# HOMOGENIZATION FOR NON-SELF-ADJOINT
# LOCALLY PERIODIC ELLIPTIC OPERATORS

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ABSTRACT. We study the homogenization problem for matrix strongly elliptic operators on $L_2(\mathbb{R}^d)^n$ of the form $\mathcal{A}^\varepsilon = -\operatorname{div} A(x, x/\varepsilon)\nabla$. The function $A$ is Lipschitz in the first variable and periodic in the second. We do not require that $A^* = A$, so $\mathcal{A}^\varepsilon$ need not be self-adjoint. In this paper, we provide, for small $\varepsilon$, two terms in the uniform approximation for $(\mathcal{A}^\varepsilon - \mu)^{-1}$ and a first term in the uniform approximation for $\nabla(\mathcal{A}^\varepsilon - \mu)^{-1}$. Primary attention is paid to proving sharp-order bounds on the errors of the approximations.

## 1. INTRODUCTION

Homogenization dates back to the late 1960s, and for more than fifty years it has become a well-established theory. In the simplest case, homogenization deals with asymptotic properties of solutions to differential equations with oscillating coefficients. Given a periodic (with period 1 in each variable) uniformly bounded and uniformly positive definite function $A \colon \mathbb{R}^d \to \mathbb{C}^{d \times d}$, consider the differential equation

$$(1.1) \qquad -\operatorname{div} A(\varepsilon^{-1}x)\nabla u_\varepsilon - \mu u_\varepsilon = f,$$

where $\varepsilon > 0$, $\mu \in \mathbb{C} \setminus \mathbb{R}_+$ and $f \in L_2(\mathbb{R}^d)$. The coefficients of the equation are $\varepsilon$-periodic and hence rapidly oscillate if $\varepsilon$ is small. In homogenization theory one is interested in studying the asymptotic behavior of $u_\varepsilon$ as $\varepsilon$ becomes smaller. It is a basic fact that, after passing to a subsequence if necessary, $u_\varepsilon$ converges to the solution $u_0$ of the differential equation

$$(1.2) \qquad -\operatorname{div} A^0 \nabla u_0 - \mu u_0 = f$$

with constant $A^0$. Since, in applications, the elliptic operator on the left side of (1.1) usually describes a physical process in a highly heterogeneous medium, this means that, in certain aspects, the process evolves very similar to that in a homogeneous medium.

It is a basic fact about homogenization theory that $u_\varepsilon$ converges to $u_0$ in $L_2(\mathbb{R}^d)$; we refer the reader to [BLP78], [BP84] or [ZhKO93] for the details. Stated differently, the resolvent of $-\operatorname{div} A(\varepsilon^{-1}x)\nabla$ converges in the strong operator topology to the resolvent of $-\operatorname{div} A^0\nabla$. In [BSu01] (see also [BSu03]), Birman and Suslina proved that, in fact, the resolvent converges in norm. Moreover, they found a sharp-order bound on the rate of convergence. Since that time there have been a

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number of interesting further results in this direction – see [Gri04], [Gri06], [Zh05], [ZhP05], [B08], [KLS12], [Su13$_1$], [Su13$_2$], [ChC16] and [ZhP16], to name a few.

Here we focus on a more general problem than the periodic one in (1.1). Let $A = \{A_{kl}\}$ with $A_{kl} \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}^{n \times n}$ being uniformly bounded functions that are Lipschitz in the first variable and periodic in the second (see Section 3 for a precise definition). Consider the operator $\mathcal{A}^\varepsilon$ on the complex space $L_2(\mathbb{R}^d)^n$ given by

$$\mathcal{A}^\varepsilon = -\operatorname{div} A(x, \varepsilon^{-1}x)\nabla = -\sum_{k,l=1}^d \partial_k A_{kl}(x, \varepsilon^{-1}x)\partial_l.$$

The coefficients now depend not only on the "fast" variable, $\varepsilon^{-1}x$, but also on the "slow" one, $x$. Assume that, for all $\varepsilon$ in some neighborhood of 0, the operator $\mathcal{A}^\varepsilon$ is coercive and furthermore the constants in the coercivity bound are independent of $\varepsilon$. Then $\mathcal{A}^\varepsilon$ is strongly elliptic for such $\varepsilon$ and there is a sector containing the spectrum of $\mathcal{A}^\varepsilon$. In this paper, we will obtain approximations for $(\mathcal{A}^\varepsilon - \mu)^{-1}$ and $\nabla(\mathcal{A}^\varepsilon - \mu)^{-1}$ (with $\mu$ outside the sector) in the operator norm and prove that

$$(1.3) \qquad \qquad \|(\mathcal{A}^\varepsilon - \mu)^{-1} - (\mathcal{A}^0 - \mu)^{-1}\|_{L_2 \to L_2} \le C\varepsilon,$$

$$(1.4) \qquad \qquad \|(\mathcal{A}^\varepsilon - \mu)^{-1} - (\mathcal{A}^0 - \mu)^{-1} - \varepsilon\mathcal{C}_\mu^\varepsilon\|_{L_2 \to L_2} \le C\varepsilon^2$$

and

$$(1.5) \qquad \|\nabla(\mathcal{A}^\varepsilon - \mu)^{-1} - \nabla(\mathcal{A}^0 - \mu)^{-1} - \varepsilon\nabla\mathcal{K}_\mu^\varepsilon\|_{L_2 \to L_2} \le C\varepsilon,$$

the estimates being sharp with respect to the order (see Theorems 6.1 and 6.2). The effective operator $\mathcal{A}^0$ is of the same form as $\mathcal{A}^\varepsilon$, but its coefficients depend only on the slow variable. In contrast, the correctors $\mathcal{K}_\mu^\varepsilon$ and $\mathcal{C}_\mu^\varepsilon$ involve rapidly oscillating functions as well. The first of these plays the role of the traditional corrector and differs from the latter in that it involves a smoothing operator. The idea of using a smoothing to regularize the traditional corrector is due to Griso, see [Gri02]. The other corrector has no analogue in classical theory and was first presented in [BSu05] for purely periodic operators. Assume for simplicity that $A^* = A$. Then $\mathcal{C}_\mu^\varepsilon$ has the form

$$\mathcal{C}_\mu^\varepsilon = (\mathcal{K}_\mu^\varepsilon - \mathcal{L}_\mu) - \mathcal{M}_\mu^\varepsilon + (\mathcal{K}_\mu^\varepsilon - \mathcal{L}_\mu)^*$$

(see Section 5). What is interesting here is that an analog of $\mathcal{C}_\mu^\varepsilon$ for periodic operators, while looking similar to this one, does not include the term $\mathcal{M}_\mu^\varepsilon$, see [Se17$_1$]. In fact, one cannot remove $\mathcal{M}_\mu^\varepsilon$ from $\mathcal{C}_\mu^\varepsilon$ if the estimate (1.4) is to remain true, see Remark 5.9 for examples. So this term is a special feature of non-periodic problems.

The results of the present paper extend the author's work [Se17$_1$] on periodic elliptic problems, where we studied non-self-adjoint scalar operators whose coefficients were periodic in some variables and Lipschitz in the others. Put differently, the fast and slow variables were separated in the sense that $A^\varepsilon(x) = A(x_1, \varepsilon^{-1}x_2)$, where $x = (x_1, x_2)$. We proved analogs of the estimates (1.3)–(1.5), yet the correctors were slightly different, see Remark 6.3 below. It should be pointed out that the operators in [Se17$_1$] were allowed to involve lower-order terms with quite general coefficients.

Previous results on uniform approximations for locally periodic elliptic operators are due to, on the one hand, Borisov and, on the other hand, Pastukhova and Tikhomirov. In [B08] Borisov established the estimates (1.3) and (1.5) for certain matrix self-adjoint operators with smooth coefficients. In the paper [PT07] of Pastukhova and Tikhomirov, similar results were proved for scalar self-adjoint

operators with rough coefficients (although their techniques also apply to non-self-adjoint problems). As far as I know, the estimate (1.4) in the locally periodic settings was not obtained even for the simplest cases.

To prove the estimates, we develop the ideas of [Se17$_1$]. In the first step we establish a variant of the resolvent identity that involves the resolvents of the original and the effective operators and a corrector (see Section 7). This combination comes as no surprise, for it is well known that the effective operator and a corrector form a first approximation to the original operator (see, e.g., [BLP78] or [ZhKO93]). When this is done, all the desired estimates will follow at once. However, we cannot use the same technique as in [Se17$_1$], so the identity is proved by different means. The point is that the technique depends heavily on the smoothing operator that has been chosen. In the case of periodic operators, the smoothing was based on the Gelfand transform; but it is not as convenient now. To my knowledge, no natural smoothing for operators with locally periodic coefficients is known, so we choose the Steklov smoothing operator, which is the most simple and has proved to be quite useful; see [Zh05] and [ZhP05], where that smoothing first appeared in the context of homogenization, as well as [PT07], [Su13$_1$] and [Su13$_2$]. We remark that a very similar smoothing had been used earlier in [Gri02] and [Gri04] (see also [Gri06]). Our technique is strongly influenced by all these works.

I believe that the same method can be of use for locally periodic problems on domains with Dirichlet or Neumann boundary conditions as well.

It is also worth noting that, once the estimates (1.3)–(1.5) are verified, a limiting argument will give similar results for operators whose coefficients are Hölder continuous in the first variable, see Remark 6.6. These results, together with the results stated here, have been announced in [Se17$_2$].

The plan of the paper is as follows. Section 2 contains basic definitions and notation. In Section 3 we introduce the original operator. We study the effective operator in Section 4 and correctors in Section 5. Section 6 states the main results. Section 7 is the core of the paper, where we first prove the identity and then complete the proofs.

## 2. Notation

The symbol $\|\cdot\|_U$ will stand for the norm on a normed space $U$. If $U$ and $V$ are Banach spaces, then $\mathbf{B}(U,V)$ is the Banach space of bounded linear operators from $U$ to $V$. When $U = V$, the space $\mathbf{B}(U) = \mathbf{B}(U,U)$ becomes a Banach algebra with the identity map $\mathcal{I}$. The norm and the inner product on $\mathbb{C}^n$ are denoted by $|\cdot|$ and $\langle \cdot, \cdot \rangle$, respectively. We shall often identify $\mathbf{B}(\mathbb{C}^n, \mathbb{C}^m)$ and $\mathbb{C}^{m \times n}$.

Let $\Sigma$ be a domain in $\mathbb{R}^d$ and $U$ a Banach space. The space $C^{0,1}(\bar{\Sigma}; U)$ consists of those uniformly continuous functions $u \colon \Sigma \to U$ for which

$$\|u\|_{C^{0,1}(\bar{\Sigma};U)} = \|u\|_{C(\bar{\Sigma};U)} + [u]_{C^{0,1}(\bar{\Sigma};U)} < \infty,$$

where $\|u\|_{C(\bar{\Sigma};U)} = \sup_{x \in \Sigma} \|u(x)\|_U$ and

$$[u]_{C^{0,1}(\bar{\Sigma};U)} = \sup_{\substack{x_1, x_2 \in \Sigma, \\ x_1 \neq x_2}} \frac{\|u(x_2) - u(x_1)\|_U}{|x_2 - x_1|}.$$

We will use the notation $\|\cdot\|_{C^{0,1}}$, $\|\cdot\|_C$ and $[\cdot]_{C^{0,1}}$ as shorthand for $\|\cdot\|_{C^{0,1}(\bar{\Sigma};U)}$, $\|\cdot\|_{C(\bar{\Sigma};U)}$ and $[\cdot]_{C^{0,1}(\bar{\Sigma};U)}$ when the context makes clear which $\Sigma$ and $U$ are meant.

The symbol $L_p(\Sigma; U)$ stands for the $L_p$-space of strongly measurable functions on $\Sigma$ with values in $U$. In case $U = \mathbb{C}^n$, we write $\|\cdot\|_{p,\Sigma}$ for the norm on $L_p(\Sigma)^n$ and $(\cdot, \cdot)_\Sigma$ for the inner product on $L_2(\Sigma)^n$. We let $W_p^m(\Sigma)^n$ denote the usual Sobolev space of $\mathbb{C}^n$-valued functions on $\Sigma$ and $(W_p^m(\Sigma)^n)^*$, its dual space under the pairing $(\cdot, \cdot)_\Sigma$. If $C_c^\infty(\Sigma)^n$ is dense in $W_p^m(\Sigma)^n$, then $W_{p^+}^{-m}(\Sigma)^n = (W_p^m(\Sigma)^n)^*$, where $p^+$ is the exponent conjugate to $p$.

Let $Q$ be the closed cube in $\mathbb{R}^d$ with center $0$ and side length $1$, sides being parallel to the axes. Then $\tilde{W}_p^m(Q)^n$ denotes the completion of $\tilde{C}^m(Q)^n$ in the $W_p^m$-norm. Here $\tilde{C}^m(Q)$ is the class of $m$-times continuously differentiable functions on $Q$ whose periodic extension to $\mathbb{R}^d$ enjoys the same smoothness. Notice that $\tilde{L}_p(Q)^n$ coincides with the space of all periodic functions in $L_{p,\mathrm{loc}}(\mathbb{R}^d)^n$. The spaces $\tilde{W}_p^m(\mathbb{R}^d \times Q)^n$ and $\tilde{C}^m(\mathbb{R}^d \times Q)^n$ are defined in a similar fashion. If $p = 2$, we write $H^m$ for $W_p^m$, $H^{-m}$ for $W_p^{-m}$, etc. The symbol $\tilde{H}_0^m(Q)^n$ will stand for the subspace of functions in $\tilde{H}^m(Q)^n$ with mean value zero. Any $u \in \tilde{H}_0^1(Q)^n$ satisfies the Poincaré inequality

$$(2.1) \qquad \|u\|_{2,Q} \leq (2\pi)^{-1}\|Du\|_{2,Q},$$

as can be seen by using Fourier series. Here and below, $D = -i\nabla$.

We will often use the notation $\alpha \lesssim \beta$ to mean that that there is a constant $C$, depending only on some fixed parameters (these are listed in Theorems 6.1 and 6.2), such that $\alpha \leq C\beta$.

## 3. Original operator

Let each $A_{kl}$ be a function in $C^{0,1}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))^{n\times n}$. Then $A = \{A_{kl}\}$ may be thought of as a bounded mapping $A\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbf{B}(\mathbb{C}^{d\times n})$ that is Lipschitz in the first variable and periodic in the second. As is well known, for any function $u\colon \mathbb{R}^d \times \mathbb{R}^d \to L_2(Q)$ satisfying the Carathéodory condition (i.e., the requirement of continuity with respect to the first variable and measurability with respect to the second) the map $\tau^\varepsilon u\colon \mathbb{R}^d \to L_2(Q)$ defined for $x \in \mathbb{R}^d$ and $z \in Q$ by

$$(3.1) \qquad \tau^\varepsilon u(x, z) = u(x, \varepsilon^{-1}x, z),$$

is measurable (here $\varepsilon > 0$). Notice that, if $v$ is another function from $\mathbb{R}^d \times \mathbb{R}^d$ to $L_2(Q)$, then $\tau^\varepsilon(uv) = (\tau^\varepsilon u)(\tau^\varepsilon v)$. We adopt the notation $u^\varepsilon = \tau^\varepsilon u$.

Consider the matrix operator $\mathcal{A}^\varepsilon\colon H^1(\mathbb{R}^d)^n \to H^{-1}(\mathbb{R}^d)^n$ given by

$$(3.2) \qquad \mathcal{A}^\varepsilon = D^* A^\varepsilon D.$$

It is easy to see that $\mathcal{A}^\varepsilon$ is bounded, with bound $C_\flat = \|A\|_C$:

$$(3.3) \qquad \|\mathcal{A}^\varepsilon u\|_{-1,2,\mathbb{R}^d} \leq C_\flat \|Du\|_{2,\mathbb{R}^d}$$

for all $u \in H^1(\mathbb{R}^d)^n$. Now we impose a condition that will render $\mathcal{A}^\varepsilon$ elliptic. Namely, we assume that $\mathcal{A}^\varepsilon$ is coercive uniformly in $\varepsilon \in \mathscr{E}$, where $\mathscr{E} = (0, \varepsilon_0]$ with $\varepsilon_0 \in (0, 1]$, that is, there are $c_A > 0$ and $C_A \geq 0$ such that

$$(3.4) \qquad \mathrm{Re}(A^\varepsilon Du, Du)_{\mathbb{R}^d} + C_A\|u\|_{2,\mathbb{R}^d}^2 \geq c_A\|Du\|_{2,\mathbb{R}^d}^2$$

for every $u \in H^1(\mathbb{R}^d)^n$. It follows that $\mathcal{A}^\varepsilon$ is $m$-sectorial with sector

$$\mathscr{S} = \left\{ z \in \mathbb{C} \colon |\mathrm{Im}\, z| \leq c_A^{-1} C_\flat (\mathrm{Re}\, z + C_A) \right\}$$

independent of $\varepsilon$. Whenever $\mu \notin \mathcal{S}$, the operator $\mathcal{A}_\mu^\varepsilon = \mathcal{A}^\varepsilon - \mu$ is an isomorphism and hence is invertible; moreover, for any $f \in H^{-1}(\mathbb{R}^d)^n$ we have

$$(3.5) \qquad \|(\mathcal{A}_\mu^\varepsilon)^{-1} f\|_{1,2,\mathbb{R}^d} \lesssim \|f\|_{-1,2,\mathbb{R}^d}.$$

Before proceeding, we make a few remarks about the coercivity condition. It follows from (3.4) (via Lemma 4.1) that $A$ satisfies the Legendre–Hadamard condition

$$(3.6) \qquad \operatorname{Re}\langle A(\cdot)\xi \otimes \eta, \xi \otimes \eta \rangle \geq c_A |\xi|^2 |\eta|^2, \qquad \xi \in \mathbb{R}^d, \eta \in \mathbb{C}^n,$$

so $\mathcal{A}^\varepsilon$ is strongly elliptic for all $\varepsilon > 0$. The Legendre–Hadamard condition does not generally imply (3.4). If we restrict our attention to the real-valued case, then for scalar operators the two statements are equivalent. But this is no longer true for matrix operators, let alone the complex-valued case. A necessary and sufficient algebraic condition on $A$ that would guarantee (3.4) is not known.

It is worthwhile to point out that we have to be able to verify the coercivity bound for all $\varepsilon$ in some interval $(0, \varepsilon_0]$, which may be rather difficult. A sufficient condition not involving $\varepsilon$ is that the operator $D^* A(x, \cdot) D$ is strongly coercive on $H^1(\mathbb{R}^d)^n$ and furthermore there is $c > 0$ so that for any $x \in \mathbb{R}^d$ and $u \in H^1(\mathbb{R}^d)^n$

$$(3.7) \qquad \operatorname{Re}(A(x, \cdot) Du, Du)_{\mathbb{R}^d} \geq c \|Du\|_{2,\mathbb{R}^d}^2.$$

This can be seen by noticing that, by change of variable, the above inequality remains true with $A(x, \varepsilon^{-1} y)$ in place of $A(x, y)$. Then a partition of unity argument will do the job, since $A$ is uniformly continuous in the first variable.

As an example of $A$ satisfying (3.7), let $b(D)$ be a matrix first-order differential operator with symbol

$$\xi \mapsto b(\xi) = \sum_{k=1}^d b_k \xi_k,$$

where $b_k \in \mathbb{C}^{m \times n}$. Suppose that the symbol has the property that, for some $\alpha > 0$,

$$b(\xi)^* b(\xi) \geq \alpha |\xi|^2, \qquad \xi \in \mathbb{R}^d.$$

Let $g$ be a function in $C^{0,1}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))^{m \times m}$ with $\operatorname{Re} g$ uniformly positive definite. Now if we take $A_{kl} = b_k^* g b_l$, then application of the Fourier transform will yield

$$\operatorname{Re}(A(x, \cdot) Du, Du)_{\mathbb{R}^d} = \operatorname{Re}(g(x, \cdot) b(D)u, b(D)u)_{\mathbb{R}^d}$$
$$\geq \alpha \|(\operatorname{Re} g)^{-1/2}\|_C^{-2} \|Du\|_{2,\mathbb{R}^d}^2.$$

Homogenization for self-adjoint operators of this type was studied by Birman and Suslina in the purely periodic setting (see, e.g., [BSu01], [BSu03], [BSu05], [BSu06], [Su13$_1$] and [Su13$_2$]) and by Borisov in the locally periodic setting (see [B08]).

Observe that the more restrictive Legendre condition, which amounts to the uniform positive definiteness of $\operatorname{Re} A$, does ensure coercivity, but excludes some strongly elliptic operators with important applications – such as certain elasticity operators.

## 4. Effective operator

Given $\xi \in \mathbb{C}^{d \times n}$ and $x \in \mathbb{R}^d$, we let $N_\xi(x, \cdot)$ be the weak solution of

$$(4.1) \qquad D^* A(x, \cdot)(DN_\xi(x, \cdot) + \xi) = 0$$

in $\tilde{H}_0^1(Q)^n$. The function $N_\xi$ is well defined, since $D^*A(x, \cdot)\xi$ is a continuous linear functional on $\tilde{H}^1(Q)^n$ and the operator $D^*A(x, \cdot)D$ is strongly coercive on $\tilde{H}^1(Q)^n$, as we shall now see.

**Lemma 4.1.** *For any $x \in \mathbb{R}^d$ and all $u \in \tilde{H}^1(Q)^n$, we have*

$$(4.2) \qquad\qquad \mathrm{Re}(A(x, \cdot)Du, Du)_Q \geq c_A\|Du\|_{2,Q}^2.$$

*Proof.* Fix $u_\varepsilon = \varepsilon u^\varepsilon \varphi$ with $u \in \tilde{C}^1(Q)^n$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$. We substitute $u_\varepsilon$ into (3.4) and let $\varepsilon$ tend to 0. Then, because $u_\varepsilon$ and $Du_\varepsilon - (Du)^\varepsilon\varphi$ converge in $L_2$ to 0,

$$\lim_{\varepsilon \to 0} \mathrm{Re} \int_{\mathbb{R}^d} \langle A^\varepsilon(x)(Du)^\varepsilon(x), (Du)^\varepsilon(x)\rangle |\varphi(x)|^2 \, dx \geq \lim_{\varepsilon \to 0} c_A \int_{\mathbb{R}^d} |(Du)^\varepsilon(x)|^2|\varphi(x)|^2 \, dx.$$

It is well known that if $f \in C_c(\mathbb{R}^d; \tilde{L}_\infty(Q))$, then

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} f^\varepsilon(x) \, dx = \int_{\mathbb{R}^d} \int_Q f(x, y) \, dx \, dy$$

(see, for instance, [A92, Lemmas 5.5 and 5.6]). As a result,

$$\mathrm{Re} \int_{\mathbb{R}^d} \int_Q \langle A(x, y)Du(y), Du(y)\rangle |\varphi(x)|^2 \, dx \, dy \geq c_A \int_{\mathbb{R}^d} \int_Q |Du(y)|^2|\varphi(x)|^2 \, dx \, dy.$$

Since $\varphi$ is an arbitrary function in $C_c^\infty(\mathbb{R}^d)$ and since $A$ is continuous in the first variable, we conclude that, for any $x \in \mathbb{R}^d$,

$$\mathrm{Re} \int_Q \langle A(x, y)Du(y), Du(y)\rangle \, dy \geq c_A \int_Q |Du(y)|^2 \, dy. \qquad \square$$

It is clear from Lemma 4.1 and Poincaré's inequality (2.1) that

$$\mathrm{Re}(A(x, \cdot)Du, Du)_Q \gtrsim \|u\|_{1,2,Q}^2$$

for every $u \in \tilde{H}_0^1(Q)^n$. Thus, the definition of $N_\xi$ makes good sense.

Denote by $N$ the map sending $\xi$ to $N_\xi$. Evidently, $N_\xi$ depends linearly on $\xi$, so $N$ is simply an operator of multiplication by a function (still denoted by $N$). The next lemma shows that $N$ has the same regularity in the first variable as $A$.

*Remark* 4.2. In what follows, we denote differentiation in the first variable by $D_1$ and differentiation in the second variable by $D_2$. When no confusion can arise, we omit the subscript and write $D$, as we did before.

**Lemma 4.3.** *We have $N \in C^{0,1}(\bar{\mathbb{R}}^d; \tilde{H}_0^1(Q))$.*

*Proof.* The identity (4.1), together with Lemma 4.1, yields

$$c_A\|DN_\xi(x, \cdot)\|_{2,Q} \leq \|A(x, \cdot)\|_{\infty,Q}|\xi|,$$

whence

$$(4.3) \qquad\qquad \|D_2 N\|_{L_\infty(\mathbb{R}^d; L_2(Q))} \leq c_A^{-1}\|A\|_C.$$

Next, by (4.1) again, for any $x_1, x_2 \in \mathbb{R}^d$ and $v \in \tilde{H}_0^1(Q)^n$

$$\big(A(x_2, \cdot)(DN_\xi(x_2, \cdot) - DN_\xi(x_1, \cdot)), Dv\big)_Q$$
$$= -\big((A(x_2, \cdot) - A(x_1, \cdot))(\xi + DN_\xi(x_1, \cdot)), Dv\big)_Q.$$

Taking $v = N_\xi(x_2, \cdot) - N_\xi(x_1, \cdot)$ and using Lemma 4.1, we obtain

$$c_A\|DN_\xi(x_2, \cdot) - DN_\xi(x_1, \cdot)\|_{2,Q} \leq \|A(x_2, \cdot) - A(x_1, \cdot)\|_{\infty,Q}\|\xi + DN_\xi(x_1, \cdot)\|_{2,Q}.$$

It now follows from (4.3) that

$$[D_2 N]_{C^{0,1}(\bar{\mathbb{R}}^d; L_2(Q))} \leq c_A^{-1}(1 + c_A^{-1}\|A\|_C)[A]_{C^{0,1}}.$$

We have proved that $D_2 N \in C^{0,1}(\bar{\mathbb{R}}^d; L_2(Q))$. But then Poincaré's inequality (2.1) implies that $N \in C^{0,1}(\bar{\mathbb{R}}^d; L_2(Q))$ as well. $\qquad\square$

Let $A^0 \colon \mathbb{R}^d \to \mathbf{B}(\mathbb{C}^{d \times n})$ be given by

$$(4.4) \qquad A^0(x) = \int_Q A(x,y)(I + D_2 N(x,y)) \, dy.$$

Since $A$ and $D_2 N$ are continuous in the first variable, so is $A^0$. In fact, we have $A^0 \in C^{0,1}(\bar{\mathbb{R}}^d)$. Indeed, the estimate

$$\|A^0\|_{C(\bar{\mathbb{R}}^d)} \leq \|A\|_C \|I + D_2 N\|_C$$

is immediate from the definition of $A^0$, and that

$$[A^0]_{C^{0,1}(\bar{\mathbb{R}}^d)} \leq \|A\|_C [D_2 N]_{C^{0,1}} + [A]_{C^{0,1}} \|I + D_2 N\|_C$$

follows by an easy calculation. Hence, $\|A^0\|_{C^{0,1}(\bar{\mathbb{R}}^d)}$ is finite.

Now we define the effective operator $\mathcal{A}^0 \colon H^1(\mathbb{R}^d)^n \to H^{-1}(\mathbb{R}^d)^n$ by setting

$$(4.5) \qquad \mathcal{A}^0 = D^* A^0 D.$$

Observe that $\mathcal{A}^0$ is bounded and coercive (recall Gårding's inequality) and thus $m$-sectorial. It can be proved that $\mathcal{A}^0$ satisfies an estimate similar to (3.4) with exactly the same constants, however the bound on its norm may be different from (3.3). Nevertheless, the sector for $\mathcal{A}^0$ remains the same as for $\mathcal{A}^\varepsilon$. We briefly sketch the argument; see [Se17$_1$, Section 2.3] for a related proof. First consider the two-scale effective system as in [A92] and check that the associated form, which is defined on $H^1(\mathbb{R}^d)^n \oplus L_2(\mathbb{R}^d; \tilde{H}_0^1(Q))^n$ by

$$u \oplus U \mapsto (A(D_1 u + D_2 U), D_1 u + D_2 U)_{\mathbb{R}^d \times Q},$$

is $m$-sectorial with sector $\mathcal{S}$. We only remark that the coercivity is obtained by substituting $u + \varepsilon U^\varepsilon$ (with sufficiently smooth $u$ and $U$) into (3.4) for $u$ and letting $\varepsilon$ tend to 0; cf. the proof of Lemma 4.1. Then notice that

$$(\mathcal{A}^0 u, u)_{\mathbb{R}^d} = (A(D_1 u + D_2 U), D_1 u + D_2 U)_{\mathbb{R}^d \times Q}$$

provided $U = N D_1 u$ (which is definitely in $L_2(\mathbb{R}^d; \tilde{H}_0^1(Q))^n$). The claim is proved.

Thus, we see that the operator $\mathcal{A}_\mu^0 = \mathcal{A}^0 - \mu$ is an isomorphism as long as $\mu$ is outside $\mathcal{S}$. In addition, standard regularity theory for strongly elliptic systems (see, e.g., [McL00, Theorem 4.16]) implies that the pre-image of $L_2(\mathbb{R}^d)^n$ under $\mathcal{A}_\mu^0$ is all of $H^2(\mathbb{R}^d)^n$ and for any $f \in L_2(\mathbb{R}^d)^n$

$$(4.6) \qquad \|(\mathcal{A}_\mu^0)^{-1} f\|_{2,2,\mathbb{R}^d} \lesssim \|f\|_{2,\mathbb{R}^d}.$$

Let us return to our discussion of coercivity at the end of the previous section. As we have seen, (4.2) follows from (3.4), which in turn is a consequence of (3.7). On the other hand, (4.2) does not generally imply (3.7), and there are examples (for $n > 1$, of course) where (4.2) holds, but (3.7) is false, see [BF15]. In such cases, a subsequence of $(\mathcal{A}_\mu^\varepsilon)^{-1}$ may still converge in the weak operator topology to $(\mathcal{A}_\mu^0)^{-1}$, but $\mathcal{A}^0$ will fail to be strongly elliptic, i.e., $A^0$ will not satisfy the Legendre–Hadamard condition.

## 5. CORRECTORS

Let the operator $\mathcal{K}_\mu\colon L_2(\mathbb{R}^d)^n \to \tilde{H}^1(\mathbb{R}^d \times Q)^n$ be given by

$$(5.1) \qquad\qquad \mathcal{K}_\mu = ND_1(\mathcal{A}_\mu^0)^{-1}.$$

Lemma 4.3, combined with the estimate (4.6), readily implies that $\mathcal{K}_\mu$ is continuous:

$$(5.2) \qquad\qquad \|\mathcal{K}_\mu f\|_{1,2,\mathbb{R}^d \times Q} \lesssim \|f\|_{2,\mathbb{R}^d}.$$

The very same argument shows that $D_1 D_2 \mathcal{K}_\mu$ is bounded on $L_2(\mathbb{R}^d)^n$ as well:

$$(5.3) \qquad\qquad \|D_1 D_2 \mathcal{K}_\mu f\|_{2,\mathbb{R}^d \times Q} \lesssim \|f\|_{2,\mathbb{R}^d}.$$

Since we do not impose any extra assumptions on the coefficients, the traditional corrector $\tau^\varepsilon \mathcal{K}_\mu$ will not even map $L_2(\mathbb{R}^d)^n$ into itself. So we must first appropriately regularize the traditional corrector, and a smoothing operator is used for exactly this purpose.

5.1. **Smoothing.** Let $\mathcal{T}^\varepsilon\colon L_2(\mathbb{R}^d \times Q) \to L_2(\mathbb{R}^d \times Q; L_2(Q))$ be the translation operator

$$(5.4) \qquad\qquad \mathcal{T}^\varepsilon u(x,y,z) = u(x + \varepsilon z, y),$$

where $(x,y) \in \mathbb{R}^d \times Q$ and $z \in Q$. Certainly, for any $u, v \in L_2(\mathbb{R}^d \times Q)$ satisfying $uv \in L_2(\mathbb{R}^d \times Q)$ we have $\mathcal{T}^\varepsilon uv = (\mathcal{T}^\varepsilon u)(\mathcal{T}^\varepsilon v)$. Next, the adjoint of $\mathcal{T}^\varepsilon$ is given by

$$(\mathcal{T}^\varepsilon)^* u(x,y) = \int_Q u(x - \varepsilon z, y, z)\, dz.$$

Note that $(\mathcal{T}^\varepsilon)^*$ is defined on $L_2(\mathbb{R}^d \times Q)$ and $L_2(\mathbb{R}^d)$ as well, by way of identifying these spaces with the corresponding subspaces of $L_2(\mathbb{R}^d \times Q; L_2(Q))$. We define the Steklov smoothing operator $\mathcal{S}^\varepsilon\colon L_2(\mathbb{R}^d \times Q) \to L_2(\mathbb{R}^d \times Q)$ to be the restriction of $(\mathcal{T}^\varepsilon)^*$ to $L_2(\mathbb{R}^d \times Q)$. In other words,

$$(5.5) \qquad\qquad \mathcal{S}^\varepsilon u(x,y) = \int_Q \mathcal{T}^\varepsilon u(x,y,z)\, dz.$$

The operator $\mathcal{S}^\varepsilon$ is plainly self-adjoint.

Here we collect some facts about $\mathcal{T}^\varepsilon$ and $\mathcal{S}^\varepsilon$.

**Lemma 5.1.** *The restriction of $\tau^\varepsilon \mathcal{T}^\varepsilon$ to $\tilde{L}_2(\mathbb{R}^d \times Q)$ is an isometry.*

*Proof.* By change of variable,

$$\|\tau^\varepsilon \mathcal{T}^\varepsilon u\|_{2,\mathbb{R}^d \times Q}^2 = \int_{\mathbb{R}^d} \int_Q |u(x, \varepsilon^{-1}x - z)|^2\, dx\, dz.$$

But since $u$ is periodic in the second variable, this equals $\|u\|_{2,\mathbb{R}^d \times Q}^2$. $\qquad\square$

A related result for $\mathcal{S}^\varepsilon$ is the following.

**Lemma 5.2.** *The restriction of $\tau^\varepsilon \mathcal{S}^\varepsilon$ to $\tilde{L}_2(\mathbb{R}^d \times Q)$ is bounded, with bound at most $1$.*

*Proof.* This is immediate from Cauchy's inequality and Lemma 5.1. $\qquad\square$

It is easy to see that both $\mathcal{T}^\varepsilon$ and $\mathcal{S}^\varepsilon$ converge in the strong operator topology to the identity operator, yet they do not converge in norm. The uniform convergence will, however, take place if we restrict them to certain Sobolev spaces.

**Lemma 5.3.** *For any $u \in C_c^\infty(\mathbb{R}^d \times Q)$ we have*

$$(5.6) \qquad \|(\mathcal{T}^\varepsilon - \mathcal{I})u\|_{2,\mathbb{R}^d \times Q \times Q} \lesssim \varepsilon \|D_1 u\|_{2,\mathbb{R}^d \times Q}.$$

*Proof.* Notice that

$$u(x + \varepsilon z, y) - u(x, y) = \varepsilon i \int_0^1 \langle D_1 u(x + \varepsilon t z, y), z \rangle \, dt.$$

Hence,

$$\|(\mathcal{T}^\varepsilon - \mathcal{I})u(\cdot, y, z)\|_{2,\mathbb{R}^d} \le \varepsilon r_Q \|D_1 u(\cdot, y)\|_{2,\mathbb{R}^d},$$

where $r_Q = 1/2 \operatorname{diam} Q$. Integrating out the $y$ and $z$ variables then yields (5.6). $\quad\square$

**Lemma 5.4.** *For any $u \in C_c^\infty(\mathbb{R}^d \times Q)$ we have*

$$(5.7) \qquad \|(\mathcal{S}^\varepsilon - \mathcal{I})u\|_{2,\mathbb{R}^d \times Q} \lesssim \varepsilon \|D_1 u\|_{2,\mathbb{R}^d \times Q},$$

$$(5.8) \qquad \|(\mathcal{S}^\varepsilon - \mathcal{I})u\|_{2,\mathbb{R}^d \times Q} \lesssim \varepsilon^2 \|D_1 D_1 u\|_{2,\mathbb{R}^d \times Q}.$$

*Proof.* The inequality (5.7) comes from (5.6). To prove (5.8), notice that

$$u(x + \varepsilon z, y) - u(x, y) = \varepsilon i \langle D_1 u(x, y), z \rangle - \varepsilon^2 \int_0^1 (1 - t) \langle D_1 D_1 u(x + \varepsilon t z, y) z, z \rangle \, dt.$$

The first term on the right-hand side has mean value zero for a.e. $x$ and $y$ (because $Q$ is centered at the origin), so

$$\|(\mathcal{S}^\varepsilon - \mathcal{I})u(\cdot, y)\|_{2,\mathbb{R}^d} \le \varepsilon^2 r_Q^2 \|D_1 D_1 u(\cdot, y)\|_{2,\mathbb{R}^d}.$$

Integrating over $Q$ completes the proof. $\quad\square$

Now we can prove the following result.

**Lemma 5.5.** *For any $u \in \tilde{C}_c^\infty(\mathbb{R}^d \times Q)$ we have*

$$\|\tau^\varepsilon \mathcal{T}^\varepsilon u - \tau^\varepsilon \mathcal{S}^\varepsilon u\|_{2,\mathbb{R}^d \times Q} \lesssim \varepsilon \|D_1 u\|_{2,\mathbb{R}^d \times Q}.$$

*Proof.* We write

$$\tau^\varepsilon \mathcal{T}^\varepsilon u - \tau^\varepsilon \mathcal{S}^\varepsilon u = \tau^\varepsilon \mathcal{T}^\varepsilon (\mathcal{I} - \mathcal{S}^\varepsilon) u + \tau^\varepsilon \mathcal{S}^\varepsilon (\mathcal{T}^\varepsilon - \mathcal{I}) u$$

(here $\mathcal{S}^\varepsilon \mathcal{T}^\varepsilon$ is understood to be defined as $\mathcal{S}^\varepsilon \mathcal{T}^\varepsilon = \mathcal{T}^\varepsilon \mathcal{S}^\varepsilon$, that is, we apply $\mathcal{S}^\varepsilon$ to $\mathcal{T}^\varepsilon u$ regarding the new variable resulting from the operator $\mathcal{T}^\varepsilon$ as a parameter). Then, it follows from Lemmas 5.1 and 5.4 that

$$\|\tau^\varepsilon \mathcal{T}^\varepsilon (\mathcal{I} - \mathcal{S}^\varepsilon) u\|_{2,\mathbb{R}^d \times Q} \lesssim \varepsilon \|D_1 u\|_{2,\mathbb{R}^d \times Q},$$

while Lemmas 5.2 and 5.3 imply that

$$\|\tau^\varepsilon \mathcal{S}^\varepsilon (\mathcal{T}^\varepsilon - \mathcal{I}) u\|_{2,\mathbb{R}^d \times Q} \lesssim \varepsilon \|D_1 u\|_{2,\mathbb{R}^d \times Q}.$$

These observations combine to give the desired estimate. $\quad\square$

*Remark* 5.6. We note that the results of Lemmas 5.1–5.5 persist if we replace the $L_2$-norms by the $L_p$-norms with $p \in [1, \infty]$. This will play a role in what follows.

5.2. **Correctors.** We define the first corrector $\mathcal{K}_\mu^\varepsilon \colon L_2(\mathbb{R}^d)^n \to H^1(\mathbb{R}^d)^n$ by

$$(5.9) \qquad\qquad\qquad\qquad \mathcal{K}_\mu^\varepsilon = \tau^\varepsilon \mathcal{S}^\varepsilon \mathcal{K}_\mu.$$

More explicitly,

$$\mathcal{K}_\mu^\varepsilon f(x) = \int_Q N(x + \varepsilon z, \varepsilon^{-1}x) \, D(\mathcal{A}_\mu^0)^{-1} f(x + \varepsilon z) \, dz.$$

Because of the smoothing $\mathcal{S}^\varepsilon$, this corrector is bounded with

$$(5.10) \qquad\qquad\qquad\qquad \|\mathcal{K}_\mu^\varepsilon f\|_{2,\mathbb{R}^d} \lesssim \|f\|_{2,\mathbb{R}^d},$$

$$(5.11) \qquad\qquad\qquad\qquad \|D\mathcal{K}_\mu^\varepsilon f\|_{2,\mathbb{R}^d} \lesssim \varepsilon^{-1}\|f\|_{2,\mathbb{R}^d}.$$

Indeed, using Lemma 5.2, we see that

$$\|\mathcal{K}_\mu^\varepsilon f\|_{2,\mathbb{R}^d} \le \|\mathcal{K}_\mu f\|_{2,\mathbb{R}^d \times Q},$$

$$\|D\mathcal{K}_\mu^\varepsilon f\|_{2,\mathbb{R}^d} \le \|D_1 \mathcal{K}_\mu f\|_{2,\mathbb{R}^d \times Q} + \varepsilon^{-1}\|D_2 \mathcal{K}_\mu f\|_{2,\mathbb{R}^d \times Q}.$$

The estimates (5.10) and (5.11) then follow from (5.2).

While the $L_2$-norm of $\mathcal{K}_\mu^\varepsilon f$ is merely uniformly bounded, the $L_2$-norm of $\mathcal{S}^\varepsilon \mathcal{K}_\mu^\varepsilon f$ turns out to be of order $\varepsilon$.

**Lemma 5.7.** *For any $\varepsilon \in \mathscr{E}$ and $f \in L_2(\mathbb{R}^d)^n$ we have*

$$\|\mathcal{S}^\varepsilon \mathcal{K}_\mu^\varepsilon f\|_{2,\mathbb{R}^d} \lesssim \varepsilon \|f\|_{2,\mathbb{R}^d}.$$

*Proof.* By definition of $\mathcal{S}^\varepsilon$ and $\mathcal{K}_\mu^\varepsilon$,

$$\mathcal{S}^\varepsilon \mathcal{K}_\mu^\varepsilon f(x) = \int_Q \int_Q \mathcal{T}^\varepsilon \mathcal{K}_\mu f(x + \varepsilon w, \varepsilon^{-1}x + z, z) \, dw \, dz.$$

Since $\mathcal{K}_\mu f(x, \cdot)$ is periodic and has mean value zero, we have

$$\int_Q \int_Q \mathcal{K}_\mu f(x + \varepsilon w, \varepsilon^{-1}x + z) \, dw \, dz = 0,$$

and hence

$$\mathcal{S}^\varepsilon \mathcal{K}_\mu^\varepsilon f(x) = \int_Q \int_Q (\mathcal{T}^\varepsilon - \mathcal{I})\mathcal{K}_\mu f(x + \varepsilon w, \varepsilon^{-1}x + z, z) \, dw \, dz.$$

Changing variables and keeping in mind that $\mathcal{K}_\mu f$ is periodic in the second variable, we find that

$$\|\mathcal{S}^\varepsilon \mathcal{K}_\mu^\varepsilon f\|_{2,\mathbb{R}^d} \le \|(\mathcal{T}^\varepsilon - \mathcal{I})\mathcal{K}_\mu f\|_{2,\mathbb{R}^d \times Q \times Q}.$$

The result is therefore immediate from Lemma 5.3 and the estimate (5.2).    □

To describe the second corrector, we need some additional notation. Let $(\mathcal{A}_\mu^\varepsilon)^+$ be the adjoint of $\mathcal{A}_\mu^\varepsilon$. Then we construct the effective operator $(\mathcal{A}_\mu^0)^+$, the corrector $(\mathcal{K}_\mu^\varepsilon)^+$ and the other objects (which will be marked with "+" as well) for $(\mathcal{A}_\mu^\varepsilon)^+$ just as we did for $\mathcal{A}_\mu^\varepsilon$. (It may be noted in passing that $(\mathcal{A}_\mu^0)^+$ is the adjoint of $\mathcal{A}_\mu^0$.) Of course, all results for $\mathcal{A}_\mu^\varepsilon$ will transfer to $(\mathcal{A}_\mu^\varepsilon)^+$. We shall not explicitly formulate these results here, but refer to them by the numbers of the corresponding statements for $\mathcal{A}_\mu^\varepsilon$ with "+" following the reference (for example, Lemma 5.7$^+$ and the estimate (5.10)$^+$).

Define $\mathcal{L}_\mu \colon L_2(\mathbb{R}^d)^n \to L_2(\mathbb{R}^d)^n$ by

$$(5.12) \qquad\qquad \mathcal{L}_\mu = (D_1 \mathcal{K}_\mu^+)^* A\big(D_1(\mathcal{A}_\mu^0)^{-1} + D_2 \mathcal{K}_\mu\big)$$

and $\mathcal{M}_\mu^\varepsilon \colon L_2(\mathbb{R}^d)^n \to L_2(\mathbb{R}^d)^n$ by

(5.13) $\quad \mathcal{M}_\mu^\varepsilon = \varepsilon^{-1}\big(\tau^\varepsilon \mathcal{T}^\varepsilon \big(D_1((\mathcal{A}_\mu^0)^+)^{-1} + D_2\mathcal{K}_\mu^+\big)\big)^* \tau^\varepsilon[A, \mathcal{T}^\varepsilon]\big(D_1(\mathcal{A}_\mu^0)^{-1} + D_2\mathcal{K}_\mu\big).$

A more convenient way of dealing with these operators is to look at their forms. If we set $u_0 = (\mathcal{A}_\mu^0)^{-1}f$, $U = \mathcal{K}_\mu f$ and $u_0^+ = ((\mathcal{A}_\mu^0)^+)^{-1}g$, $U^+ = \mathcal{K}_\mu^+ g$, then

$$(\mathcal{L}_\mu f, g)_{\mathbb{R}^d} = \big(A(D_1 u_0 + D_2 U), D_1 U^+\big)_{\mathbb{R}^d \times Q}$$

and

$$(\mathcal{M}_\mu^\varepsilon f, g)_{\mathbb{R}^d} = \varepsilon^{-1}\big(\tau^\varepsilon[A, \mathcal{T}^\varepsilon](D_1 u_0 + D_2 U), \tau^\varepsilon \mathcal{T}^\varepsilon(D_1 u_0^+ + D_2 U^+)\big)_{\mathbb{R}^d \times Q}.$$

Both $\mathcal{L}_\mu$ and $\mathcal{M}_\mu^\varepsilon$ are bounded. Indeed,

$$|(\mathcal{L}_\mu f, g)_{\mathbb{R}^d}| \leq \|A(D_1 u_0 + D_2 U)\|_{2,\mathbb{R}^d \times Q}\|D_1 U^+\|_{2,\mathbb{R}^d \times Q},$$

and so, according to the estimates (4.6), (5.2) and $(5.2)^+$,

(5.14) $$\|\mathcal{L}_\mu f\|_{2,\mathbb{R}^d} \lesssim \|f\|_{2,\mathbb{R}^d}.$$

Likewise, observing that $\tau^\varepsilon[A, \mathcal{T}^\varepsilon] = \tau^\varepsilon(\mathcal{I} - \mathcal{T}^\varepsilon)A \cdot \tau^\varepsilon \mathcal{T}^\varepsilon$ (by the multiplicativity of $\tau^\varepsilon$ and $\mathcal{T}^\varepsilon$), we conclude that

$$|(\mathcal{M}_\mu^\varepsilon f, g)_{\mathbb{R}^d}| \leq r_Q[A]_{C^{0,1}}\|\tau^\varepsilon \mathcal{T}^\varepsilon(D_1 u_0 + D_2 U)\|_{2,\mathbb{R}^d \times Q}\|\tau^\varepsilon \mathcal{T}^\varepsilon(D_1 u_0^+ + D_2 U^+)\|_{2,\mathbb{R}^d \times Q}.$$

This, together with Lemma 5.1 and the estimates (4.6), (5.2) and $(4.6)^+$, $(5.2)^+$, yields that

(5.15) $$\|\mathcal{M}_\mu^\varepsilon f\|_{2,\mathbb{R}^d} \lesssim \|f\|_{2,\mathbb{R}^d}.$$

Now we introduce the second corrector $\mathcal{C}_\mu^\varepsilon \colon L_2(\mathbb{R}^d)^n \to L_2(\mathbb{R}^d)^n$ by

(5.16) $$\mathcal{C}_\mu^\varepsilon = (\mathcal{K}_\mu^\varepsilon - \mathcal{L}_\mu) - \mathcal{M}_\mu^\varepsilon + ((\mathcal{K}_\mu^\varepsilon)^+ - \mathcal{L}_\mu^+)^*.$$

Then (5.10), (5.14), (5.15) and $(5.10)^+$, $(5.14)^+$ imply that $\mathcal{C}_\mu^\varepsilon$ is continuous:

(5.17) $$\|\mathcal{C}_\mu^\varepsilon f\|_{2,\mathbb{R}^d} \lesssim \|f\|_{2,\mathbb{R}^d}.$$

*Remark* 5.8. From (5.15) we know that the operator norm of $\mathcal{M}_\mu^\varepsilon$ is bounded uniformly in $\varepsilon$. In some situations, we can go further and prove that

(5.18) $$\|\mathcal{M}_\mu^\varepsilon f\|_{2,\mathbb{R}^d} \lesssim \varepsilon\|f\|_{2,\mathbb{R}^d}.$$

The term $\mathcal{M}_\mu^\varepsilon$ can then be removed from $\mathcal{C}_\mu^\varepsilon$, because, in the context where the corrector $\mathcal{C}_\mu^\varepsilon$ is needed (see Theorem 6.2 below), this term will be absorbed to the error.

The estimate (5.18) is true, for instance, if $A \in C^{1,1}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))$. To see this, we notice that if $u, v \in \tilde{H}^1(\mathbb{R}^d \times Q)^{d \times n}$, then $uv \in \tilde{W}_1^1(\mathbb{R}^d \times Q)$ and, by an $L_1$-variant of Lemma 5.5 (see Remark 5.6),

$$\big|\big(\tau^\varepsilon(\mathcal{I} - \mathcal{T}^\varepsilon)A \cdot \tau^\varepsilon \mathcal{T}^\varepsilon u, \tau^\varepsilon \mathcal{T}^\varepsilon v\big)_{\mathbb{R}^d \times Q} - \big(\tau^\varepsilon(\mathcal{I} - \mathcal{S}^\varepsilon)A \cdot \tau^\varepsilon \mathcal{T}^\varepsilon u, \tau^\varepsilon \mathcal{T}^\varepsilon v\big)_{\mathbb{R}^d \times Q}\big|$$
$$\leq \varepsilon r_Q[A]_{C^{0,1}}\|\tau^\varepsilon \mathcal{T}^\varepsilon u\bar{v} - \tau^\varepsilon \mathcal{S}^\varepsilon u\bar{v}\|_{1,\mathbb{R}^d \times Q} \lesssim \varepsilon^2 \|D_1 u\bar{v}\|_{1,\mathbb{R}^d \times Q}$$

(we have reversed the order of integration to pass from $\mathcal{S}^\varepsilon$ to $\mathcal{T}^\varepsilon$ in the second term on the left). This means that we may replace the function $\tau^\varepsilon(\mathcal{I} - \mathcal{T}^\varepsilon)A$ in $\mathcal{M}_\mu^\varepsilon$ by $\tau^\varepsilon(\mathcal{I} - \mathcal{S}^\varepsilon)A$ with error being of order $\varepsilon$. But since $A \in C^{1,1}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))$, an $L_\infty$-variant of the second estimate in Lemma 5.4 (again see Remark 5.6) will imply that $\mathcal{M}_\mu^\varepsilon$ itself is of order $\varepsilon$. Another example when (5.18) holds is the case where the fast and slow variables are separated, that is, $A^\varepsilon(x) = A(x_1, \varepsilon^{-1}x_2)$

with $x = (x_1, x_2)$. Since only the rapid oscillations must be regularized, we may choose $\mathcal{T}^\varepsilon$ to be the translation operator in the variable $x_2$:

$$\mathcal{T}^\varepsilon u(x, y)(z_2) = u(x_1, x_2 + \varepsilon z_2, y).$$

Then $(\mathcal{I} - \mathcal{T}^\varepsilon)A$ is identically zero. Operators with such coefficients have been studied in $[\mathrm{Se17}_1]$.

*Remark* 5.9. Given the previous remark, it may be tempting to conjecture that (5.18) holds for all $A \in C^{0,1}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))$. However, this is not the case, as the following example shows. Define

$$\chi(x) = \sum_{k \in \mathbb{N}} k^{-2} \cos 2^k \pi x.$$

Then $\chi$ is uniformly continuous, but does not satisfy a Hölder condition of any order at all points (see [H16, Section 4] for details). Let $A_1$ be a uniformly positive definite Lipschitz function on $\mathbb{R}$ whose derivative equals $\chi$ on $(0, 1)$ and is $0$ off $(0, 1)$, and let $A_2(y) = 4\pi^{1/2}(2 + \sin 2\pi y)^{-1}$. Set $A(x, y) = A_1(x)A_2(y)$. Select an $f \in L_2(\mathbb{R})$ in such a way that $|Du_0|^2 = 1$ on $(0, 1)$. It is a straightforward, yet tedious, calculation to see that

$$(\mathcal{M}_\mu^{\varepsilon_k} f, f)_\mathbb{R} = (\log_2 \varepsilon_k^{-1})^{-2} + O(\varepsilon_k), \qquad k \to \infty,$$

where $\varepsilon_k = 2^{-k}$. In fact, for any monotone function $\zeta \in C([0, 1])$ that satisfies $\zeta(0) = 0$ and $\zeta(\varepsilon) \geq \varepsilon$, we can construct a uniformly elliptic operator $\mathcal{A}^\varepsilon$ on $H^1(\mathbb{R})$ and find a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ converging to $0$ such that

$$(\mathcal{M}_\mu^{\varepsilon_k} f, f)_\mathbb{R} = \zeta(\varepsilon_k) + O(\varepsilon_k), \qquad k \to \infty,$$

for some $f \in L_2(\mathbb{R})$. The idea is to adjust gaps in the Fourier series for $\chi$.

*Remark* 5.10. We observe that $\mathcal{L}_\mu$ can be written in the form

$$\mathcal{L}_\mu = (\mathcal{A}_\mu^0)^{-1} D^* \mathcal{L} D (\mathcal{A}_\mu^0)^{-1},$$

where $\mathcal{L} \colon H^1(\mathbb{R}^d)^n \to L_2(\mathbb{R}^d)^n$ is a first-order differential operator with bounded coefficients:

$$\mathcal{L} = \int_Q N^+(\cdot, y)^* D_1^* A(\cdot, y)(I + D_2 N(\cdot, y)) \, dy$$

(cf. $[\mathrm{Se17}_1, \text{Remark 2.6}]$). Likewise, we can write $\mathcal{M}_\mu^\varepsilon$ as

$$\mathcal{M}_\mu^\varepsilon = (\mathcal{A}_\mu^0)^{-1} D^* M_\varepsilon D (\mathcal{A}_\mu^0)^{-1}$$

where $M_\varepsilon$ is the bounded function given by

$$M_\varepsilon(x) = \varepsilon^{-1} \int_Q (I + D_2 N^+(x, \varepsilon^{-1}x + z))^* \Delta_{\varepsilon z} A(x, \varepsilon^{-1}x + z)(I + D_2 N(x, \varepsilon^{-1}x + z)) \, dz$$

with $\Delta_{\varepsilon z} A(x, y) = A(x + \varepsilon z, y) - A(x, y)$.

## 6. Main results

Now we formulate the main results of the paper.

**Theorem 6.1.** *If $\mu \notin \mathscr{S}$, then for any $\varepsilon \in \mathscr{E}$ and $f \in L_2(\mathbb{R}^d)^n$ we have*

$$(6.1) \qquad \|(\mathcal{A}_\mu^\varepsilon)^{-1} f - (\mathcal{A}_\mu^0)^{-1} f\|_{2, \mathbb{R}^d} \lesssim \varepsilon \|f\|_{2, \mathbb{R}^d},$$

$$(6.2) \qquad \|D(\mathcal{A}_\mu^\varepsilon)^{-1} f - D(\mathcal{A}_\mu^0)^{-1} f - \varepsilon D \mathcal{K}_\mu^\varepsilon f\|_{2, \mathbb{R}^d} \lesssim \varepsilon \|f\|_{2, \mathbb{R}^d}.$$

*The estimates are sharp with respect to the order, and the constants depend only on the parameters $d$, $n$, $\mu$, the norm $\|A\|_{C^{0,1}}$ and the constants $c_A$ and $C_A$ in the coercivity bound.*

**Theorem 6.2.** *If $\mu \notin \mathscr{S}$, then for any $\varepsilon \in \mathscr{E}$ and $f \in L_2(\mathbb{R}^d)^n$ it holds that*

$$(6.3) \qquad \|(\mathcal{A}_\mu^\varepsilon)^{-1}f - (\mathcal{A}_\mu^0)^{-1}f - \varepsilon \mathcal{C}_\mu^\varepsilon f\|_{2,\mathbb{R}^d} \lesssim \varepsilon^2 \|f\|_{2,\mathbb{R}^d}.$$

*The estimate is sharp with respect to the order, and the constant depends only on the parameters $d$, $n$, $\mu$, the norm $\|A\|_{C^{0,1}}$ and the constants $c_A$ and $C_A$ in the coercivity bound.*

*Remark* 6.3. These results should be compared with those in [Se17$_1$]. Suppose that $A^\varepsilon$ is periodic, that is, $A^\varepsilon(x) = A(x_1, \varepsilon^{-1}x_2)$, where $x = (x_1, x_2)$. In [Se17$_1$] we proved estimates similar to (6.1)–(6.3), but with different correctors in (6.2) and (6.3). The difference stems from the smoothing operator. As mentioned earlier, in the periodic case we may reduce $\mathcal{T}^\varepsilon$ to the translation operator in the variable $x_2$. Then $\mathcal{S}^\varepsilon$ will involve averaging over $\varepsilon Q$, with $Q$ being the basic cell for the lattice of periods (not necessarily of full rank). The Gelfand transform provides another smoothing that is, in a sense, dual to the first one and involves averaging over the dual cell $\varepsilon^{-1}Q^*$ in the reciprocal space. (Here $Q^*$ is the Wigner–Seitz cell in the dual lattice.) It is this last smoothing that appeared in [Se17$_1$]. One can verify directly that either of these may be used in the corrector $\mathcal{K}_\mu^\varepsilon$. As for $\mathcal{L}_\mu$ and $\mathcal{M}_\mu^\varepsilon$, the former does not depend on smoothing and is just the same as in [Se17$_1$], and the latter is zero by the choice of $\mathcal{T}^\varepsilon$ (see Remark 5.8).

*Remark* 6.4. Theorems 6.1 and 6.2 can be extended to allow all $\mu \notin \operatorname{spec} \mathcal{A}^0$, though it may be necessary to replace $\mathscr{E}$ by a smaller set $\mathscr{E}_\mu$ depending on $\mu$. Indeed, the proofs of the theorems go over without change to the case $\mu \notin \operatorname{spec} \mathcal{A}^0$ provided we establish estimates similar to (3.5) and (4.6). By the first resolvent identity, this amounts to checking that $\mathcal{A}_\mu^\varepsilon$ as an operator on $L_2(\mathbb{R}^d)^n$ has a uniformly bounded inverse. Suppose that $\mu \in \mathscr{S}$ (otherwise $\mathscr{E}_\mu = \mathscr{E}$). We know from Theorem 6.1 that if $\nu \notin \mathscr{S}$, then

$$\|(\mathcal{A}_\nu^\varepsilon)^{-1}f - (\mathcal{A}_\nu^0)^{-1}f\|_{2,\mathbb{R}^d} \leq C_\nu \varepsilon \|f\|_{2,\mathbb{R}^d}$$

for all $\varepsilon \in \mathscr{E}$ and $f \in L_2(\mathbb{R}^d)^n$. Therefore, using the identity

$$(\mathcal{A}_\mu^\varepsilon)^{-1} - (\mathcal{A}_\mu^0)^{-1} = \left( \mathcal{I} - (\mu - \nu)\mathcal{A}_\nu^0(\mathcal{A}_\mu^0)^{-1}\big((\mathcal{A}_\nu^\varepsilon)^{-1} - (\mathcal{A}_\nu^0)^{-1}\big) \right)^{-1}$$
$$\times \mathcal{A}_\nu^0(\mathcal{A}_\mu^0)^{-1}\big((\mathcal{A}_\nu^\varepsilon)^{-1} - (\mathcal{A}_\nu^0)^{-1}\big)\mathcal{A}_\nu^0(\mathcal{A}_\mu^0)^{-1},$$

we see that $(\mathcal{A}_\mu^\varepsilon)^{-1}$ is bounded on $L_2(\mathbb{R}^d)^n$ uniformly in $\varepsilon \leq \varepsilon_{\mu,\nu} \wedge \varepsilon_0$, where

$$\varepsilon_{\mu,\nu} < \frac{\operatorname{dist}(\mu, \operatorname{spec} \mathcal{A}^0)}{C_\nu |\mu - \nu|\big(\operatorname{dist}(\mu, \operatorname{spec} \mathcal{A}^0) + |\mu - \nu|\big)}.$$

It follows that we can set $\mathscr{E}_\mu = (0, \varepsilon_{\mu,\nu} \wedge \varepsilon_0]$.

*Remark* 6.5. We note that the operator $D(\mathcal{A}_\mu^\varepsilon)^{-1}$ converges in the uniform topology if and only if $D_2^* A(x, \cdot)\xi = 0$ on $\tilde{H}^1(Q)^n$ for every $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^{d \times n}$, in which case $N$ is zero and hence so is $\mathcal{K}_\mu^\varepsilon$. Notice also that the effective coefficients are then obtained by ordinary averaging over $Q$.

*Remark* 6.6. By keeping track of $[A]_{C^{0,1}}$ in estimates, we can find that the constants on the right of (6.1) and (6.2) depend linearly on $[A]_{C^{0,1}}$, while the constant on

the right of (6.3), quadratically. These observations play a role in proving results similar to Theorems 6.1 and 6.2 when the coefficients are Hölder continuous, or even continuous, in the slow variable. The key idea is to use mollification to replace $A$ with a function $A_\delta$ that is Lipschitz in the first variable. In the case of Hölder continuous coefficients, we are able to control both the convergence rate of $A_\delta$ to $A$ in a Hölder seminorm and the growth rate of $[A_\delta]_{C^{0,1}}$ in terms of $\delta$ as $\delta \to 0$. In the end, this allows us to obtain the desired operator estimates. However, if the coefficients are only continuous, such an approach yields the convergence of the resolvent, but not the rate. These results have been announced in [Se17$_2$]; detailed proofs will appear elsewhere.

## 7. Proof of the main results

Our first task is to obtain an identity involving $(\mathcal{A}_\mu^\varepsilon)^{-1}$, $(\mathcal{A}_\mu^0)^{-1}$ and $\mathcal{K}_\mu^\varepsilon$ that will play a crucial role in the proofs.

Fix $f \in L_2(\mathbb{R}^d)^n$ and $g \in H^{-1}(\mathbb{R}^d)^n$. Let $u_0 = (\mathcal{A}_\mu^0)^{-1}f$, $U = \mathcal{K}_\mu f$, $U_\varepsilon = \mathcal{K}_\mu^\varepsilon f$ and $u_\varepsilon^+ = ((\mathcal{A}_\mu^\varepsilon)^+)^{-1}g$. Then we have
$$(7.1)$$
$$\left((\mathcal{A}_\mu^\varepsilon)^{-1}f - (\mathcal{A}_\mu^0)^{-1}f - \varepsilon\mathcal{K}_\mu^\varepsilon f, g\right)_{\mathbb{R}^d} = (\mathcal{A}^0 u_0, u_\varepsilon^+)_{\mathbb{R}^d} - (\mathcal{A}^\varepsilon(\mathcal{S}^\varepsilon u_0 + \varepsilon U_\varepsilon), u_\varepsilon^+)_{\mathbb{R}^d}$$
$$- (\mathcal{A}^\varepsilon(\mathcal{I} - \mathcal{S}^\varepsilon)u_0, u_\varepsilon^+)_{\mathbb{R}^d} + \varepsilon\mu(U_\varepsilon, u_\varepsilon^+)_{\mathbb{R}^d}.$$

Let us look at the first two terms on the right. By the definition of the effective coefficients,
$$(\mathcal{A}^0 u_0, u_\varepsilon^+)_{\mathbb{R}^d} = \left(A(D_1 u_0 + D_2 U), D_1 u_\varepsilon^+\right)_{\mathbb{R}^d \times Q}.$$

Then Lemma 5.1 yields that
$$(7.2) \qquad (\mathcal{A}^0 u_0, u_\varepsilon^+)_{\mathbb{R}^d} = \left(\tau^\varepsilon \mathcal{T}^\varepsilon A(D_1 u_0 + D_2 U), \mathcal{T}^\varepsilon D_1 u_\varepsilon^+\right)_{\mathbb{R}^d \times Q}$$

(notice here that $u_\varepsilon^+$ does not depend on the second variable). On the other hand,
$$(7.3) \qquad \begin{aligned} (\mathcal{A}^\varepsilon(\mathcal{S}^\varepsilon u_0 + \varepsilon U_\varepsilon), u_\varepsilon^+)_{\mathbb{R}^d} &= \left(\tau^\varepsilon A\mathcal{T}^\varepsilon(D_1 u_0 + D_2 U), D_1 u_\varepsilon^+\right)_{\mathbb{R}^d \times Q} \\ &\quad + \varepsilon\left(\tau^\varepsilon A\mathcal{T}^\varepsilon D_1 U, D_1 u_\varepsilon^+\right)_{\mathbb{R}^d \times Q}. \end{aligned}$$

Commuting $\mathcal{T}^\varepsilon$ past $A$ in the first term on the right and combining the resulting identity with (7.2), we conclude that
$$(7.4) \qquad \begin{aligned} &(\mathcal{A}^0 u_0, u_\varepsilon^+)_{\mathbb{R}^d} - (\mathcal{A}^\varepsilon(\mathcal{S}^\varepsilon u_0 + \varepsilon U_\varepsilon), u_\varepsilon^+)_{\mathbb{R}^d} \\ &\quad = \left(\tau^\varepsilon \mathcal{T}^\varepsilon A(D_1 u_0 + D_2 U), (\mathcal{T}^\varepsilon - \mathcal{I})D_1 u_\varepsilon^+\right)_{\mathbb{R}^d \times Q} \\ &\qquad - \left(\tau^\varepsilon [A, \mathcal{T}^\varepsilon](D_1 u_0 + D_2 U), D_1 u_\varepsilon^+\right)_{\mathbb{R}^d \times Q} \\ &\qquad - \varepsilon\left(\tau^\varepsilon A\mathcal{T}^\varepsilon D_1 U, D_1 u_\varepsilon^+\right)_{\mathbb{R}^d \times Q}. \end{aligned}$$

We would like to be able to prove that the norm of the operator corresponding to the left-hand side is of order $\varepsilon$. It is clear from the previous discussion that the last two terms on the right satisfy the desired estimate. The same would be true for the first term if we could integrate by parts and transfer $D_1$ from $(\mathcal{T}^\varepsilon - \mathcal{I})u_\varepsilon^+$ to $A(D_1 u_0 + D_2 U)$. The following technical result will be useful for this purpose.

**Lemma 7.1.** *Let $F \in C^{0,1}(\bar{\mathbb{R}}^d; \tilde{L}_2(Q))^{d \times n}$ be such that $D_2^* F(x, \cdot) = 0$ on $\tilde{H}^1(Q)^n$ for each $x \in \mathbb{R}^d$. Then $D_1^* \tau^\varepsilon \mathcal{T}^\varepsilon F = \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* F$ on $C_c^1(\mathbb{R}^d; C(Q))^n$ for any $\varepsilon > 0$.*

*Proof.* It suffices to check the assertion for $\varepsilon = 1$, because the general result will then follow from this special case applied to the function $(x, y) \mapsto F(\varepsilon x, y)$. After a change of variables, we must show that, for any $\varphi \in C_c^1(\mathbb{R}^d; C(Q))^n$,

$$(7.5) \quad \int_{\mathbb{R}^d} \int_Q \langle F(x, x+y), D_1\varphi(x, y) \rangle \, dx \, dy = \int_{\mathbb{R}^d} \int_Q \langle D_1^* F(x, x+y), \varphi(x, y) \rangle \, dx \, dy.$$

Were $F$ smooth, this would be nothing but the usual integration by parts formula. But we can find a sequence of divergence free smooth functions that converges, in a certain sense, to the function $F$, which will yield the desired conclusion.

If $e_k(y) = e^{2\pi i \langle y, k \rangle}$, where $k \in \mathbb{Z}^d$, then we let $F_K(x, \cdot)$ denote the partial sum of the Fourier series for $F(x, \cdot)$:

$$F_K(x, \cdot) = \sum_{|k| \leq K} \hat{F}_k(x) e_k.$$

By hypothesis, $D_2^* F(x, \cdot) = 0$ on $\tilde{H}^1(Q)^n$, so

$$\langle \hat{F}_k(x), k \rangle = (2\pi)^{-1} \int_Q \langle F(x, y), De_k(y) \rangle \, dy = 0$$

for each $k \in \mathbb{Z}^d$. An integration by parts then gives
(7.6)
$$\int_{\mathbb{R}^d} \int_Q \langle F_K(x, x+y), D_1\varphi(x, y) \rangle \, dx \, dy = \int_{\mathbb{R}^d} \int_Q \langle D_1^* F_K(x, x+y), \varphi(x, y) \rangle \, dx \, dy$$

(notice here that $D\hat{F}_k(x)$ are exactly the Fourier coefficients of $D_1 F(x, \cdot)$).

Our goal now is to pass from (7.6) to (7.5). Let $f$ be a function in $C^{0,1}(\bar{\mathbb{R}}^d; \tilde{L}_2(Q))$ and let $f_K(x, \cdot)$ be the partial sum of the Fourier series for $f(x, \cdot)$. We claim that $f_K \to f$ in the weak-$*$ topology on $C_c(\mathbb{R}^d \times Q)^*$ as $K \to \infty$. Indeed, given any $\psi \in C_c(\mathbb{R}^d \times Q)$, the sequence of functions $x \mapsto (f_K(x, \cdot), \psi(x, \cdot))_Q$ converges pointwise to the function $x \mapsto (f(x, \cdot), \psi(x, \cdot))_Q$, because $f_K(x, \cdot) \to f(x, \cdot)$ in $L_2$. In addition, all the functions in the sequence are supported in a compact set and are uniformly bounded, since

$$|(f_K(x, \cdot), \psi(x, \cdot))_Q| \leq \|f(x, \cdot)\|_{2,Q} \|\psi(x, \cdot)\|_{2,Q} \leq \|f\|_C \|\psi\|_C.$$

We see that $(f_K, \psi)_{\mathbb{R}^d \times Q} \to (f, \psi)_{\mathbb{R}^d \times Q}$ by the Lebesgue dominated convergence theorem, and the claim follows.

The proof is completed now by letting $K \to \infty$ in (7.6).                    $\square$

By definition, we have $A(D_1u_0 + D_2U) = A(I + D_2N)Du_0$. Assume for the moment that $u_0, u_\varepsilon^+ \in C_c^\infty(\mathbb{R}^d)^n$. We recall that, for each $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^{d \times n}$, $D_2^* A(x, \cdot)(I + D_2N(x, \cdot))\xi = 0$ on $\tilde{H}^1(Q)^n$, so Lemma 7.1 applies to show that

$$(7.7) \quad \begin{aligned} &\left( \tau^\varepsilon \mathcal{T}^\varepsilon A(D_1u_0 + D_2U), (\mathcal{T}^\varepsilon - \mathcal{I})D_1u_\varepsilon^+ \right)_{\mathbb{R}^d \times Q} \\ &= \left( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1u_0 + D_2U), (\mathcal{T}^\varepsilon - \mathcal{I})u_\varepsilon^+ \right)_{\mathbb{R}^d \times Q} \end{aligned}$$

for every $u_0, u_\varepsilon^+ \in C_c^\infty(\mathbb{R}^d)^n$. Moreover, since the form

$$(u_0, u_\varepsilon^+) \mapsto \left( \tau^\varepsilon \mathcal{T}^\varepsilon A(D_1u_0 + D_2U), (\mathcal{T}^\varepsilon - \mathcal{I})D_1u_\varepsilon^+ \right)_{\mathbb{R}^d \times Q}$$

is continuous on $H^1(\mathbb{R}^d)^n \times H^1(\mathbb{R}^d)^n$ and since the form

$$(u_0, u_\varepsilon^+) \mapsto \left( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1u_0 + D_2U), (\mathcal{T}^\varepsilon - \mathcal{I})u_\varepsilon^+ \right)_{\mathbb{R}^d \times Q}$$

is continuous on $H^2(\mathbb{R}^d)^n \times L_2(\mathbb{R}^d)^n$, the last equality holds for any $u_0 \in H^2(\mathbb{R}^d)^n$ and $u_\varepsilon^+ \in H^1(\mathbb{R}^d)^n$.

Now that we have this result, (7.4) becomes

$$
\begin{aligned}
(\mathcal{A}^0 u_0, &\, u_\varepsilon^+)_{\mathbb{R}^d} - (\mathcal{A}^\varepsilon(\mathcal{S}^\varepsilon u_0 + \varepsilon U_\varepsilon), u_\varepsilon^+)_{\mathbb{R}^d} \\
&= \left( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), (\mathcal{T}^\varepsilon - \mathcal{I}) u_\varepsilon^+ \right)_{\mathbb{R}^d \times Q} \\
&\quad - \left( \tau^\varepsilon [A, \mathcal{T}^\varepsilon](D_1 u_0 + D_2 U), D_1 u_\varepsilon^+ \right)_{\mathbb{R}^d \times Q} \\
&\quad - \varepsilon \left( \tau^\varepsilon A \mathcal{T}^\varepsilon D_1 U, D_1 u_\varepsilon^+ \right)_{\mathbb{R}^d \times Q}.
\end{aligned}
\tag{7.8}
$$

Putting (7.8) into (7.1), we finally obtain the desired identity:

$$
\begin{aligned}
\big( (\mathcal{A}_\mu^\varepsilon)^{-1} f &- (\mathcal{A}_\mu^0)^{-1} f - \varepsilon \mathcal{K}_\mu^\varepsilon f, g \big)_{\mathbb{R}^d} \\
&= \left( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), (\mathcal{T}^\varepsilon - \mathcal{I}) u_\varepsilon^+ \right)_{\mathbb{R}^d \times Q} \\
&\quad - \left( \tau^\varepsilon [A, \mathcal{T}^\varepsilon](D_1 u_0 + D_2 U), D_1 u_\varepsilon^+ \right)_{\mathbb{R}^d \times Q} \\
&\quad - \varepsilon \left( \tau^\varepsilon A \mathcal{T}^\varepsilon D_1 U, D_1 u_\varepsilon^+ \right)_{\mathbb{R}^d \times Q} \\
&\quad - (\mathcal{A}^\varepsilon (\mathcal{I} - \mathcal{S}^\varepsilon) u_0, u_\varepsilon^+)_{\mathbb{R}^d} + \varepsilon \mu (U_\varepsilon, u_\varepsilon^+)_{\mathbb{R}^d}.
\end{aligned}
\tag{7.9}
$$

We are now in a position to prove the theorems.

*Proof of Theorem* 6.1. We estimate each term in (7.9). By Lemmas 5.1 and 5.3,

$$
\begin{aligned}
\big| \big( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* &A(D_1 u_0 + D_2 U), (\mathcal{T}^\varepsilon - \mathcal{I}) u_\varepsilon^+ \big)_{\mathbb{R}^d \times Q} \big| \\
&\leq \| \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U) \|_{2, \mathbb{R}^d \times Q} \| (\mathcal{T}^\varepsilon - \mathcal{I}) u_\varepsilon^+ \|_{2, \mathbb{R}^d \times Q} \\
&\lesssim \varepsilon \big( \| D u_0 \|_{1,2,\mathbb{R}^d} + \| D_1 D_2 U \|_{2,\mathbb{R}^d \times Q} + \| D_2 U \|_{2,\mathbb{R}^d \times Q} \big) \| D u_\varepsilon^+ \|_{2,\mathbb{R}^d}.
\end{aligned}
\tag{7.10}
$$

Using Lemma 5.1 again, we see that

$$
\begin{aligned}
\big| \big( \tau^\varepsilon [A, \mathcal{T}^\varepsilon] &(D_1 u_0 + D_2 U), D_1 u_\varepsilon^+ \big)_{\mathbb{R}^d \times Q} \big| \\
&\leq \varepsilon r_Q [A]_{C^{0,1}} \| \tau^\varepsilon \mathcal{T}^\varepsilon (D_1 u_0 + D_2 U) \|_{2,\mathbb{R}^d \times Q} \| D_1 u_\varepsilon^+ \|_{2,\mathbb{R}^d \times Q} \\
&\lesssim \varepsilon \big( \| D u_0 \|_{2,\mathbb{R}^d} + \| D_2 U \|_{2,\mathbb{R}^d \times Q} \big) \| D u_\varepsilon^+ \|_{2,\mathbb{R}^d}
\end{aligned}
\tag{7.11}
$$

(recall that $r_Q = 1/2 \operatorname{diam} Q$) and

$$
\begin{aligned}
\varepsilon \big| \big( \tau^\varepsilon A \mathcal{T}^\varepsilon D_1 U, D_1 u_\varepsilon^+ \big)_{\mathbb{R}^d \times Q} \big| &\leq \varepsilon \| A \|_C \| \tau^\varepsilon \mathcal{T}^\varepsilon D_1 U \|_{2,\mathbb{R}^d \times Q} \| D_1 u_\varepsilon^+ \|_{2,\mathbb{R}^d \times Q} \\
&\lesssim \varepsilon \| D_1 U \|_{2,\mathbb{R}^d \times Q} \| D u_\varepsilon^+ \|_{2,\mathbb{R}^d}.
\end{aligned}
\tag{7.12}
$$

Next, it follows from the estimate (3.3) and Lemma 5.4 that

$$
\begin{aligned}
|(\mathcal{A}^\varepsilon (\mathcal{I} - \mathcal{S}^\varepsilon) u_0, u_\varepsilon^+)_{\mathbb{R}^d}| &\lesssim \| (\mathcal{I} - \mathcal{S}^\varepsilon) u_0 \|_{1,2,\mathbb{R}^d} \| u_\varepsilon^+ \|_{1,2,\mathbb{R}^d} \\
&\lesssim \varepsilon \| D u_0 \|_{1,2,\mathbb{R}^d} \| u_\varepsilon^+ \|_{1,2,\mathbb{R}^d}.
\end{aligned}
$$

Finally, Lemma 5.2 yields

$$
\varepsilon |(U_\varepsilon, u_\varepsilon^+)_{\mathbb{R}^d}| \leq \varepsilon \| U_\varepsilon \|_{2,\mathbb{R}^d} \| u_\varepsilon^+ \|_{2,\mathbb{R}^d} \lesssim \varepsilon \| U \|_{2,\mathbb{R}^d \times Q} \| u_\varepsilon^+ \|_{2,\mathbb{R}^d}.
$$

In summary, we have found that

$$
\begin{aligned}
\big| \big( (\mathcal{A}_\mu^\varepsilon)^{-1} f &- (\mathcal{A}_\mu^0)^{-1} f - \varepsilon \mathcal{K}_\mu^\varepsilon f, g \big)_{\mathbb{R}^d} \big| \\
&\lesssim \varepsilon \big( \| D u_0 \|_{1,2,\mathbb{R}^d} + \| D_1 D_2 U \|_{2,\mathbb{R}^d \times Q} + \| U \|_{1,2,\mathbb{R}^d \times Q} \big) \| u_\varepsilon^+ \|_{1,2,\mathbb{R}^d}.
\end{aligned}
$$

Now suppose that $g \in L_2(\mathbb{R}^d)^n$. Then from (4.6), (5.2), (5.3), (5.10) and $(3.5)^+$,

$$
\big| \big( (\mathcal{A}_\mu^\varepsilon)^{-1} f - (\mathcal{A}_\mu^0)^{-1} f, g \big)_{\mathbb{R}^d} \big| \lesssim \varepsilon \| f \|_{2,\mathbb{R}^d} \| g \|_{2,\mathbb{R}^d},
$$

which proves (6.1). On the other hand, setting $g = D^* h$ where $h \in L_2(\mathbb{R}^d)^{d \times n}$ and using (4.6), (5.2), (5.3) and $(3.5)^+$, we obtain

$$\left| \left( (\mathcal{A}_\mu^\varepsilon)^{-1} f - (\mathcal{A}_\mu^0)^{-1} f - \varepsilon \mathcal{K}_\mu^\varepsilon f, D^* h \right)_{\mathbb{R}^d} \right| \lesssim \varepsilon \| f \|_{2, \mathbb{R}^d} \| h \|_{2, \mathbb{R}^d},$$

which proves (6.2). □

*Proof of Theorem* 6.2. Let $u_0^+ = ((\mathcal{A}_\mu^0)^+)^{-1} g$, $U^+ = \mathcal{K}_\mu^+ g$ and $U_\varepsilon^+ = (\mathcal{K}_\mu^\varepsilon)^+ g$. As a first step, we rewrite the corrector $\mathcal{C}_\mu^\varepsilon$ dropping, as we may, terms with operator norm of order $\varepsilon$.

By the very definition of $\mathcal{C}_\mu^\varepsilon$,

$$(\mathcal{C}_\mu^\varepsilon f, g)_{\mathbb{R}^d} = (\mathcal{K}_\mu^\varepsilon f, g)_{\mathbb{R}^d} - (\mathcal{L}_\mu f, g)_{\mathbb{R}^d} - (\mathcal{M}_\mu^\varepsilon f, g)_{\mathbb{R}^d} + (f, (\mathcal{K}_\mu^\varepsilon)^+ g)_{\mathbb{R}^d} - (f, \mathcal{L}_\mu^+ g)_{\mathbb{R}^d}.$$

We claim that

$$
\begin{aligned}
(7.13) \qquad & -\varepsilon(\mathcal{L}_\mu f, g)_{\mathbb{R}^d} - \varepsilon(\mathcal{M}_\mu^\varepsilon f, g)_{\mathbb{R}^d} + \varepsilon(f, (\mathcal{K}_\mu^\varepsilon)^+ g)_{\mathbb{R}^d} - \varepsilon(f, \mathcal{L}_\mu^+ g)_{\mathbb{R}^d} \\
& \approx \left( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), (\mathcal{T}^\varepsilon - \mathcal{I})(u_0^+ + \varepsilon U_\varepsilon^+) \right)_{\mathbb{R}^d \times Q} \\
& \quad - \left( \tau^\varepsilon [A, \mathcal{T}^\varepsilon](D_1 u_0 + D_2 U), D_1(u_0^+ + \varepsilon U_\varepsilon^+) \right)_{\mathbb{R}^d \times Q} \\
& \quad - \varepsilon \left( \tau^\varepsilon A \mathcal{T}^\varepsilon D_1 U, D_1(u_0^+ + \varepsilon U_\varepsilon^+) \right)_{\mathbb{R}^d \times Q},
\end{aligned}
$$

where the symbol $\approx$ is used to indicate equality up to terms that will eventually be absorbed into the error.

Indeed, by Lemma 5.1 we have

$$(\mathcal{L}_\mu f, g)_{\mathbb{R}^d} = \left( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), \tau^\varepsilon \mathcal{T}^\varepsilon U^+ \right)_{\mathbb{R}^d \times Q}.$$

Now observe that $\tau^\varepsilon \mathcal{T}^\varepsilon U^+$ may be replaced by $\tau^\varepsilon \mathcal{S}^\varepsilon U^+$. This is so because

$$
\begin{aligned}
& \left| \left( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), \tau^\varepsilon \mathcal{T}^\varepsilon U^+ - \tau^\varepsilon \mathcal{S}^\varepsilon U^+ \right)_{\mathbb{R}^d \times Q} \right| \\
& \qquad \leq \| \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U) \|_{2, \mathbb{R}^d \times Q} \| \tau^\varepsilon \mathcal{T}^\varepsilon U^+ - \tau^\varepsilon \mathcal{S}^\varepsilon U^+ \|_{2, \mathbb{R}^d \times Q},
\end{aligned}
$$

whence, by Lemmas 5.1 and 5.5 and the estimates (4.6), (5.2), (5.3) and $(5.2)^+$,

$$\left| \left( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), \tau^\varepsilon \mathcal{T}^\varepsilon U^+ - \tau^\varepsilon \mathcal{S}^\varepsilon U^+ \right)_{\mathbb{R}^d \times Q} \right| \lesssim \varepsilon \| f \|_{2, \mathbb{R}^d} \| g \|_{2, \mathbb{R}^d}.$$

Recalling that $U_\varepsilon^+ = \tau^\varepsilon \mathcal{S}^\varepsilon U^+$, we see that

$$(7.14) \qquad (\mathcal{L}_\mu f, g)_{\mathbb{R}^d} \approx \left( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), U_\varepsilon^+ \right)_{\mathbb{R}^d \times Q}.$$

We next want to show that

$$(7.15) \qquad (f, \mathcal{L}_\mu^+ g)_{\mathbb{R}^d} \approx \left( \tau^\varepsilon A \mathcal{T}^\varepsilon D_1 U, D_1(u_0^+ + \varepsilon U_\varepsilon^+) \right)_{\mathbb{R}^d \times Q}.$$

According to Lemma 5.1,

$$(f, \mathcal{L}_\mu^+ g)_{\mathbb{R}^d} = \left( \tau^\varepsilon \mathcal{T}^\varepsilon A D_1 U, \tau^\varepsilon \mathcal{T}^\varepsilon (D_1 u_0^+ + D_2 U^+) \right)_{\mathbb{R}^d \times Q}.$$

We commute $\mathcal{T}^\varepsilon$ through $A$ and use Lemma 5.1 and the estimates (5.2) and $(4.6)^+$, $(5.2)^+$ to get

$$(f, \mathcal{L}_\mu^+ g)_{\mathbb{R}^d} \approx \left( \tau^\varepsilon A \mathcal{T}^\varepsilon D_1 U, \tau^\varepsilon \mathcal{T}^\varepsilon (D_1 u_0^+ + D_2 U^+) \right)_{\mathbb{R}^d \times Q}$$

(notice here that $\tau^\varepsilon [A, \mathcal{T}^\varepsilon] = \tau^\varepsilon (\mathcal{I} - \mathcal{T}^\varepsilon) A \cdot \tau^\varepsilon \mathcal{T}^\varepsilon$). A similar argument using Lemma 5.3 shows that $\tau^\varepsilon \mathcal{T}^\varepsilon D_1 u_0^+$ (which is, of course, equal to $\mathcal{T}^\varepsilon D_1 u_0^+$) may be replaced by $D_1 u_0^+$. With a little extra care we can pass from $\tau^\varepsilon \mathcal{T}^\varepsilon D_2 U^+$ to $\varepsilon D_1 U_\varepsilon^+$, as well. Indeed, $\varepsilon D_1 U_\varepsilon^+ = \varepsilon \tau^\varepsilon \mathcal{S}^\varepsilon D_1 U^+ + \tau^\varepsilon \mathcal{S}^\varepsilon D_2 U^+$, where $\varepsilon \tau^\varepsilon \mathcal{S}^\varepsilon D_1 U^+$ creates another error term and $\tau^\varepsilon \mathcal{S}^\varepsilon D_2 U^+$ is handled exactly as above, by Lemma 5.5. Hence (7.15) is proved.

Repeating these last arguments for $\tau^\varepsilon \mathcal{T}^\varepsilon (D_1 u_0^+ + D_2 U^+)$, we find also that

$$(7.16) \qquad (\mathcal{M}_\mu^\varepsilon f, g)_{\mathbb{R}^d} \approx \varepsilon^{-1} \big( \tau^\varepsilon [A, \mathcal{T}^\varepsilon](D_1 u_0 + D_2 U), D_1(u_0^+ + \varepsilon U_\varepsilon^+) \big)_{\mathbb{R}^d \times Q}.$$

Let us turn to the term involving $(\mathcal{K}_\mu^\varepsilon)^+$. By the definition of $u_0$ and $U_\varepsilon^+$,

$$(f, (\mathcal{K}_\mu^\varepsilon)^+ g)_{\mathbb{R}^d} = (\mathcal{A}^0 u_0, U_\varepsilon^+)_{\mathbb{R}^d} - \mu(u_0, U_\varepsilon^+)_{\mathbb{R}^d}.$$

Applying Lemmas 5.4 and $5.7^+$ and the estimates (4.6) and $(5.10)^+$ yields

$$|(u_0, U_\varepsilon^+)_{\mathbb{R}^d}| \le |((\mathcal{S}^\varepsilon - \mathcal{I})u_0, U_\varepsilon^+)_{\mathbb{R}^d}| + |(u_0, \mathcal{S}^\varepsilon U_\varepsilon^+)_{\mathbb{R}^d}| \lesssim \varepsilon \|f\|_{2,\mathbb{R}^d} \|g\|_{2,\mathbb{R}^d},$$

so

$$(f, (\mathcal{K}_\mu^\varepsilon)^+ g)_{\mathbb{R}^d} \approx (\mathcal{A}^0 u_0, U_\varepsilon^+)_{\mathbb{R}^d}.$$

Thus, from Lemma 5.1 and the definition of the effective coefficients, we have

$$(7.17) \qquad (f, (\mathcal{K}_\mu^\varepsilon)^+ g)_{\mathbb{R}^d} \approx \big( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), \mathcal{T}^\varepsilon U_\varepsilon^+ \big)_{\mathbb{R}^d \times Q}.$$

To summarize: by (7.14)–(7.17), (7.13) reduces to showing that

$$(7.18) \qquad \big| \big( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), (\mathcal{T}^\varepsilon - \mathcal{I})u_0^+ \big)_{\mathbb{R}^d \times Q} \big| \lesssim \varepsilon^2 \|f\|_{2,\mathbb{R}^d} \|g\|_{2,\mathbb{R}^d}.$$

Let us prove (7.18). From Lemma 7.1, we know that

$$\big( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), (\mathcal{T}^\varepsilon - \mathcal{I})u_0^+ \big)_{\mathbb{R}^d \times Q}$$
$$= \big( \tau^\varepsilon \mathcal{T}^\varepsilon A(D_1 u_0 + D_2 U), (\mathcal{T}^\varepsilon - \mathcal{I})D_1 u_0^+ \big)_{\mathbb{R}^d \times Q}$$

(cf. (7.7)). Lemmas 5.3 and 5.5 and the estimates (4.6), (5.2), (5.3) and $(4.6)^+$ enable us to replace $\tau^\varepsilon \mathcal{T}^\varepsilon A(D_1 u_0 + D_2 U)$ with $\tau^\varepsilon \mathcal{S}^\varepsilon A(D_1 u_0 + D_2 U)$. Reversing the order of integration to switch $\mathcal{T}^\varepsilon$ and $\mathcal{S}^\varepsilon$ and again using Lemma 7.1, we get

$$\big( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), (\mathcal{T}^\varepsilon - \mathcal{I})u_0^+ \big)_{\mathbb{R}^d \times Q}$$
$$\approx \big( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), (\mathcal{S}^\varepsilon - \mathcal{I})u_0^+ \big)_{\mathbb{R}^d \times Q}.$$

It then follows from Lemmas 5.1 and 5.4 and the estimates (4.6), (5.2), (5.3) and $(4.6)^+$ that

$$\big| \big( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), (\mathcal{S}^\varepsilon - \mathcal{I})u_0^+ \big)_{\mathbb{R}^d \times Q} \big| \lesssim \varepsilon^2 \|f\|_{2,\mathbb{R}^d} \|g\|_{2,\mathbb{R}^d}.$$

We have verified (7.18), and therefore the claim is established.

Now we subtract (7.13) from (7.9) to obtain

$$\big( (\mathcal{A}_\mu^\varepsilon)^{-1} f - (\mathcal{A}_\mu^0)^{-1} f - \varepsilon \mathcal{C}_\mu^\varepsilon f, g \big)_{\mathbb{R}^d}$$
$$\approx \big( \tau^\varepsilon \mathcal{T}^\varepsilon D_1^* A(D_1 u_0 + D_2 U), (\mathcal{T}^\varepsilon - \mathcal{I})(u_\varepsilon^+ - u_0^+ - \varepsilon U_\varepsilon^+) \big)_{\mathbb{R}^d \times Q}$$
$$- \big( \tau^\varepsilon [A, \mathcal{T}^\varepsilon](D_1 u_0 + D_2 U), D_1(u_\varepsilon^+ - u_0^+ - \varepsilon U_\varepsilon^+) \big)_{\mathbb{R}^d \times Q}$$
$$- \varepsilon \big( \tau^\varepsilon A \mathcal{T}^\varepsilon D_1 U, D_1(u_\varepsilon^+ - u_0^+ - \varepsilon U_\varepsilon^+) \big)_{\mathbb{R}^d \times Q}$$
$$- (\mathcal{A}^\varepsilon (\mathcal{I} - \mathcal{S}^\varepsilon)u_0, u_\varepsilon^+)_{\mathbb{R}^d} + \varepsilon \mu(U_\varepsilon, u_\varepsilon^+)_{\mathbb{R}^d}.$$

Using the inequalities (7.10), (7.11) and (7.12) with $u_\varepsilon^+ - u_0^+ - \varepsilon U_\varepsilon^+$ in place of $u_\varepsilon^+$ and then applying the estimates (4.6), (5.2), (5.3) and $(6.2)^+$, we see that the norms of the operators associated with the first three forms on the right are of order $\varepsilon^2$. As for the last two forms, we write

$$(\mathcal{A}^\varepsilon (\mathcal{I} - \mathcal{S}^\varepsilon)u_0, u_\varepsilon^+)_{\mathbb{R}^d} = \big( (\mathcal{I} - \mathcal{S}^\varepsilon)u_0, g + \bar{\mu} u_\varepsilon^+ \big)_{\mathbb{R}^d}$$

and

$$\varepsilon(U_\varepsilon, u_\varepsilon^+)_{\mathbb{R}^d} = \varepsilon(\mathcal{S}^\varepsilon U_\varepsilon, u_\varepsilon^+)_{\mathbb{R}^d} + \varepsilon(U_\varepsilon, (\mathcal{I} - \mathcal{S}^\varepsilon)u_\varepsilon^+)_{\mathbb{R}^d}.$$

Then, by Lemma 5.4 and the estimates (4.6) and (3.5)$^+$,

$$|(\mathcal{A}^\varepsilon(\mathcal{I} - \mathcal{S}^\varepsilon)u_0, u_\varepsilon^+)_{\mathbb{R}^d}| \leq \|(\mathcal{I} - \mathcal{S}^\varepsilon)u_0\|_{2,\mathbb{R}^d}\big(\|g\|_{2,\mathbb{R}^d} + |\mu|\|u_\varepsilon^+\|_{2,\mathbb{R}^d}\big)$$
$$\lesssim \varepsilon^2\|f\|_{2,\mathbb{R}^d}\|g\|_{2,\mathbb{R}^d},$$

while, by Lemmas 5.4 and 5.7 and the estimates (5.10) and (3.5)$^+$,

$$\varepsilon|(U_\varepsilon, u_\varepsilon^+)_{\mathbb{R}^d}| \leq \varepsilon\|\mathcal{S}^\varepsilon U_\varepsilon\|_{2,\mathbb{R}^d}\|u_\varepsilon^+\|_{2,\mathbb{R}^d} + \varepsilon\|U_\varepsilon\|_{2,\mathbb{R}^d}\|(\mathcal{I} - \mathcal{S}^\varepsilon)u_\varepsilon^+\|_{2,\mathbb{R}^d}$$
$$\lesssim \varepsilon^2\|f\|_{2,\mathbb{R}^d}\|g\|_{2,\mathbb{R}^d}.$$

The proof is complete. $\qquad\square$

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