + O\Big(\frac{|\mathcal{F}_3|}{\omega}\Big),\\
&\mathcal{C}_2(\omega) = -\pi\,\frac{\mathcal{F}_0\mathcal{F}_1}{\omega} + \gamma\pi\,\frac{\mathcal{F}_0\mathcal{F}_2+\mathcal{F}_1^2}{\omega} + O\Big(\frac{|\mathcal{F}_0\mathcal{F}_3|+|\mathcal{F}_1\mathcal{F}_2|}{\omega}\Big),\\
&\mathcal{I}_1(\omega) = \frac{1}{4\omega} + \frac{\pi^3}{8}\,\frac{\mathcal{F}_1\mathcal{F}_2}{\omega^2} + O\Big(\frac{|\mathcal{F}_1\mathcal{F}_3|+\mathcal{F}_2^2}{\omega^2}\Big),\\
&\mathcal{I}_2(\omega) = \frac{3}{4\omega} + \frac{\pi^3}{2}\,\frac{\mathcal{F}_0^2\mathcal{F}_1\mathcal{F}_2+\mathcal{F}_0\mathcal{F}_1^3}{\omega^2} + O\Big(\frac{|\mathcal{F}_0^2\mathcal{F}_1\mathcal{F}_3|+|\mathcal{F}_0\mathcal{F}_1^2\mathcal{F}_2|}{\omega^2}\Big).
\end{aligned}
\tag{20}
$$

Here γ is the Euler constant and all functions $\mathcal{F}_n$ are taken at the point $1/\omega$. We substitute (20) into formulas (5) and (6) and obtain

$$
\begin{aligned}
D_1(\omega) &= \frac{\pi^2}{2\omega^3}\,\mathcal{F}_1^2\Big[\sin\Big(\frac{\omega}{2}\Big) - 2\gamma\sin\Big(\frac{\omega}{2}\Big)\frac{\mathcal{F}_2}{\mathcal{F}_1} - \pi\cos\Big(\frac{\omega}{2}\Big)\frac{\mathcal{F}_2}{\mathcal{F}_1}\\
&\qquad + O\Big(\frac{|\mathcal{F}_3\mathcal{F}_1|+\mathcal{F}_2^2}{\mathcal{F}_1^2}\Big)\Big];\\
D_2(\omega) &= \frac{\pi^2}{\omega^3}\Big(\mathcal{F}_0\mathcal{F}_1\Big)^2\Big[-\cos\Big(\frac{\omega}{2}\Big) + 2\gamma\cos\Big(\frac{\omega}{2}\Big)\frac{\mathcal{F}_0\mathcal{F}_2+\mathcal{F}_1^2}{\mathcal{F}_0\mathcal{F}_1}\\
&\qquad - \pi\sin\Big(\frac{\omega}{2}\Big)\frac{\mathcal{F}_0\mathcal{F}_2+\mathcal{F}_1^2}{\mathcal{F}_0\mathcal{F}_1}\\
&\qquad + O\Big(\frac{(\mathcal{F}_0\mathcal{F}_2)^2+\mathcal{F}_1^4+|\mathcal{F}_0^2\mathcal{F}_1\mathcal{F}_3|+|\mathcal{F}_0\mathcal{F}_1^2\mathcal{F}_2|}{\mathcal{F}_0^2\mathcal{F}_1^2}\Big)\Big].
\end{aligned}
\tag{21}
$$

We recall that $\omega_{2k-1}^{(1)}$ are roots of the equation $D_1(\omega)=0$. Taking into account (21) we rewrite this equation as follows:

$$
\mathrm{tg}\Big(\frac{\omega}{2}\Big) = \pi\,\frac{\mathcal{F}_2}{\mathcal{F}_1} + O\Big(\frac{|\mathcal{F}_3\mathcal{F}_1|+\mathcal{F}_2^2}{\mathcal{F}_1^2}\Big).
\tag{22}
$$

Here all functions $\mathcal{F}_n$ are taken at the $1/\omega$. Since the right hand side of (22) tends to zero as $\omega\to\infty$, there is exactly one root of this equation in a neighbourhood of the point $2\pi k$ for all k sufficiently large.

It was noticed in the Introduction that $\lambda_{2k}^{(1)} = \lambda_{2k}$ and $\lambda_{2k-1}^{(1)} \leqq \lambda_{2k-1}$. Thus $\omega_{2k}^{(1)} = 2\pi k$ and $\omega_{2k-1}^{(1)} \geqq \pi(2k-1)$. These facts and the interlacing property for eigenvalues imply the interval $[(2k-1)\pi,(2k+1)\pi)$ contains just two roots, $\omega_{2k-1}^{(1)}$ and $\omega_{2k}^{(1)}$. Thus, it is the root $\omega_{2k-1}^{(1)}$ of the equation (22) that lies in a neighbourhood of the point $2\pi k$.

Using standard argument, we obtain the asymptotics of $\omega_{2k-1}^{(1)}$ as $k\to\infty$:

$$
\omega_{2k-1}^{(1)} = 2\pi k + 2\pi\,\frac{\mathcal{F}_2}{\mathcal{F}_1} + O\Big(\frac{|\mathcal{F}_3\mathcal{F}_1|+\mathcal{F}_2^2}{\mathcal{F}_1^2}\Big).
$$

Here all functions $\mathcal{F}_n$ are taken at the point $1/2\pi k$. The relations (43)–(46) imply

$$
\begin{aligned}
&\mathcal{F}_1 = \frac{1}{\mathcal{F}_0} + O\Big(\frac{1}{|\mathcal{F}_0|^3}\Big), \quad \mathcal{F}_2 = -\frac{1}{\mathcal{F}_0^3} + O\Big(\frac{1}{|\mathcal{F}_0|^5}\Big),\\
&\mathcal{F}_3 = O\Big(\frac{1}{|\mathcal{F}_0|^5}\Big), \qquad \omega\to\infty,
\end{aligned}
\tag{23}
$$

where all functions $\mathcal{F}_n$ are taken at the point $1/\omega$. Hence, we have

$$\omega_{2k-1}^{(1)} = 2\pi k + \frac{2\pi}{\mathcal{F}_0^2(1/2\pi k)} + O\Big(\frac{1}{\mathcal{F}_0^4(1/2\pi k)}\Big).$$

It is known (see, e.g., [32]) that

$$\mathcal{F}(x) = \Phi^{-1}(x) = -\sqrt{-2\ln x}\cdot\Big[1 + O\Big(\frac{\ln(-\ln x)}{\ln x}\Big)\Big], \qquad x \to 0.$$

Therefore

$$\omega_{2k-1}^{(1)} = 2\pi k + \frac{\pi}{\ln k} + O\Big(\frac{\ln(\ln k)}{\ln^2 k}\Big). \tag{24}$$

Further, $\omega_{2k}^{(2)}$ are roots of the equation $D_2(\omega) = 0$. Taking into account (21) we rewrite this equation as follows:

$$\operatorname{ctg}\Big(\frac{\omega}{2}\Big) = -\pi\,\frac{\mathcal{F}_0\mathcal{F}_2 + \mathcal{F}_1^2}{\mathcal{F}_0\mathcal{F}_1} + O\Big(\frac{(\mathcal{F}_0\mathcal{F}_2)^2 + \mathcal{F}_1^4 + \mathcal{F}_0^2|\mathcal{F}_1\mathcal{F}_3| + |\mathcal{F}_0\mathcal{F}_1^2\mathcal{F}_2|}{\mathcal{F}_0^2\mathcal{F}_1^2}\Big).$$

Here all functions $\mathcal{F}_n$ are taken at the point $1/\omega$. Arguing as above, we see that there is exactly one root $\omega_{2k}^{(2)}$ of this equation in a neighbourhood of $2\pi k + \pi$ for all k sufficiently large. The asymptotics of these roots is as follows:

$$\omega_{2k}^{(2)} = \pi + 2\pi k - \pi\,\frac{\mathcal{F}_0\mathcal{F}_2 + \mathcal{F}_1^2}{\mathcal{F}_0\mathcal{F}_1} + O\Big(\frac{(\mathcal{F}_0\mathcal{F}_2)^2 + \mathcal{F}_1^4 + \mathcal{F}_0^2|\mathcal{F}_1\mathcal{F}_3| + |\mathcal{F}_0\mathcal{F}_1^2\mathcal{F}_2|}{\mathcal{F}_0^2\mathcal{F}_1^2}\Big),$$

Here all functions $\mathcal{F}_n$ are taken at the point $1/\pi(2k+1)$. Finally, by (23) we have

$$\omega_{2k}^{(2)} = \pi + 2\pi k + O\Big(\frac{1}{\mathcal{F}_0^4(1/\pi(2k+1))}\Big) = \pi(2k+1) + O\Big(\frac{1}{\ln^2 k}\Big). \tag{25}$$

4. Small ball asymptotics for the process $X^{(2)}$. Now we apply the obtained results to the problem of small ball asymptotics in L_2-norm for the Gaussian processes $X^{(1)}, X^{(2)}, X^{(3)}$ with zero mean and covariance functions (2)–(4).

The problem for $X^{(2)}$ is the simplest one. We apply the Wenbo Li comparison principle (Proposition 1). As an approximation we take a process for which the eigenvalues of the corresponding integral operator are $\gamma_k = [(2k+1)\pi/2]^{-2}$, $k \in \mathbf{N}$.

It is well known ([8, Theorem 3.4], see also [11, Theorem 6.2]) that

$$\mathbf{P}\Big\{\sum_{k=1}^{\infty}\gamma_k\xi_k^2 < \varepsilon^2\Big\} \sim \frac{4}{\pi^{3/2}}\,\varepsilon^{-1}\exp\Big(-\frac{1}{8\varepsilon^2}\Big). \tag{26}$$

To calculate the constant from (1), we define

$$\mu_1 := \pi^{-2}, \quad \mu_{2k} = \mu_{2k+1} := \big[(2k+1)\pi\big]^{-2}.$$

Then we have

$$\left(\prod_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k^{(2)}}\right)^{1/2}=\left(\prod_{k=1}^{\infty}\frac{\gamma_k}{\mu_k}\right)^{1/2}\left(\prod_{k=1}^{\infty}\frac{\mu_k}{\lambda_k^{(2)}}\right)^{1/2}=\left(\prod_{k=1}^{\infty}\frac{\gamma_k}{\mu_k}\right)^{1/2}\left(\prod_{k=1}^{\infty}\frac{\mu_{2k}}{\lambda_{2k}^{(2)}}\right)^{1/2}.$$

The first infinite product can be easily calculated by the Stirling formula:

$$\left(\prod_{k=1}^{\infty}\frac{\gamma_k}{\mu_k}\right)^{1/2}=\lim_{k\to\infty}\frac{\pi\cdot(3\pi)^2\cdot(5\pi)^2\cdots((2k+1)\pi)^2}{\frac{3\pi}{2}\cdot\frac{5\pi}{2}\cdots\frac{(4k+3)\pi}{2}}=\frac{1}{\sqrt{2}}. \tag{27}$$

Next, we notice that $\mu_{2k}^{-1/2}$ and $(\lambda_{2k}^{(2)})^{-1/2}=\omega_{2k}^{(2)}$ are roots of the following entire functions:

$$M(\omega):=\frac{\cos(\omega/2)}{1-(\omega/\pi)^2}\quad\text{and}\quad\mathcal{D}(\omega):=\omega D_2(\omega),\qquad\omega\in\mathbf{C}.$$

Moreover, these roots are rather close to each other. Note also that $M(0)=\mathcal{D}(0)=1$. Using Lemma 1 (see Appendix), we arrive at

$$\prod_{k=1}^{\infty}\frac{\mu_{2k}}{\lambda_{2k}^{(2)}}=\lim_{\substack{|\omega|=2\pi k,\\ k\to\infty}}\frac{M(\omega)}{\mathcal{D}(\omega)},$$

and we can pass to the limit along the real axis. So using formulas (21) and (23), we have

$$\prod_{k=1}^{\infty}\frac{\mu_{2k}}{\lambda_{2k}^{(2)}}=\lim_{\substack{\omega=2\pi k\\ k\to\infty}}\frac{M(\omega)}{\omega D_2(\omega)}=\lim_{\substack{\omega=2\pi k\\ k\to\infty}}\left(\mathcal{F}_0\left(\frac{1}{\omega}\right)\mathcal{F}_1\left(\frac{1}{\omega}\right)\right)^2=1. \tag{28}$$

Finally, from formulas (1), (26)–(28) we obtain

$$\begin{aligned}\mathbf{P}\{\|X^{(2)}\|<\varepsilon\}&=\mathbf{P}\Big\{\sum_{k=1}^{\infty}\lambda_k^{(2)}\xi_k^2<\varepsilon^2\Big\}\\ &\sim\frac{2\sqrt{2}}{\pi^{3/2}}\,\varepsilon^{-1}\exp\left(-\frac{1}{8\varepsilon^2}\right),\qquad\varepsilon\to 0.\end{aligned} \tag{29}$$

5. Small ball asymptotics for the processes $X^{(1)}$ and $X^{(3)}$. We cannot calculate the sharp constant in small ball asymptotics for the processes $X^{(1)}$ and $X^{(3)}$. To calculate this asymptotics up to a constant we use the following proposition, which is a particular case of [9, Theorem 3.1].

PROPOSITION 3. *Let $\phi(t)$ be a positive, logarithmically convex, twice differentiable and summable function definded on $[1,\infty)$. Then*

$$\mathbf{P}\Big\{\sum_{k=1}^{\infty}\phi(k)\xi_k^2\leqq r\Big\}\sim C\sqrt{\frac{f(u\phi(1))}{I_2(u)}}\exp(I_0(u)+ur),\qquad r\to 0.$$

Here $f(t)=(1+2t)^{-1/2}$ and $u=u(r)$ is a function satisfying

$$\lim_{r\to 0}\frac{I_1(u)+ur}{\sqrt{I_2(u)}}=0, \tag{30}$$

while the functions I_0, I_1, I_2 are defined by formulas

$$I_0(u) = \int_1^\infty \ln f(u\phi(t))\,dt, \quad I_1(u) = \int_1^\infty u\phi(t)(\ln f(u\phi(t)))'\,dt,$$
$$I_2(u) = \int_1^\infty (u\phi(t))^2(\ln f(u\phi(t)))''\,dt.$$

We apply this proposition with the function

$$\phi(t) = (t+\delta+F(t))^{-d}, \qquad d>1.$$

Here $F(t)$ is a function slowly varying and monotonically tending to zero as $t\to\infty$. If $\int_1^\infty F(t)/t\,dt<\infty$, then the product $\prod \phi(k)/(k+\delta)^{-d}$ converges. In this case the asymptotics up to a constant follows from [11, Theorem 6.2]. So we may assume that the integral $\int_1^\infty F(t)/t\,dt$ diverges.

We transform the integral I_1 as follows:

$$\begin{aligned}
I_1 &= -u\int_1^\infty \frac{dt}{2u+(t+\delta+F(t))^d}\\
&= -\frac{1}{2}\int_{1+\delta+F(1)}^\infty \frac{dx}{1+(x/(2u)^{1/d})^d} + \frac{1}{2}\int_1^\infty \frac{F'(t)\,dt}{1+((t+\delta+F(t))/(2u)^{1/d})^d}\\
&= -\frac{\pi(2u)^{1/d}}{2d\sin(\pi/d)} + \frac{1}{2}\int_0^{1+\delta+F(1)} \frac{dx}{1+(x/(2u)^{1/d})^d}\\
&\quad + \frac{1}{2}\int_1^\infty \frac{F'(t)\,dt}{1+((t+\delta+F(t))/(2u)^{1/d})^d}.
\end{aligned}$$

Since both integrands are majorazed by summable functions, we obtain by the Lebesgue Dominated Convergence Theorem that, as $u\to\infty$,

$$\frac{1}{2}\int_0^{1+\delta+F(1)} \frac{dx}{1+(x/(2u)^{1/d})^d} + \frac{1}{2}\int_1^\infty \frac{F'(t)\,dt}{1+((t+\delta+F(t))/(2u)^{1/d})^d} \to \frac{1+\delta}{2}.$$

Thus,

$$I_1(u) = -\frac{\pi(2u)^{1/d}}{2d\sin(\pi/d)} + \frac{\delta+1}{2} + o(1), \qquad u\to\infty.$$

Repeating the same arguments for $I_2(u)$, we obtain, as $u\to\infty$,

$$I_2(u) = 2u^2\int_1^\infty \frac{dt}{(2u+(t+\delta+F(t))^d)^2} = \frac{(d-1)\pi(2u)^{1/d}}{2d^2\sin(\pi/d)} + O(1).$$

Note that the asymptotics of $I_1(u)$ and $I_2(u)$ coincides with the asymptotics of the corresponding integrals in [11]. Therefore, we can choose

(31) $$u = u(r) := \frac{1}{2}\Big(\frac{\pi r^{-1}}{d\sin(\pi/d)}\Big)^{d/d-1}$$

(This function satisfies the assumption (30)).

Consider now $I_0(u)$. Integrating by parts, we get

$$\begin{aligned}
I_0(u) &= -\frac{1}{2}\int_1^\infty \ln\left(1+\frac{2u}{(t+\delta+F(t))^d}\right)dt \\
&= \frac{(1+\delta)}{2}\ln\left(1+\frac{2u}{(1+\delta+F(1))^d}\right) \\
&\quad - ud\int_1^\infty \frac{(t+\delta)(1+F'(t))\,dt}{(2u+(t+\delta+F(t))^d)(t+\delta+F(t))}.
\end{aligned}$$

The last integral can be represented as the sum of four integrals:

$$\begin{aligned}
&-ud\int_1^\infty \frac{dt}{2u+(t+\delta+F(t))^d} - ud\int_1^\infty \frac{F'(t)\,dt}{2u+(t+\delta+F(t))^d} \\
&\quad + ud\int_1^\infty \frac{F(t)\,dt}{(2u+(t+\delta+F(t))^d)(t+\delta+F(t))} \\
&\quad + ud\int_1^\infty \frac{F(t)F'(t)\,dt}{(2u+(t+\delta+F(t))^d)(t+\delta+F(t))} \\
&=: K_1+K_2+K_3+K_4.
\end{aligned}$$

The integral $K_1 = -d\cdot I_1$, so we have

$$K_1 = \frac{\pi(2u)^{1/d}}{2\sin(\pi/d)} + \mathrm{const} + o(1), \qquad u\to\infty.$$

The integrands in K_2 and K_4 are majorized by summable functions, hence, we obtain as $u\to\infty$

$$\begin{aligned}
K_2 &= \frac{d}{2}\int_1^\infty \frac{F'(t)\,dt}{1+((t+\delta+F(t))/(2u)^{1/d})^d} \\
&\to \frac{d}{2}\int_1^\infty F'(t)\,dt = -\frac{d}{2}\cdot F(1) = \mathrm{const}; \\
K_4 &= \frac{d}{2}\int_1^\infty \frac{F(t)F'(t)\,dt}{(1+((t+\delta+F(t))/(2u)^{1/d})^d)(t+\delta+F(t))} \\
&\to \frac{d}{2}\int_1^\infty \frac{F(t)F'(t)\,dt}{(t+\delta+F(t))} = \mathrm{const}.
\end{aligned}$$

Next, we represent the integral K_3 as a sum of four integrals:

$$\begin{aligned}
K_3 &= \frac{d}{2}\int_1^{(2u)^{1/d}} \frac{F(t)}{t}\,dt - \frac{d}{2}\int_1^{(2u)^{1/d}} \frac{F(t)(\delta+F(t))}{t(t+\delta+F(t))}\,dt \\
&\quad - \frac{1}{2u}\,\frac{d}{2}\int_1^{(2u)^{1/d}} \frac{F(t)(t+\delta+F(t))^{d-1}}{1+((t+\delta+F(t))/(2u)^{1/d})^d}\,dt \\
&\quad + \frac{d}{2}\int_{(2u)^{1/d}}^\infty \frac{F(t)\,dt}{(1+((t+\delta+F(t))/(2u)^{1/d})^d)(t+\delta+F(t))} \\
&=: \frac{d}{2}F_{-1}\left((2u)^{1/d}\right) - K_{31} - K_{32} + K_{33},
\end{aligned}$$

where $F_{-1}(x) = \int_1^x (F(t)/t)\,dt$. By the Lebesgue theorem we have

$$K_{31} \to \frac{d}{2}\int_1^\infty \frac{F(t)(\delta+F(t))}{t(t+\delta+F(t))}\,dt = \mathrm{const}.$$

In the integrals K_{32} and K_{33} we change variable $t = (2u)^{1/d} \cdot z$ and obtain:

$$K_{32} = \frac{d}{2} \int_{1/(2u)^{1/d}}^{1} \frac{F((2u)^{1/d}z)\left(z + \frac{\delta + F((2u)^{1/d}z)}{(2u)^{1/d}}\right)^{d-1}}{1 + \left(z + \frac{\delta + F((2u)^{1/d}z)}{(2u)^{1/d}}\right)^{d}}\, dz,$$

$$K_{33} = \int_1^\infty \frac{F((2u)^{1/d}z)\, dz}{\left(1 + \left(z + \frac{\delta + F((2u)^{1/d}z)}{(2u)^{1/d}}\right)^{d}\right)\left(z + \frac{\delta + F((2u)^{1/d}z)}{(2u)^{1/d}}\right)}.$$

The integrands here are bounded, so by the Lebesgue theorem $K_{32} \to 0$ and $K_{33} \to 0$ as $u \to \infty$. Thus, we arrive at

$$K_3 = \frac{d}{2} F_{-1}\big((2u)^{1/d}\big) + \mathrm{const} + o(1), \qquad \text{as } u \to \infty.$$

Whence,

$$\begin{aligned} I_0(u) = {} & \frac{(1+\delta)}{2} \ln\left(1 + \frac{2u}{(1+\delta+F(1))^d}\right) - \frac{\pi(2u)^{1/d}}{2\sin(\pi/d)} \\ & + \frac{d}{2} F_{-1}\left((2u)^{1/d}\right) + \mathrm{const} + o(1). \end{aligned} \tag{32}$$

Note that the function $\exp(F_{-1}(t))$ is slowly varying [31, Theorem 1.2]. Hence, taking into account the relation (31) we have

$$\exp\left(F_{-1}((2u)^{1/d})\right) = \exp\left(F_{-1}(r^{-1/d-1}) + o(1)\right), \qquad r \to 0. \tag{33}$$

We apply Proposition 3 taking into account (31)–(33) and rescaling. This gives the following result.

THEOREM 4. *Consider the form* $\sum_{k=0}^\infty \Lambda_k \xi_k^2$ *with* $\Lambda_k = (\vartheta(k+\delta+F(k)))^{-d}$*, where* $\vartheta > 0$, $\delta > -1$ *and* $d > 1$ *are some constants. Then we have, as* $\varepsilon \to 0$,

$$\begin{aligned} \mathbf{P}\left\{\sum_{k=0}^\infty \Lambda_k \xi_k^2 < \varepsilon^2\right\} \sim C \cdot \varepsilon^\gamma \cdot \exp\bigg(& -\frac{d-1}{2}\left(\frac{\pi}{d\vartheta \sin(\pi/d)}\right)^{d/(d-1)} \cdot \varepsilon^{-2/(d-1)} \\ & + \frac{d}{2} \cdot F_{-1}(\varepsilon^{-2/(d-1)})\bigg), \end{aligned}$$

where

$$\gamma = \frac{2-d-2d\delta}{2(d-1)}, \quad C = C(\vartheta, \delta, d, F) = \mathrm{const}.$$

For the process $X^{(1)}$ we set

$$\Lambda_k := \left[\pi\left(k + \frac{1}{2} + \frac{1}{2\ln(k+1)}\right)\right]^{-2}.$$

Let us check that $\prod \lambda_k^{(1)}/\Lambda_k$ converges. To prove this we define

$$\tau_{2k} := (2\pi k)^{-2}, \quad \tau_{2k-1} := \left[\pi\left(2k + \frac{1}{\ln(k+1)}\right)\right]^{-2}, \qquad k \in \mathbf{N}.$$

We have

$$\left(\prod_{k=1}^{\infty}\frac{\lambda_k^{(1)}}{\Lambda_k}\right)^{1/2}=\left(\prod_{k=1}^{\infty}\frac{\lambda_k^{(1)}}{\tau_k}\right)^{1/2}\left(\prod_{k=1}^{\infty}\frac{\tau_k}{\Lambda_k}\right)^{1/2}$$
$$=\left(\prod_{k=1}^{\infty}\frac{\lambda_{2k-1}^{(1)}}{\tau_{2k-1}}\right)^{1/2}\left(\prod_{k=1}^{\infty}\frac{\tau_k}{\Lambda_k}\right)^{1/2}. \tag{34}$$

The first product in (34) converges, since the relation (24) gives

$$\left(\prod_{k=1}^{\infty}\frac{\lambda_{2k-1}^{(1)}}{\tau_{2k-1}}\right)^{1/2}=\prod_{k=1}^{\infty}\left(1+O\left(\frac{\ln\ln k}{k\ln^2 k}\right)\right)<\infty.$$

The second product can be represented as follows:

$$\left(\prod_{k=1}^{\infty}\frac{\tau_{2k}\tau_{2k-1}}{\Lambda_{2k}\Lambda_{2k-1}}\right)^{1/2}$$
$$=\prod_{k=1}^{\infty}\frac{(2k+1/2+1/(2\ln(2k+1)))(2k-1/2+1/(2\ln(2k))}{2k(2k+1/\ln(k+1))}$$
$$=\prod_{k=1}^{\infty}\left(1+\frac{1}{4k}\left(\frac{1}{\ln(2k+1)}+\frac{1}{\ln(2k)}-\frac{2}{\ln(k+1)}\right)+O\left(\frac{1}{k^2}\right)\right)$$
$$=\prod_{k=1}^{\infty}\left(1+O\left(\frac{1}{k\ln^2 k}\right)\right)<\infty.$$

Applying the Wenbo Li comparison principle and Theorem 4, we obtain

$$\mathbf{P}\left\{\|X^{(1)}\|<\varepsilon\right\}=\mathbf{P}\left\{\sum_{k=1}^{\infty}\lambda_k^{(1)}\xi_k^2<\varepsilon^2\right\}\sim C\cdot\mathbf{P}\left\{\sum_{k=1}^{\infty}\Lambda_k\xi_k^2<\varepsilon^2\right\}$$
$$\sim C\varepsilon^{-1}\ln^{1/2}\left(\frac{1}{\varepsilon}\right)\exp\left(-\frac{1}{8\varepsilon^2}\right),\qquad \varepsilon\to 0. \tag{35}$$

For the process $X^{(3)}$ we set

$$\widetilde{\Lambda}_k:=\left[\pi\left(k+1+\frac{1}{2\ln(k+1)}\right)\right]^{-2},\quad k\in\mathbf{N}.$$

Similarly to (35), we get

$$\mathbf{P}\left\{\|X^{(3)}\|<\varepsilon\right\}=\mathbf{P}\left\{\sum_{k=1}^{\infty}\lambda_k^{(3)}\xi_k^2<\varepsilon^2\right\}\sim C\mathbf{P}\left\{\sum_{k=1}^{\infty}\widetilde{\Lambda}_k\xi_k^2<\varepsilon^2\right\}$$
$$\sim C\varepsilon^{-2}\ln^{1/2}\left(\frac{1}{\varepsilon}\right)\exp\left(-\frac{1}{8\varepsilon^2}\right),\qquad \varepsilon\to 0. \tag{36}$$

6. Appendix.

6.1. The following lemma strengthens the result of [18, Lemma 1.2].

LEMMA 1. *Let sequences ω_k and ρ_k have the same two-term asymptotics*

$$\omega_k\sim c(k+\delta)+a_k,\quad \rho_k\sim c(k+\delta)+b_k,\qquad k\to\infty,$$

where $a_k, b_k \to 0$, *and* $|a_k - b_k|$ *decreases monotonically as* $k \to \infty$. *Suppose, moreover, that*

$$\sum_{k=1}^{\infty} \frac{|a_k - b_k|}{k} < \infty. \tag{37}$$

Then the functions

$$f(\zeta) = \prod_{k=1}^{\infty} \left(1 - \frac{\zeta^2}{\omega_k^2}\right), \quad g(\zeta) = \prod_{k=1}^{\infty} \left(1 - \frac{\zeta^2}{\rho_k^2}\right)$$

have the same behavior up to a constant at infinity. Namely, for $|\zeta| = c(n + \delta + 1/2)$ *we have, as* $n \to \infty$,

$$\frac{f(\zeta)}{g(\zeta)} \rightrightarrows \mathrm{const} = \prod_{k=1}^{\infty} \frac{\rho_k^2}{\omega_k^2}. \tag{38}$$

Proof. We have

$$\frac{f(\zeta)}{g(\zeta)} = \prod_{k=1}^{\infty} \frac{\rho_k^2}{\omega_k^2} \cdot \prod_{k=1}^{\infty} \left(1 + \frac{\omega_k - \rho_k}{\rho_k + \zeta}\right) \cdot \prod_{k=1}^{\infty} \left(1 + \frac{\omega_k - \rho_k}{\rho_k - \zeta}\right). \tag{39}$$

The convergence of the first product in (39) is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty} \frac{|a_k - b_k|}{c(k + \delta) + b_k},$$

which converges by assumption (37). Let $\Re(\zeta) \geqq 0$. Then the second product in (39) converges uniformly. The third product converges uniformly if so does the series

$$\sum_{k=1}^{\infty} \frac{|\omega_k - \rho_k|}{|\rho_k - R|}, \quad \text{where} \quad R = c(n + \delta + 1/2). \tag{40}$$

Note that $|\rho_k - R| \geqq c|n - k + \delta_1|$, where $\delta_1 > 0$. Therefore,

$$\sum_{k=1}^{\infty} \frac{|\omega_k - \rho_k|}{|\rho_k - R|} \leqq \left(\sum_{k \leqq \frac{2}{3}n} + \sum_{\frac{2}{3}n \leqq k \leqq \frac{4}{3}n} + \sum_{k \geqq \frac{4}{3}n} \right) \frac{|a_k - b_k|}{c|n - k + \delta_1|}. \tag{41}$$

The third sum in (41) tends to zero as $n \to \infty$:

$$\sum_{k \geqq \frac{4}{3}n}^{\infty} \frac{|a_k - b_k|}{c|n - k + \delta_1|} \leqq C \sum_{k \geqq \frac{4}{3}n} \frac{|a_k - b_k|}{k} \to 0.$$

The first sum is majorized by the converging series

$$\sum_{k \leqq \frac{2}{3}n} \frac{|a_k - b_k|}{c|n - k + \delta_1|} \leqq C \sum_{k=1}^{\infty} \frac{|a_k - b_k|}{k}.$$

Hence, it converges uniformly and we obtain as $n \to \infty$

$$\sum_{k \leqq \frac{2}{3}n} \frac{|a_k - b_k|}{c|n-k+\delta_1|} \to 0.$$

Since $|a_k - b_k|$ decreases monotonically, we have

$$\sum_{\frac{2}{3}n \leqq k \leqq \frac{4}{3}n} \frac{|a_k - b_k|}{c|n-k+\delta_1|} \leqq 2 \sum_{\frac{2}{3}n \leqq k \leqq n} \frac{|a_k - b_k|}{c|n-k+\delta_1|} \leqq \frac{2}{c} \sum_{m \leqq \frac{1}{3}n} \frac{|a_{m+\frac{n}{3}} - b_{m+\frac{n}{3}}|}{|m+\delta_1|}.$$

The last sum is majorized by the converging series, therefore, it converges uniformly. Thus, the series (40) converges uniformly and we can pass to the limit in (39) as $|\zeta| = c(n+\delta+1/2) \to \infty$. This gives (38) for $\Re(\zeta) \geqq 0$. The proof for $\Re(\zeta) \leqq 0$ is similar, and Lemma 1 follows.

6.2. The properties of $\Phi^{-1}(x)$. We introduce the notation:

$$x = \Phi(y), \qquad y = F_0(x) := \Phi^{-1}(x),$$

and construct the sequence of functions:

$$F_{N+1}(x) := xF_N'(x), \quad N \geqq 0. \tag{42}$$

Denote $\widetilde{F}_N(y) := F_N(x(y))$ and notice that

$$\begin{aligned} \widetilde{F}_1(y) &= x(y)\,\frac{dF(x(y))}{dx} = x(y)\,\frac{dy}{dx}, \\ \widetilde{F}_{N+1}(y) &= x(y)\,\frac{dF_N(x)}{dx} = x(y)\,\frac{dy}{dx}\,\frac{d\widetilde{F}_N(y)}{dy} = \widetilde{F}_1(y)\widetilde{F}_N'(y). \end{aligned} \tag{43}$$

We study the behavior of the functions $\widetilde{F}_N(y)$. First, we consider the function $\widetilde{F}_1(y)$:

$$\begin{aligned} \widetilde{F}_1(y) = x(y)\cdot\frac{dy}{dx} &= \frac{x(y)}{dx/dy} = \frac{\int_{-\infty}^{y} \exp(-t^2/2)\,dt}{\exp(-y^2/2)} \\ &= \exp\left(\frac{y^2}{2}\right)\int_{-\infty}^{y} \exp\left(-\frac{t^2}{2}\right)dt \\ &= \int_0^{\infty} \exp\left(yz - \frac{z^2}{2}\right)dz = -\frac{1}{y}\int_0^{\infty} \exp\left(-u - \frac{u^2}{2y^2}\right)du. \end{aligned}$$

We introduce the auxiliary functions:

$$e_N(y) := \int_0^{\infty} \exp\left(-u - \frac{u^2}{2y^2}\right)u^{2N-2}\,du.$$

LEMMA 2. *The following relations hold*:

$$1. \quad e_N'(y) = \frac{e_{N+1}(y)}{y^3}; \tag{44}$$

$$2. \quad (2N-2)!\left(1 - \frac{N(2N-1)}{y^2}\right) < e_N(y) < (2N-2)!. \tag{45}$$

Proof. 1. It is checked by direct calculation.

2. Taking into account that

$$1-\frac{u^2}{2y^2}<\exp\Big(-\frac{u^2}{2y^2}\Big)<1,$$

we obtain

$$\int_0^\infty \exp(-u)\Big(1-\frac{u^2}{2y^2}\Big)u^{2N-2}\,du<e_N(y)<\int_0^\infty \exp(-u)u^{2N-2}\,du,$$
$$(2N-2)!-\frac{(2N)!}{2y^2}<e_N(y)<(2N-2)!,$$

which provides (45), and Lemma 2 follows.

LEMMA 3. *The following identity holds for the N-th derivative of $\widetilde{F}_1(y)$:*

$$\widetilde{F}_1^{(N)}(y)=\frac{(-1)^{N+1}N!e_1(y)}{y^{N+1}}+\frac{c_2e_2(y)}{y^{N+3}}+\dots+\frac{c_{N+1}e_{N+1}(y)}{y^{3N+1}}, \tag{46}$$

where

$$c_2=c_2(N),\quad c_3=c_3(N),\quad \dots,\quad c_{N+1}=c_{N+1}(N)\ \textit{are some constants.}$$

To prove we proceed by induction using (44).

COROLLARY 1. *The following relation holds as $y\to-\infty$:*

$$\widetilde{F}_1^{(N)}(y)\sim\frac{(-1)^{N+1}N!}{y^{N+1}}. \tag{47}$$

Proof follows from (45) and (46).

LEMMA 4. *$\widetilde{F}_N(y)$ can be represented in the following form:*

$$\widetilde{F}_N(y)=\sum_{\{n_1,\dots,n_N\}}c_{n_1,\cdots,n_N}\widetilde{F}_N^{n_1,\dots,n_N}(y), \tag{48}$$

where

$$\widetilde{F}_N^{n_1,\dots,n_N}(y):=(\widetilde{F}_1(y))^{n_1}(\widetilde{F}_1'(y))^{n_2}\cdots(\widetilde{F}_1^{(N-1)}(y))^{n_N}. \tag{49}$$

Here

$$n_1,\dots,n_N\in\mathbf{N}_0=\{0,1,\dots\};$$
$$1\cdot n_1+2\cdot n_2+\dots+N\cdot n_N=2N-1, \tag{50}$$

and the coefficients in (48) *satisfy the following inequalities*:

$$c_{n_1,\dots,n_N}\geqq 0,\qquad \sum_{\{n_1,\dots,n_N\}}c_{n_1,\dots,n_N}>0. \tag{51}$$

Proof. We proceed by induction on N.

Base case $N=1$: $\widetilde{F}_1(y)=c_{n_1}\widetilde{F}_1^{n_1}$, $n_1=1, c_{n_1}=1$. Properties (49)–(51) evidently hold.

Inductive step: suppose that the statement holds for $\widetilde{F}_N(y)$. We write (50) as follows:

$$1\cdot n_1+\dots+N\cdot n_N+(N+1)\cdot n_{N+1}=2N-1,$$

where $n_{N+1}=0$. By (43) and (48) we have

$$\widetilde{F}_{N+1}(y)=\sum_{\{n_1,\dots,n_N\}} c_{n_1,\dots,n_N}\widetilde{F}_1(y)\frac{d}{dy}\Big[(\widetilde{F}_1(y))^{n_1}(\widetilde{F}_1'(y))^{n_2}\cdots(\widetilde{F}_1^{(N-1)}(y))^{n_N}\Big].$$

Here the coefficients $c_{n_1,\dots,n_N}$ satisfy the condition (51). Differentiating we get

$$\sum_{k=0}^{N} c_{n_1,\dots,n_N} n_k(\widetilde{F}_1)^{n_1+1}\cdots(\widetilde{F}_1^{(k-1)})^{n_k-1}(\widetilde{F}_1^{(k)})^{n_{k+1}+1}\cdots(\widetilde{F}_1^{(N-1)})^{n_N}. \tag{52}$$

Consider a term in the sum (52) corresponding to some k. Let $n_1', n_2', \dots, n_{N+1}'$ be the powers of the corresponding derivatives of $\widetilde{F}_1$ in this term.

If $k=1$, then $n_2'=n_2+1$ and $n_i'=n_i$ for all other i.

If $k\geqq 2$, then $n_1'=n_1+1$, $n_k'=n_k-1$, $n_{k+1}'=n_{k+1}+1$ and $n_i'=n_i$ for all other i.

Therefore, the property (50) holds. Next, all coefficients in (52) are evidently non-negative, and moreover,

$$\sum_{\substack{k=1,\dots,N\\ \{n_1,\dots,n_N\}}} c_{n_1,\dots,n_N}\cdot n_k>0.$$

Thus, the inequalities (51) hold, and Lemma 4 follows.

LEMMA 5. *As $y\to-\infty$ we have*

$$\widetilde{F}_N(y)\sim-\frac{C}{y^{2N-1}}, \tag{53}$$

where $C=C(N)>0$ is a constant.

Proof. For each term of the form (49) with non-zero coefficient $c_{n_1,\dots,n_N}$ we obtain by (47) and (50)

$$\begin{aligned}(\widetilde{F}_1(y))^{n_1}&(\widetilde{F}_1'(y))^{n_2}\cdots(\widetilde{F}_1^{(N-1)}(y))^{n_N}\\ &\sim\left(\frac{(-1)0!}{y^1}\right)^{n_1}\left(\frac{(-1)^2 1!}{y^2}\right)^{n_2}\cdots\left(\frac{(-1)^N(N-1)!}{y^N}\right)^{n_N}\\ &=\underbrace{0!^{n_1}1!^{n_2}\cdots(N-1)!^{n_N}}_{c_N}\frac{(-1)^{n_1+2n_2+\dots+Nn_N}}{y^{n_1+2n_2+\dots+Nn_N}}\\ &=c_N\frac{(-1)^{2N-1}}{y^{2N-1}}=-\frac{c_N}{y^{2N-1}}.\end{aligned}$$

Hence, the relation (53) holds with

$$C=c_N\sum_{\{n_1,\dots,n_N\}} c_{n_1,\cdots,n_N},$$

and Lemma 5 follows.

Definition 1 (see [31, Section 1]). A measurable function $L(x)$ is called slowly varying at infinity, if it is of constant sign on $[A, \infty)$, for some $A > 0$, and for arbitrary $\lambda > 0$ we have

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1.$$

A function $L(x)$ is called slowly varying at zero, if the function $L(1/x)$ is slowly varying at infinity.

For example, the functions $\ln^\alpha x$, $\alpha \in \mathbf{R}$, are slowly varying at infinity.

Theorem 5. *$F_N(x)$ are slowly varying functions at zero for all $N \geqq 0$.*

Proof. According to [31, Section 1] it is sufficient to prove, that $F_N(x)$ is of constant sign in a neighbourhood of $x = 0$ (or equivalently, $\widetilde{F}_N(y)$ is of constant sign in a neighbourhood of $y = -\infty$), and

$$\lim_{y \to -\infty} \frac{\widetilde{F}_{N+1}(y)}{\widetilde{F}_N(y)} = 0.$$

Both statements easily follow from Lemma 5. This completes the proof of Theorem 5.

Remark 2. Theorem 5 remains valid for the sequence (42) constructed from the function $F_0(x) = (\Phi^{-1}(x))^n$, $n \in \mathbf{N}$.

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