HOMOGENIZATION FOR LOCALLY PERIODIC ELLIPTIC PROBLEMS ON A DOMAIN

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1. Introduction

Let Ω be a bounded domain, and let $A\colon \Omega\times\mathbb{R}^d\to\mathbb{C}^{d\times d}$ be a uniformly elliptic function which is smooth in the first variable and periodic in the second. A classical result in homogenization theory tells us that, for any f in $H^{-1}(\Omega)$, the dual of the Sobolev space $\mathring{H}^1(\Omega)$, the solution $u_\varepsilon\in\mathring{H}^1(\Omega)$ of the Dirichlet problem

$$\begin{aligned} -\operatorname{div} A(x,\varepsilon^{-1}x)\nabla u_\varepsilon &= f \qquad \text{in } \Omega,\\ u_\varepsilon &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

converges, as $\varepsilon\to 0$, to the solution u_0 of a similar problem

$$\begin{aligned} -\operatorname{div} A^0(x)\nabla u_0 &= f \qquad \text{in } \Omega,\\ u_0 &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $A^0\colon\Omega\to\mathbb{C}^{d\times d}$ is a smooth function. In applications, this usually is interpreted as approximation of a highly heterogeneous medium, described by the rapidly oscillating locally periodic function $x\mapsto A(x,\varepsilon^{-1}x)$, with a homogeneous one, described by the slowly varying function $x\mapsto A^0(x)$.

There are various ways to prove the convergence. Among the first were the method of asymptotic expansions, using on powerful tools of asymptotic analysis (see [BLP78] or [BP84]), and the energy method, based on compensated compactness phenomenon (see [MT97]). Another way of dealing with the problem (1) is to use the two-scale convergence technique (see, e.g., [A92]). In any case, one finds that u_ε converges to u_0 weakly in the Sobolev space $\mathring{H}^1(\Omega)$, and therefore strongly in the Lebesgue space $L_2(\Omega)$. The latter can be phrased as saying that the resolvent of the operator $-\operatorname{div} A(x,\varepsilon^{-1}x)\nabla$ converges to the resolvent of the operator $-\operatorname{div} A^0(x)\nabla$ in the respective strong operator topology. A simple argument, see [AC98], using a compact embedding theorem then shows that the resolvent converges in the uniform operator topology on $L_2(\Omega)$, the strongest operator topology on $L_2(\Omega)$. However, this says nothing about the rate of convergence, nor does it apply to the case of unbounded Ω (or quasi-bounded, to be precise; see [AF03]).

A sharp-order bound on the rate was found in the pioneering paper [BSu01] (see also [BSu03]) for a purely periodic problem (when the coefficients depend

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on x/ε only) on $\mathbb{R}^d$. Uniform operator approximations in homogenization theory have attracted considerable attention since then, with a number of interesting results – see [Gri04], [Gri06], [Zh05], [ZhP05], [B08], [KLS12], [Su13$_1$], [Su13$_2$], [ChC16] and [ZhP16], to name just a few.

As each weakly convergent sequence of operators is bounded, one may ask whether a sequence of the resolvents of $-\operatorname{div} A(x, \varepsilon^{-1}x)\nabla$ converge in the uniform operator topology on $L_p(\Omega)$ provided that it is bounded in the operator norm from $W_p^{-1}(\Omega)$ to $\mathring{W}_p^1(\Omega)$ for p other than 2. Another question that naturally arises in this context is which domains and boundary conditions are allowed to still yield the convergence of the resolvent in the uniform operator topology on $L_p(\Omega)$. The answer we give in this paper is somewhat implicit, for it is formulated in terms of, e.g., boundary regularity results, but we provide some examples as well.

Let Ω be a uniformly Lipschitz domain (possibly unbounded). For fixed $p \in (1, \infty)$, let $\mathscr{W}_p^1(\Omega; \mathbb{C}^n)$ be a subspace of the Sobolev space $W_p^1(\Omega)^n$ that contains $\mathring{W}_p^1(\Omega)^n$, and let $\mathscr{W}_{p^+}^1(\Omega; \mathbb{C}^n)$ be defined similarly for the exponent p^+ conjugate to p. Let A_{kl} be $\mathbb{C}^{n\times n}$-matrix-valued mappings on $\bar{\Omega} \times \mathbb{R}^d$ that are Lipschitz in the first variable and periodic in the second and set $A = \{A_{kl}\}_{k,l=1}^d$. We will study the matrix operator

$$\mathcal{A}^\varepsilon = -\operatorname{div} A(x, \varepsilon^{-1}x)\nabla$$

acting between $\mathscr{W}_p^1(\Omega; \mathbb{C}^n)$ and $\mathscr{W}_p^{-1}(\Omega; \mathbb{C}^n)$, the dual of $\mathscr{W}_{p^+}^1(\Omega; \mathbb{C}^n)$. We point out that a function in $\mathscr{W}_p^1(\Omega; \mathbb{C}^n)$ may satisfy mixed boundary conditions and even different components of this function may satisfy different boundary conditions.

Suppose that, for some $\mu \in \mathbb{C}$ and all sufficiently small ε, the operators $\mathcal{A}^\varepsilon - \mu$ are isomorphisms with uniformly bounded (in ε) inverses. This condition is obviously necessary for the sequence $(\mathcal{A}^\varepsilon - \mu)^{-1}$ to have a limit even in the weak operator topology and thus is not related to homogenization; the next two definitely are. We assume that, for each $x \in \Omega$, the cell problem

$$-\operatorname{div} A(x, \cdot\,)(\nabla N(x, \cdot\,) + I) = 0$$

has a unique solution in $W_p^1(\mathbb{T}^d)$ which is Lipschitz in x, and the resolvent $(\mathcal{A}^0 - \mu)^{-1}$ of the effective operator $\mathcal{A}^0$ is continuous from $L_p(\Omega)^n$ to $\mathscr{W}_p^1(\Omega)^n \cap W_p^{1+s}(\Omega)^n$ for some $s \in (0, 1]$.

The basic examples are the Dirichlet and the Neumann problems for strongly elliptic operators $\mathcal{A}^\varepsilon$ on a bounded $C^{1,1}$ domain. In this case, there is a sector $\mathscr{S}$ in the complex plain and an open neighborhood $\mathscr{P}_0$ of the exponent 2 such that our assumptions hold for any $\mu \notin \mathscr{S}$ and $p \in \mathscr{P}_0$. Moreover, $\mathscr{P}_0 = (1, \infty)$ as long as the function A belongs to the VMO space in the “periodic” variable. See Section 7 for details.

In this paper we prove that

$$\|(\mathcal{A}^\varepsilon - \mu)^{-1} - (\mathcal{A}^0 - \mu)^{-1}\|_{L_p(\Omega)\to L_p(\Omega)} \le C\varepsilon^{s/p}, \tag{3}$$

$$\|\nabla(\mathcal{A}^\varepsilon - \mu)^{-1} - \nabla(\mathcal{A}^0 - \mu)^{-1} - \varepsilon\nabla\mathcal{K}_\mu^\varepsilon\|_{L_p(\Omega)\to L_p(\Omega)} \le C\varepsilon^{s/p}, \tag{4}$$

where $\mathcal{K}_\mu^\varepsilon$ is a so-called corrector, see Theorem 6.1. If, in addition, the adjoint of $\mathcal{A}^\varepsilon$ satisfies similar assumptions as $\mathcal{A}^\varepsilon$, then

$$\|(\mathcal{A}^\varepsilon - \mu)^{-1} - (\mathcal{A}^0 - \mu)^{-1}\|_{L_p(\Omega)\to L_p(\Omega)} \le C\varepsilon^{s}, \tag{5}$$

see Theorem 6.3. For $s = 1$, the convergence rate in (5) is the same as in the whole space case, which is known to be sharp with respect to the order. If we have a uniform Caccioppoli-type inequality for $\mathcal{A}^\varepsilon$, then the estimate (4) can be improved as well, but only away from the boundary. Thus, for a subdomain Σ with closure in Ω,

$$\|\nabla(\mathcal{A}^\varepsilon-\mu)^{-1}-\nabla(\mathcal{A}^0-\mu)^{-1}-\varepsilon\nabla\mathcal{K}_\mu^\varepsilon\|_{L_p(\Omega)\to L_p(\Sigma)}\le C\varepsilon^s, \tag{6}$$

see Corollary 6.5. We note that, in the whole space case, one can also find the second term in the approximation (5) so that the rate becomes of order ε^2, see [Se17$_1$], where the case $p=2$ was handled.

Purely periodic homogenization problems on a bounded domain are thoroughly studied. By using the unfolding method, Griso [Gri04], [Gri06] established uniform approximations (3)–(6) in the Hilbert-space case $p=2$ for scalar problems on $C^{1,1}$ domains with Dirichlet or Neumann boundary conditions, as well as on $C^{0,1}$ domains with mixed boundary conditions. In the case when $s=1$ and $p=2$, Zhikov [Zh05] and Zhikov with Pastukhova [ZhP05] (see also the survey paper [ZhP16] and the references therein) proved (3)–(4) for scalar problems and the linear elasticity system on sufficiently smooth domains with Dirichlet or Neumann boundary conditions. In [KLS12], the authors considered self-adjoint Dirichlet and Neumann problems on $C^{0,1}$ domains with Hölder continuous coefficients and, for $p=2$, obtained the approximation (5) with error of order $\varepsilon|\ln\varepsilon|^\sigma$ for any $\sigma>1/2$. They also improved the rate to ε if $s=1$. Quite general self-adjoint strongly elliptic systems on $C^{1,1}$ domains with Dirichlet or Neumann boundary conditions were studied by Suslina [Su13$_1$], [Su13$_2$], where the estimates (3)–(6) were proved for $s=1$ and $p=2$.

To prove the results, we study a first-order approximation, involving the resolvents of the original and the effective operators and the corrector. First-order approximations are well-known in homogenization theory, see, e.g., [BLP78] or [ZhKO93]. The one we use here differs from the classical one in that the corrector is now regularized. The idea of using a smoothing to regularize the classical corrector is due to Cioranescu, Damlamian and Griso, see [CDG02]. Besides the standard mollification, we employ the Steklov smoothing operator, which is the most simple and which had already proved to be quite useful for both linear and non-linear problems; see [Zh05] and [ZhP05], where that smoothing first appeared in the context of homogenization, as well as [PT07], [Su13$_1$] and [Su13$_2$]. We adopt the technique related to the Steklov smoothing operator from these papers.

For the first-order approximation, we derive an operator representation that splits the problem into interior and boundary parts, see (69). The interior part is treated in the same way as for the whole space case, cf. [Se17$_1$]. On the other hand, the boundary part, being supported in a small neighborhood of the boundary, is small as well, no matter what the boundary conditions.

We note that, once the estimates (3)–(6) are obtained, a limiting argument will lead to similar results for locally periodic operators whose coefficients are Hölder continuous in the first variable, see [Se17$_3$] for some details. We also mention the paper [Se20], where the elliptic bounds (4)–(5) for the Dirichlet problem with $s=1$ and $p=2$ were carried over to the parabolic semigroup by keeping track of the rate dependence on both the small parameter ε and spectral parameter μ.

2. Notation

The symbol $\|\cdot\|_U$ will stand for the norm on a normed space U. If U and V are Banach spaces, then $\mathbf{B}(U,V)$ is the Banach space of bounded linear operators from U to V. When $U=V$, the space $\mathbf{B}(U)=\mathbf{B}(U,U)$ becomes a Banach algebra with the identity map $\mathcal{I}$. The norm and the inner product on $\mathbb{C}^n$ are denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. We shall often identify $\mathbf{B}(\mathbb{C}^n,\mathbb{C}^m)$ and $\mathbb{C}^{m\times n}$.

Let Σ be a domain in $\mathbb{R}^d$ and U a Banach space. The space $C^{0,1}(\bar\Sigma;U)$ consists of those uniformly continuous functions $u\colon\Sigma\to U$ for which

$$\|u\|_{C^{0,1}(\bar\Sigma;U)}=\|u\|_{C(\bar\Sigma;U)}+[u]_{C^{0,1}(\bar\Sigma;U)}<\infty,$$

where $\|u\|_{C(\bar\Sigma;U)}=\sup_{x\in\Sigma}\|u(x)\|_U$ and

$$[u]_{C^{0,1}(\bar\Sigma;U)}=\sup_{\substack{x_1,x_2\in\Sigma,\\ x_1\neq x_2}}\frac{\|u(x_2)-u(x_1)\|_U}{|x_2-x_1|}.$$

We will use the notation $\|\cdot\|_{C^{0,1}}$, $\|\cdot\|_C$ and $[\cdot]_{C^{0,1}}$ as shorthand for $\|\cdot\|_{C^{0,1}(\bar\Sigma;U)}$, $\|\cdot\|_{C(\bar\Sigma;U)}$ and $[\cdot]_{C^{0,1}(\bar\Sigma;U)}$ when the context makes clear which Σ and U are meant. The corresponding modulus of continuity will be denoted by ω_u:

$$\omega_u(r)=\sup_{\substack{x_1,x_2\in\Sigma,\\ |x_1-x_2|<r}}\|u(x_2)-u(x_1)\|_U.$$

Let $L_0(\Sigma;U)$ be the vector space of all strongly measurable functions on Σ with values in U. The symbol $L_p(\Sigma;U)$, $p\in[1,\infty]$, stands for the L_p-space of $L_0(\Sigma;U)$-functions. For finite p and $s>0$, we let $W_p^s(\Sigma;U)$ denote the usual Sobolev space or Sobolev–Slobodetskii space of $L_0(\Sigma;U)$-functions on Σ with norm

$$\|u\|_{W_p^s(\Sigma;U)}=\Biggl(\sum_{|\alpha|=0}^{m}\|D^\alpha u\|_{L_p(\Sigma;U)}^p\Biggr)^{1/p}$$

if $s=m\in\mathbb{N}$ and

$$\|u\|_{W_p^s(\Sigma;U)}=\Biggl(\sum_{|\alpha|=m}\|D_{\Sigma,U}^{r,p}D^\alpha u\|_{L_p(\Sigma;U)}^p+\|u\|_{W_p^m(\Sigma;U)}^p\Biggr)^{1/p}$$

if $s=m+r$ with $m\in\mathbb{N}_0$ and $r\in(0,1)$. Here $D=-i\nabla$ and $D_{\Sigma,U}^{r,p}$ is the fractional derivative of order r given by

$$D_{\Sigma,U}^{r,p}u(x)=\biggl(\int_{-x+\Sigma}|h|^{-d-rp}\|\varDelta_h u(x)\|_U^p\,dh\biggr)^{1/p},$$

where $\varDelta_h u(x)=u(x+h)-u(x)$ and $x\in\Sigma$. In case $U=\mathbb{C}^n$, we write $\|\cdot\|_{p,\Sigma}$ and $\|\cdot\|_{s,p,\Sigma}$ for the norms on $L_p(\Sigma)^n=L_p(\Sigma;U)$ and $W_p^s(\Sigma)^n=W_p^s(\Sigma;U)$ and $(\cdot,\cdot)_\Sigma$ for the inner product on $L_2(\Sigma)^n$. When it is clear from the context which Σ and U are meant, we will write $D^{r,p}$ instead of $D_{\Sigma,U}^{r,p}$. The dual space of $W_p^s(\Sigma)^n$ under the pairing $(\cdot,\cdot)_\Sigma$ is denoted by $(W_p^s(\Sigma)^n)^*$, with $\|\cdot\|_{-s,p^+,\Sigma}^*$ standing for the norm. Here p^+ is the exponent conjugate to p, that is, $1/p^+=1-1/p$. The closure of $C_c^\infty(\Sigma)^n$ in $W_p^s(\Sigma)^n$ is $\mathring{W}_p^s(\Sigma)^n$, and $W_{p^+}^{-s}(\Sigma)^n$ is its dual, with norm $\|\cdot\|_{-s,p^+,\Sigma}$. The space $(W_p^s(\Sigma)^n)^*$ is continuously embedded in $W_{p^+}^{-s}(\Sigma)^n$.

Let Q be the closed cube in $\mathbb{R}^d$ with center 0 and side length 1, sides being parallel to the axes. Then $\tilde W_p^m(Q)^n$ denotes the completion of $\tilde C^m(Q)^n$ in the W_p^m-norm.

Here $\tilde{C}^m(Q)$ is the class of m-times continuously differentiable functions on Q whose periodic extension to $\mathbb{R}^d$ enjoys the same smoothness. Notice that $\tilde{L}_p(Q)^n$ can be identified with the space of all periodic functions in $L_{p,\mathrm{loc}}(\mathbb{R}^d)^n$. In a similar fashion, we define $\tilde{W}_p^m(\mathbb{R}^d \times Q)^n$ and $\tilde{C}^m(\mathbb{R}^d \times Q)^n$. The dual of $\tilde{W}_p^m$ is denoted by $\tilde{W}_{p^+}^{-m}$.

Let B be the open unit ball in $\mathbb{R}^d$ centered at the origin, and let B_+ be the open unit half-ball with $x_d \in (0,1)$. We say that Σ satisfies the uniform weak Lipschitz condition if there is a uniformly locally-finite open covering $\{W_k\}$ of $\partial\Sigma$ and a sequence of bi-Lipschitz transformations $\omega_k \colon W_k \to B$ so that (1) $\omega_k(W_k \cap \Sigma) = B_+$ and $\omega_k(W_k \cap \partial\Sigma) = \partial B_+ \setminus \partial B$; (2) $\sup_k [\omega_k]_{C^{0,1}(\bar{W}_k)}$ and $\sup_k [\omega_k^{-1}]_{C^{0,1}(\bar{B})}$ are finite; and (3) for some $\delta > 0$, any open ball $B_\delta(x)$ with $x \in \partial\Sigma$ is contained in a coordinate patch W_k. The last two conditions are automatically satisfied provided that the boundary of Σ is compact. Notice that the domain $\mathbb{R}^d \setminus \bar{\Sigma}$ is uniformly weakly Lipschitz whenever Σ is.

For such Σ, there exists a C^∞-partition of unity $\{\varphi_k\}$ subordinate to $\{W_k\}$ with the property that $\sup_k \|D^\alpha \varphi_k\|_{\infty, W_k}$ is finite for any α, see [Ste70, Chapter 6, Section 3]. Then there is an extension operator from $W_p^s(\Sigma)^n$ to $W_p^s(\mathbb{R}^d)^n$. It follows that standard density and embedding results which hold for $\mathbb{R}^d$ must also hold for Σ. In particular, the Sobolev theorem states that $W_p^1(\Sigma)^n$ is continuously embedded in $L_q(\Sigma)^n$ for any $q \in [p, p^*]$. Here p^* is the Sobolev conjugate to p given by $1/p^* = 1/p - 1/d$ if $p < d$; p^* is any finite number greater than or equal to p if $p = d$; and $p^* = \infty$ if $p > d$. By p_* we denote an exponent such that $W_{p_*}^1(\Sigma)^n$ is embedded in $L_p(\Sigma)^n$; more precisely, $p_* = 1$ if $p \in [1, d^+)$, p_* satisfies $1/p_* = 1/p + 1/d$ if $p \in [d^+, \infty)$ and p_* is any number greater than d if $p = \infty$.

If $p = 2$, we write H^s for W_p^s, H^{-s} for W_p^{-s}, etc.

For a set $\Xi \subset \mathbb{R}^d$, we let Ξ_δ denote a neighborhood of Ξ:

$$\Xi_\delta = \big\{x \in \Xi \colon \operatorname{dist}(x, \Xi) < r_Q \delta\big\},$$

where $2r_Q = \operatorname{diam} Q = d^{1/2}$. Thus, $\Xi + \delta Q \subset \Xi_\delta$.

We shall also need the $\mathrm{BMO}(\mathbb{R}^d)$ and $\mathrm{VMO}(\mathbb{R}^d)$ spaces. The former consists of all $u \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\|u\|_{\mathrm{BMO}} = \sup_{B_R} \fint_{B_R} |u(x) - m_{B_R}(u)|\, dx < \infty,$$

where $B_R \subset \mathbb{R}^d$ is a ball of radius R and $m_{B_R}(u) = \fint_{B_R} u(y)\, dy$ is the mean value of u over B_R. The latter is the subspace in $\mathrm{BMO}(\mathbb{R}^d)$ of all functions u for which the VMO-modulus, given by

$$\eta_u(r) = \sup_{B_R \colon R < r} \fint_{B_R} |u(x) - m_{B_R}(u)|\, dx,$$

tends to zero as $r \to 0$. We refer the reader to [Gra14$_2$] and [Gar07] for more on this matter.

We will often use the notation $\alpha \lesssim \beta$ (which is the same as saying that $\beta \gtrsim \alpha$) to mean that there is a positive constant C depending only on some fixed parameters (listed in Theorem 6.1–Corollary 6.6) such that $\alpha \le C\beta$.

Finally, $\alpha \wedge \beta$ and $\alpha \vee \beta$ are, respectively, the smaller and the larger of α and β.

3. Original operator

Let $\Omega \subset \mathbb{R}^d$ be a (possibly unbounded) domain satisfying the uniform weak Lipschitz condition. Define the operation τ^ε, $\varepsilon > 0$, as follows: given a function $u\colon \Omega \times \mathbb{R}^d \to L_0(Q)$, we set $\tau^\varepsilon u\colon \Omega \to L_0(Q)$ to be

$$\tau^\varepsilon u(x,z) = u(x, \varepsilon^{-1}x, z), \tag{7}$$

where $x \in \Omega$ and $z \in Q$. Obviously, τ^ε is a homomorphism of the respective algebras; in other words, for any two functions u and v from $\Omega \times \mathbb{R}^d$ to $L_0(Q)$

$$\tau^\varepsilon(u+v) = \tau^\varepsilon u + \tau^\varepsilon v, \qquad \tau^\varepsilon uv = \tau^\varepsilon u \cdot \tau^\varepsilon v \tag{8}$$

(the $\cdot$ denotes the pointwise product of functions). We adopt the notation $u^\varepsilon = \tau^\varepsilon u$.

Let A_{kl}, with $1 \le k, l \le d$, be a function in $C^{0,1}(\bar{\Omega}; \tilde{L}_\infty(Q))^{n\times n}$. Then $A = \{A_{kl}\}$ can be thought of as a bounded mapping $A\colon \bar{\Omega} \times \mathbb{R}^d \to \mathbf{B}(\mathbb{C}^{d\times n})$ which is Lipschitz in the first variable and periodic in the second. It follows that A satisfies a Carathéodory-type condition, i.e., $A(\,\cdot\,, y)$ is continuous on $\bar{\Omega}$ for almost every $y \in Q$ uniformly with respect to y and $A(x, \cdot\,)$ is measurable on $\mathbb{R}^d$ for each $x \in \bar{\Omega}$ (see, e.g., the proof of Lemma 5.6 in [A92]). Therefore, A^ε is measurable and uniformly bounded.

Fix $p \in (1,\infty)$. Let $\mathscr{W}_p^1(\Omega;\mathbb{C}^n)$ and $\mathscr{W}_{p^+}^1(\Omega;\mathbb{C}^n)$ be subspaces of, respectively, $W_p^1(\Omega)^n$ and $W_{p^+}^1(\Omega)^n$ that contain all functions in $C_c^\infty(\Omega)^n$; for instance,

$$\mathring{W}_p^1(\Omega)^n \subseteq \mathscr{W}_p^1(\Omega;\mathbb{C}^n) \subseteq W_p^1(\Omega)^n. \tag{9}$$

By $\mathscr{W}_p^{-1}(\Omega;\mathbb{C}^n)$ and $|\!|\!|\cdot|\!|\!|_{-1,p,\Omega}$ we denote the dual of $\mathscr{W}_{p^+}^1(\Omega;\mathbb{C}^n)$ (under the L_2-pairing) and the associated norm. Since $\mathscr{W}_p^{-1}(\Omega;\mathbb{C}^n)$ is isometrically isomorphic to the quotient space $(W_{p^+}^1(\Omega)^n)^*/(\mathscr{W}_{p^+}^1(\Omega;\mathbb{C}^n))^\perp$, where $(\mathscr{W}_{p^+}^1(\Omega;\mathbb{C}^n))^\perp$ is the subspace of all functionals on $W_{p^+}^1(\Omega)^n$ vanishing on $\mathscr{W}_{p^+}^1(\Omega;\mathbb{C}^n)$, the natural projection

$$\pi\colon f \mapsto f + (\mathscr{W}_{p^+}^1(\Omega;\mathbb{C}^n))^\perp \tag{10}$$

can be thought of as a continuous epimorphism of $(W_{p^+}^1(\Omega)^n)^*$ onto $\mathscr{W}_p^{-1}(\Omega;\mathbb{C}^n)$:

$$|\!|\!|\pi f|\!|\!|_{-1,p,\Omega} \le \|f\|_{-1,p,\Omega}^*. \tag{11}$$

Consider the matrix operator $\mathcal{A}^\varepsilon\colon \mathscr{W}_p^1(\Omega;\mathbb{C}^n) \to \mathscr{W}_p^{-1}(\Omega;\mathbb{C}^n)$ given by

$$\mathcal{A}^\varepsilon = D^* A^\varepsilon D, \tag{12}$$

that is, $\mathcal{A}^\varepsilon$ sends each $u \in \mathscr{W}_p^1(\Omega;\mathbb{C}^n)$ to the functional $v \mapsto (A^\varepsilon Du, Dv)_\Omega$ in $\mathscr{W}_p^{-1}(\Omega;\mathbb{C}^n)$. It is plain that $\mathcal{A}^\varepsilon$ is bounded uniformly with respect to ε:

$$|\!|\!|\mathcal{A}^\varepsilon u|\!|\!|_{-1,p,\Omega} \le \|A\|_{L_\infty} \|Du\|_{p,\Omega}, \qquad u \in \mathscr{W}_p^1(\Omega;\mathbb{C}^n). \tag{13}$$

We further assume that, for some $\mu \in \mathbb{C}$, there is $\varepsilon_\mu \in (0,1]$ so that the operators $\mathcal{A}_\mu^\varepsilon = \mathcal{A}^\varepsilon - \mu$ are isomorphisms for any $\varepsilon \in \mathscr{E}_\mu = (0, \varepsilon_\mu]$ and, moreover, have uniformly bounded inverses:

$$\|(\mathcal{A}_\mu^\varepsilon)^{-1} f\|_{1,p,\Omega} \lesssim |\!|\!|f|\!|\!|_{-1,p,\Omega}, \qquad f \in \mathscr{W}_p^{-1}(\Omega;\mathbb{C}^n). \tag{14}$$

Let $(\mathcal{A}_\mu^\varepsilon)^+$ be the adjont of $\mathcal{A}_\mu^\varepsilon$. The corresponding objects and results related to $(\mathcal{A}_\mu^\varepsilon)^+$, will be marked with "+" too. Notice that $(\mathcal{A}_\mu^\varepsilon)^+$ obeys (13^+) and (14^+), with the same constants, in fact.

Remark 3.1. Basic examples to keep in mind are the extreme cases where $\mathscr{W}_p^1(\Omega;\mathbb{C}^n)$ coincides with either $\mathring{W}_p^1(\Omega)^n$ or $W_p^1(\Omega)^n$. The first case corresponds to the homogeneous Dirichlet problem, and the second, to the homogeneous Neumann problem. Notice that components of u in $\mathscr{W}_p^1(\Omega;\mathbb{C}^n)$ may satisfy different boundary conditions.

Lemma 3.2. *Let $v=\chi(u-\xi)$, where $\chi\in C_c^{0,1}(\mathbb{R}^d)$, $\xi\in\mathbb{C}^n$ and $u\in\mathscr{W}_p^1(\Omega;\mathbb{C}^n)$. If $v\in\mathscr{W}_p^1(\Omega;\mathbb{C}^n)$, then*

$$\|v\|_{1,p,\Omega}\lesssim \||D\chi|\,Du\|_{p_*\vee 1,\Omega}+\||D\chi|\,(u-\xi)\|_{p,\Omega}+|\xi|\,\|\chi\|_{p,\Omega}+|||\chi\mathcal{A}_\mu^\varepsilon u|||_{-1,p,\Omega}, \tag{15}$$

where the constant depends only on p, n, μ, Ω, $\|A\|_{L_\infty}$ and the constant in the bound (14)*.*

Proof. A simple calculation yields the identity

$$\mathcal{A}_\mu^\varepsilon v=-(D\bar{\chi})^*\cdot A^\varepsilon Du+D^*A^\varepsilon(D\chi\cdot(u-\xi))+\mu\chi\xi+\chi\mathcal{A}_\mu^\varepsilon u.$$

Observe that $(p^+)^*=(p_*)^+$ and therefore $L_{p_*\vee 1}(\Omega)^n\subset\mathscr{W}_p^{-1}(\Omega;\mathbb{C}^n)$. Thus, using (14), we obtain (15). □

As a consequence of Lemma 3.2, we prove the well-known (weak) reverse Hölder inequality for the Dirichlet and Neumann problems. In what follows, Q_R will denote a closed cube in $\mathbb{R}^d$ having side length R, sides parallel to the axes, and αQ_R will denote the α-fold dilate of Q_R (with the same center).

Lemma 3.3. *Suppose that $d\ge 2$, $p\ge d^+$ and $\mathscr{W}_p^1(\Omega;\mathbb{C}^n)$ is either $\mathring{W}_p^1(\Omega)^n$ or $W_p^1(\Omega)^n$. Let $\mathcal{A}_\mu^\varepsilon u_\varepsilon=f+D^*F$ with $f\in L_p(\Omega)^n$ and $F\in L_p(\Omega)^{dn}$; we regard f and F as being identically zero outside Ω. Then there is $R_\Omega>0$, depending on d and Ω, such that, for any $Q_R\subset\mathbb{R}^d$ with $R\le R_\Omega$, one has*

$$\|u_\varepsilon\|_{1,p,Q_R\cap\Omega}\lesssim R^{-1}\big(\|Du_\varepsilon\|_{p_*,2Q_R\cap\Omega}+\|u_\varepsilon\|_{p_*,2Q_R\cap\Omega}\big)+\|f\|_{p_*,2Q_R}+\|F\|_{p,2Q_R}, \tag{16}$$

where the constant depends only on d, p, n, μ, Ω, $\|A\|_{L_\infty}$ and the constant in the bound (14)*.*

Proof. We intend to apply Lemma 3.2. Take $\chi\in C_c^{0,1}(2Q_R)$ such that $0\le\chi\le 1$ and $|D\chi(x)|\le 4/R$, with $\chi=1$ on Q_R and $\chi=0$ outside $3/2Q_R$. Notice that

$$|||\chi\mathcal{A}_\mu^\varepsilon u_\varepsilon|||_{-1,p,\Omega}\lesssim\|f\|_{p_*,2Q_R}+R^{-1}\|F\|_{p_*,2Q_R}+\|F\|_{p,2Q_R},$$

where we have used the fact that $L_{p_*}(\Omega)^n\subset\mathscr{W}_p^{-1}(\Omega;\mathbb{C}^n)$ as long as $p\ge d^+$. Since

$$\|F\|_{p_*,2Q_R}\le|2Q_R|^{1/d}\|F\|_{p,2Q_R}=2R\|F\|_{p,2Q_R}$$

by Hölder's inequality, we see that

$$|||\chi\mathcal{A}_\mu^\varepsilon u_\varepsilon|||_{-1,p,\Omega}\lesssim\|f\|_{p_*,2Q_R}+\|F\|_{p,2Q_R}. \tag{17}$$

We first consider the case when $\mathscr{W}_p^1(\Omega;\mathbb{C}^n)=\mathring{W}_p^1(\Omega)^n$. Extend u_ε by 0 outside Ω.

Suppose that $3/2Q_R\subset\bar{\Omega}$. According to Lemma 3.2, with $\xi=m_{2Q_R}(u_\varepsilon)$, and the estimate (17),

$$\begin{aligned}\|u_\varepsilon\|_{1,p,Q_R}&\le\|u_\varepsilon-m_{2Q_R}(u_\varepsilon)\|_{1,p,Q_R}+|2Q_R|^{1/p}|m_{2Q_R}(u_\varepsilon)|\\&\lesssim R^{-1}\big(\|Du_\varepsilon\|_{p_*,2Q_R}+\|u_\varepsilon-m_{2Q_R}(u_\varepsilon)\|_{p,2Q_R}\big)\\&\quad+|2Q_R|^{1/p}|m_{2Q_R}(u_\varepsilon)|+\|f\|_{p_*,2Q_R}+\|F\|_{p,2Q_R}.\end{aligned}$$

By the Hölder and the Sobolev–Poincaré inequalities, we have

$$|m_{2Q_R}(u_\varepsilon)| \le |2Q_R|^{-1/p_*}\|u_\varepsilon\|_{p_*,2Q_R}$$

and

$$\|u_\varepsilon - m_{2Q_R}(u_\varepsilon)\|_{p,2Q_R} \lesssim \|Du_\varepsilon\|_{p_*,2Q_R},$$

respectively. Thus, (16) follows.

On the other hand, if $3/2Q_R$ intersects $\mathbb{R}^d\setminus\bar\Omega$, then using Lemma 3.2, with $\xi = 0$, and keeping in mind (17), we obtain

$$\|u_\varepsilon\|_{1,p,Q_R} \lesssim R^{-1}\big(\|Du_\varepsilon\|_{p_*,2Q_R} + \|u_\varepsilon\|_{p,2Q_R}\big) + \|f\|_{p_*,2Q_R} + \|F\|_{p,2Q_R}.$$

Now find a point $x_0 \in 3/2Q_R \cap \partial\Omega$ in such a way that $1/2Q_R(x_0) \subset 2Q_R$. For uniformly weakly Lipschitz Ω, there are constants $c_\Omega > 0$ and $R_\Omega > 0$ so that for any cube $Q_r(x)$ with $x \in \partial\Omega$ and $r \le R_\Omega$

$$|Q_r(x) \setminus \Omega| \ge c_\Omega |Q_r(x)|. \tag{18}$$

Notice also that, if a function u vanishes on a set $\Sigma_0 \subset \Sigma$,

$$|\Sigma|\,|m_\Sigma(u)| \le \int_{\Sigma\setminus\Sigma_0} |u - m_\Sigma(u)|\,dx + |\Sigma\setminus\Sigma_0|\,|m_\Sigma(u)|. \tag{19}$$

Since $u_\varepsilon = 0$ on $(1/2Q_R(x_0)) \setminus \Omega$, (18) and (19) yield

$$\begin{aligned} |m_{2Q_R}(u_\varepsilon)| &\le \frac{|2Q_R|}{|(1/2Q_R(x_0))\setminus\Omega|} \fint_{2Q_R} |u_\varepsilon - m_{2Q_R}(u_\varepsilon)|\,dx \\ &\lesssim |2Q_R|^{-1/p}\|u_\varepsilon - m_{2Q_R}(u_\varepsilon)\|_{p,2Q_R} \end{aligned}$$

provided $R \le R_\Omega$. It then follows from the Sobolev–Poincaré inequality that

$$\|u_\varepsilon\|_{p,2Q_R} \le \|u_\varepsilon - m_{2Q_R}(u_\varepsilon)\|_{p,2Q_R} + |2Q_R|^{1/p}|m_{2Q_R}(u_\varepsilon)| \lesssim \|Du_\varepsilon\|_{p_*,2Q_R}.$$

As a result,

$$\|u_\varepsilon\|_{1,p,Q_R} \lesssim R^{-1}\|Du_\varepsilon\|_{p_*,2Q_R} + \|f\|_{p_*,2Q_R} + \|F\|_{p,2Q_R},$$

which implies (16).

Finally, if $\mathscr{W}^1_p(\Omega;\mathbb{C}^n) = W^1_p(\Omega)^n$, then $\chi(u_\varepsilon - m_{2Q_R}(u_\varepsilon)) \in \mathscr{W}^1_p(\Omega;\mathbb{C}^n)$ for any Q_R and we can repeat the argument used for the "interior" case $3/2Q_R \subset \bar\Omega$ above. □

Remark 3.4. The reverse Hölder inequality is a first step in proving higher integrability of the solution, together with its gradient, to an elliptic equation with Dirichlet or Neumann boundary conditions, see, e.g., [GiM79] or [Gia83]. If $d \ge 2$ and $p \ge d^+$, then from Lemma 3.3 one deduces that, given $f \in C_c^\infty(\Omega)^n$ and $F \in C_c^\infty(\Omega)^{dn}$,

$$\begin{aligned} \fint_{Q_R\cap\Omega} \big(|Du_\varepsilon(x)|^p + |u_\varepsilon(x)|^p\big)\,dx &\lesssim \bigg(\fint_{2Q_R\cap\Omega} \big(|Du_\varepsilon(x)|^p + |u_\varepsilon(x)|^p\big)^{p_*/p}\,dx\bigg)^{p/p_*} \\ &\quad + \fint_{2Q_R} |f(x)|^p\,dx + \fint_{2Q_R} |F(x)|^p\,dx \end{aligned}$$

whenever $R \le R_\Omega$. Therefore, by the generalization of Gehring's lemma due to Giaquinta and Modica [GiM79, Proposition 5.1], there is $p_\mu > p$, depending only on p, d and the constant in the previous estimate, such that the inequality

$$\left(\fint_{Q_R\cap\Omega}\left(|Du_\varepsilon(x)|^q+|u_\varepsilon(x)|^q\right)dx\right)^{1/q}\lesssim\left(\fint_{2Q_R\cap\Omega}\left(|Du_\varepsilon(x)|^p+|u_\varepsilon(x)|^p\right)dx\right)^{1/p}$$
$$+\left(\fint_{2Q_R}|f(x)|^q\,dx\right)^{1/q}+\left(\fint_{2Q_R}|F(x)|^q\,dx\right)^{1/q}$$

holds for each $q\in[p,p_\mu)$. Then an argument using a partition of unity shows that

$$\|u_\varepsilon\|_{1,q,\Omega}\lesssim\|f\|_{p\cap q,\Omega}+\|F\|_{p\cap q,\Omega}$$

(here $\|\cdot\|_{p\cap q,\Omega}$ is the norm on $L_p(\Omega)\cap L_q(\Omega)$). If Ω is bounded, this, together with a duality argument (notice that $p^+>d\geq d^+$ if $p<d^+$), implies that the range of $q>1$ for which $u_\varepsilon\in W^1_q(\Omega)^n$, is open, and moreover (14) is valid with any $q\in(p^+_\mu,p_\mu)$ in place of p. This can be viewed as a special case of the extrapolation result due to Shneiberg [Shn74] (see also [Agr13, Section 17.2]).

We close this section with another consequence of Lemma 3.2 , which will prove useful in the context of interior estimates.

Lemma 3.5. *Suppose that the inverse of $\mathcal{A}^\varepsilon_\mu$ is also bounded from $(W^1_{q^+}(\Omega)^n)^*$ to $W^1_q(\Omega)^n$ for some $q\in[1,p)$. Assume further that, given any $\chi\in C^{0,1}_c(\Omega)$, there is $\chi'\in C^{0,1}_c(\Omega)$, with $\operatorname{supp}\chi\subset\operatorname{supp}\chi'$, so that*

$$\|D\chi u\|_{q,\operatorname{supp}\chi}\lesssim\|u\|_{q,\operatorname{supp}\chi'}+|||\chi'\mathcal{A}^\varepsilon_\mu u|||_{-1,q,\Omega}\tag{20}$$

for all $u\in C^1_c(\Omega)^n$. Then a similar result holds with q replaced by p.

Proof. Fix $\chi\in C^{0,1}_c(\Omega)$ and choose a sequence of cutoff functions $\chi_k\in C^{0,1}_c(\Omega)$, where $0\leq k\leq m=\lceil d(1/q-1/p)\rceil$ (here $\lceil\cdot\rceil$ is the ceiling function), in such a way that $\chi_0=\chi$, $\operatorname{supp}\chi_k\subset\operatorname{supp}\chi_{k+1}$ and $\chi_{k+1}=1$ on $\operatorname{supp}\chi_k$. Let $q_0=p$ and $q_{k+1}=(q_k)_*$. Notice that (14) implies that the inverse of $\mathcal{A}^\varepsilon_\mu$ is bounded from $(W^1_{p^+}(\Omega)^n)^*$ to $W^1_p(\Omega)^n$ (according to (11)), and hence also from $(W^1_{q_k^+}(\Omega)^n)^*$ to $W^1_{q_k}(\Omega)^n$ for all $q_k\geq q$, via interpolation. Then, by Lemma 3.2 and Hölder's inequality,

$$\|D\chi_k u\|_{q_k,\operatorname{supp}\chi_k}\lesssim\|Du\|_{q_{k+1}\vee q,\operatorname{supp}\chi_k}+\|u\|_{p,\operatorname{supp}\chi_k}+|||\chi_k\mathcal{A}^\varepsilon_\mu u|||_{-1,p,\Omega}.$$

Iterating this and using the fact that $\chi_{k+1}=1$ on $\operatorname{supp}\chi_k$, we obtain

$$\|D\chi u\|_{p,\operatorname{supp}\chi}\lesssim\|Du\|_{q_m\vee q,\operatorname{supp}\chi_{m-1}}+\|u\|_{p,\operatorname{supp}\chi_{m-1}}+|||\chi_{m-1}\mathcal{A}^\varepsilon_\mu u|||_{-1,p,\Omega}.\tag{21}$$

Now note that $q_m\leq q$, so the hypothesis and Hölder's inequality show that

$$\|Du\|_{q_m\vee q,\operatorname{supp}\chi_{m-1}}\lesssim\|u\|_{p,\operatorname{supp}\chi'_m}+|||\chi'_m\mathcal{A}^\varepsilon_\mu u|||_{-1,p,\Omega}.$$

Substituting this to (21) gives the desired estimate with any $\chi'\in C^{0,1}_c(\Omega)$ which is 1 on $\operatorname{supp}\chi'_m$. □

4. Effective operator

As usual, the coefficients of the effective operator are described by the solution of the so-called cell problem. Let $D^{r,q}_1$ and $D^{r,q}_2$ stand for differentiation in the first variable and the second variable, respectively. Then the cell problem is as follows:

for each $\xi \in \mathbb{C}^{d\times n}$ and $x \in \Omega$, find $N_\xi(x,\cdot) \in \tilde{W}_p^1(Q)^n$, with $\fint_Q N_\xi(x,y)\,dy = 0$, satisfying

$$D_2^* A(x,\cdot)(D_2 N_\xi(x,\cdot) + \xi) = 0 \tag{22}$$

on $\tilde{W}_{p^+}^1(Q)^n$. We assume that such an N_ξ exists, is unique and is Lipschitz on $\bar{\Omega}$ with values in $\tilde{W}_p^1(Q)$. Since N_ξ depends linearly on ξ, the map assigning N_ξ to each ξ is simply an operator of multiplication by a function, which we denote by N. Thus,

$$N \in C^{0,1}(\bar{\Omega}; \tilde{W}_p^1(Q)). \tag{23}$$

A standard sufficient condition is this:

Lemma 4.1. *For any $x \in \Omega$, let $\mathcal{A}(x) = D_2^* A(x,\cdot) D_2$ be an isomorphism of $\tilde{W}_p^1(Q)^n/\mathbb{C}$ onto $\tilde{W}_p^{-1}(Q)^n$ with uniformly bounded (in x) inverse. Then the problem* (22) *has a unique solution, satisfying* (23).

Proof. By assumption,

$$N_\xi(x,\cdot) + \mathbb{C} = -\mathcal{A}(x)^{-1} D_2^* A(x,\cdot)\xi$$

is a unique solution of (22) and

$$\|D_2 N_\xi(x,\cdot)\|_{p,Q} \lesssim \|D_2^* A(x,\cdot)\xi\|_{-1,p,Q},$$

and therefore

$$\|D_2 N\|_{L_\infty(\Omega; L_p(Q))} \lesssim \|A\|_{L_\infty}.$$

Let $\mathcal{T}_h$, $h \in \mathbb{R}^d$, be the translation operator defined by $\mathcal{T}_h u(x,y) = u(x+h,y)$, where $u \in L_0(\mathbb{R}^d \times \mathbb{R}^d)$, and let $\Delta_h = \mathcal{T}_h - \mathcal{I}$. Obviously,

$$\Delta_h uv = \Delta_h u \cdot v + \mathcal{T}_h u \cdot \Delta_h v$$

for any $u, v \in L_0(\mathbb{R}^d \times \mathbb{R}^d)$. It follows that if $x, x+h \in \Omega$, then

$$\Delta_h N_\xi(x,\cdot) = -\mathcal{A}(x+h)^{-1} D_2^*(\Delta_h A(x,\cdot)\cdot(D_2 N_\xi(x,\cdot) + \xi)).$$

Hence,

$$\|D_2 \Delta_h N_\xi(x,\cdot)\|_{p,Q} \lesssim \|D_2^*(\Delta_h A(x,\cdot)\cdot(D_2 N_\xi(x,\cdot)+\xi))\|_{-1,p,Q},$$

and as a result

$$\|D_1 D_2 N\|_{L_\infty(\Omega; L_p(Q))} \lesssim \|D_1 A\|_{L_\infty} \|I + D_2 N\|_{L_\infty(\Omega; L_p(Q))}.$$

We have verified that $D_2 N \in C^{0,1}(\bar{\Omega}; \tilde{L}_p(Q))$. It is then immediate from the Poincaré inequality that $N \in C^{0,1}(\bar{\Omega}; \tilde{L}_p(Q))$ as well. □

Now define the effective operator $\mathcal{A}^0 \colon \mathscr{W}_p^1(\Omega; \mathbb{C}^n) \to \mathscr{W}_p^{-1}(\Omega; \mathbb{C}^n)$ by setting

$$\mathcal{A}^0 = D^* A^0 D, \tag{24}$$

where $A^0 \colon \bar{\Omega} \to \mathbf{B}(\mathbb{C}^{d\times n})$ is given by

$$A^0(x) = \int_Q A(x,y)(I + D_2 N(x,y))\,dy. \tag{25}$$

Notice that since A and $D_2 N$ are uniformly continuous in the first variable, so is A^0. In fact, we have $A^0 \in C^{0,1}(\bar{\Omega})$. Indeed, an easy calculation shows that

$$\|A^0\|_{L_\infty} \le \|A\|_{L_\infty} \|I + D_2 N\|_{L_\infty(\Omega; L_p(Q))}$$

and

$$\|D_1A^0\|_{L_\infty} \le \|A\|_{L_\infty}\|D_1D_2N\|_{L_\infty(\Omega;L_p(Q))} + \|D_1A\|_{L_\infty}\|I+D_2N\|_{L_\infty(\Omega;L_p(Q))}.$$

Thus, by (23), $\|A^0\|_{C^{0,1}(\bar\Omega)}$ is finite.

We suppose that there is $s\in(0,1]$ such that the operator $\mathcal{A}^0_\mu = \mathcal{A}^0-\mu$, with the same μ as in (14), has a continuous inverse from $L_p(\Omega)^n$ to $W_p^{1+s}(\Omega)^n$:

$$\|(\mathcal{A}^0_\mu)^{-1}f\|_{1+s,p,\Omega} \lesssim \|f\|_{p,\Omega}, \qquad f\in L_p(\Omega)^n. \tag{26}$$

Remark 4.2. Usually, one starts with an isomorphism $\mathcal{A}^0_\mu\colon \mathscr{W}^1_p(\Omega;\mathbb{C}^n)\to\mathscr{W}^{-1}_p(\Omega;\mathbb{C}^n)$, while additional regularity as in (26) requires that both the boundary of Ω and the boundary conditions be more regular as well. For the Dirichlet or the Neumann problems on uniformly $C^{1,1}$-regular domains, we have $s=1$, see, e.g., [McL00, Chapter 4]; the same holds under a weaker assumption that each coordinate map ω_k is a $(p,2)$-diffeomorphism with multiplier norm uniformly bounded in k, see [MSh09, Chapter 14]. In the case of mixed Dirichlet–Neumann problems, one cannot hope that s will be "too large" even for very regular domains and coefficients, as $|1+s-2/p|<1/2$ for the Laplacian on a half-space with mixed boundary conditions, see [Sha68]. We refer the reader also to [Grv11] for more on this matter.

5. Corrector

Fix an extension operator $\mathcal{E}$ that maps the spaces $W^1_p(\Omega)$ and $W^{1+s}_p(\Omega)$ continuously into, respectively, $W^1_p(\mathbb{R}^d)$ and $W^{1+s}_p(\mathbb{R}^d)$. We also extend the function N to $\mathbb{R}^d\times Q$ in such a way that $N\in C^{0,1}(\bar{\mathbb{R}}^d;\tilde W^1_p(Q))$ (e.g., by doing a reflection in the boundary). Define the operator $\mathcal{K}_\mu\colon L_p(\Omega)^n\to W^s_p(\mathbb{R}^d;\tilde W^1_p(Q)^n)$ to be

$$\mathcal{K}_\mu = ND_1\mathcal{E}(\mathcal{A}^0_\mu)^{-1}. \tag{27}$$

From the assumptions (23) and (26) we immediately conclude that $\mathcal{K}_\mu$ is bounded:

$$\begin{aligned}\|D_1^{s,p}D_2\mathcal{K}_\mu f\|_{p,\mathbb{R}^d\times Q} &+ \|D_2\mathcal{K}_\mu f\|_{p,\mathbb{R}^d\times Q}\\ &+\|D_1^{s,p}\mathcal{K}_\mu f\|_{p,\mathbb{R}^d\times Q}+\|\mathcal{K}_\mu f\|_{p,\mathbb{R}^d\times Q}\lesssim\|f\|_{p,\Omega}.\end{aligned} \tag{28}$$

The image of $\mathcal{K}_\mu$ is contained in the space $W^1_p(\mathbb{R}^d;\tilde W^1_p(Q)^n)$ only if $s=1$. For the other cases, we will use mollification to regularize the operator $\mathcal{E}(\mathcal{A}^0_\mu)^{-1}$ in $\mathcal{K}_\mu$.

Fix a non-negative function $J\in C_c^\infty(B_1(0))$ with $\int_{\mathbb{R}^d}J(x)\,dx=1$. For $\delta>0$, let $\mathcal{J}_\delta$ be the standard operator of mollification, that is, $\mathcal{J}_\delta u = J_\delta * u$, where $J_\delta(x)=\delta^{-d}J(\delta^{-1}x)$. Obviously, the operator $\mathcal{J}_\delta$ maps $W^s_p(\mathbb{R}^d)$ into $W^1_p(\mathbb{R}^d)$, but for $s<1$ its norm blows up as $\delta\to0$. It is also known that $\mathcal{J}_\delta$ converges, as $\delta\to0$, to $\mathcal{I}$ in the operator norm on $L_p(\mathbb{R}^d)$. The next two lemmas provide the rates of blow-up and convergence, respectively.

Lemma 5.1. *Let $0<s\le r\le1$ and $q\in[1,\infty)$. Then for any $\delta>0$ and $u\in C_c^\infty(\mathbb{R}^d)$, we have*

$$\|D^{r,q}\mathcal{J}_\delta u\|_{q,\mathbb{R}^d}\lesssim\delta^{-(r-s)}\|D^{s,q}u\|_{q,\mathbb{R}^d}. \tag{29}$$

Proof. We will prove that, for $r<1$,

$$\|D^{r,q}\mathcal{J}_\delta u\|_{q,\mathbb{R}^d}\lesssim\delta^{-(r-s)}(1-r)^{-1/q}\|D^{s,q}u\|_{q,\mathbb{R}^d}, \tag{30}$$

where the constant does not depend on r. It then follows from the formula

$$\lim_{r\to 1}(1-r)^{1/q}\|D^{r,q}u\|_{q,\mathbb{R}^d}=C_{d,q}\|Du\|_{q,\mathbb{R}^d},$$

see [BBM01], that (29) holds for $r=1$ as well.

Suppose first that $|h|\le\delta$. It is easy to see that

$$\varDelta_h\mathcal{J}_\delta u(x)=-\int_{\mathbb{R}^d}\varDelta_h J_\delta(x-\hat{x})\,\varDelta_{x-\hat{x}}u(\hat{x})\,d\hat{x},$$

where the integration is, in fact, running over $B_{2\delta}(x)$. Then, by Hölder's inequality,

$$\begin{aligned}&\int_{B_\delta(0)}|h|^{-d-rq}|\varDelta_h\mathcal{J}_\delta u(x)|^q\,dh\\&\qquad\le\int_{B_\delta(0)}|h|^{-d-rq}\biggl(\int_{B_{2\delta}(0)}|\varDelta_h J_\delta(\hat{x})|^{q^+}\,d\hat{x}\biggr)^{q-1}dh\int_{B_{2\delta}(0)}|\varDelta_{\hat{h}}u(x)|^q\,d\hat{h}.\end{aligned}$$

Since

$$\int_{B_{2\delta}(0)}|\varDelta_h J_\delta(\hat{x})|^{q^+}\,d\hat{x}\le\delta^{-q^++(1-q^+)d}|h|^{q^+}[J]_{C^{0,1}}^{q^+}|B_1(0)|,$$

the first integral on the right is estimated, up to a constant, by

$$\delta^{-d-q}\int_{B_\delta(0)}|h|^{-d+(1-r)q}\,dh\lesssim(1-r)^{-1}\delta^{-d-rq}.$$

The other integral is obviously bounded by $(2\delta)^{d+sq}|D^{s,q}u(x)|^q$. As a result,

$$\int_{\mathbb{R}^d}\int_{B_\delta(0)}|h|^{-d-rq}|\varDelta_h\mathcal{J}_\delta u(x)|^q\,dx\,dh\lesssim(1-r)^{-1}\delta^{-(r-s)q}\|D^{s,q}u\|_{q,\mathbb{R}^d}^q.\tag{31}$$

On the other hand, if $|h|>\delta$, then using the identity

$$\varDelta_h\mathcal{J}_\delta u(x)=\int_{\mathbb{R}^d}J_\delta(\hat{x})\,\varDelta_h u(x-\hat{x})\,d\hat{x}$$

and applying Hölder's inequality yield

$$\begin{aligned}&\int_{\mathbb{R}^d\setminus B_\delta(0)}|h|^{-d-rq}|\varDelta_h\mathcal{J}_\delta u(x)|^q\,dh\\&\qquad\lesssim\delta^{-(r-s)q}\int_{B_\delta(0)}\biggl(\int_{B_\delta(0)}|J_\delta(\hat{x})|^{q^+}\,d\hat{x}\biggr)^{q-1}|D^{s,q}u(x-\hat{x})|^q\,d\hat{x}.\end{aligned}$$

Now

$$\int_{B_\delta(0)}|J_\delta(\hat{x})|^{q^+}\,d\hat{x}\le\delta^{(1-q^+)d}\|J\|_{L_\infty}|B_1(0)|,$$

and therefore

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\setminus B_\delta(0)}|h|^{-d-rq}|\varDelta_h\mathcal{J}_\delta u(x)|^q\,dx\,dh\lesssim\delta^{-(r-s)q}\|D^{s,q}u\|_{q,\mathbb{R}^d}^q.\tag{32}$$

Combining (31) and (32), we obtain (30). □

Lemma 5.2. *Let $r\in(0,1)$ and $q\in[1,\infty)$. Then for any $\delta>0$ and $u\in C_c^\infty(\mathbb{R}^d)$, we have*

$$\|(\mathcal{J}_\delta-\mathcal{I})u\|_{q,\mathbb{R}^d}\lesssim\delta^r\|D^{r,q}u\|_{q,\mathbb{R}^d}.\tag{33}$$

Proof. We write

$$(\mathcal{J}_\delta-\mathcal{I})u(x)=\int_{\mathbb{R}^d} J_\delta(\hat{x})\,\Delta_{-\hat{x}}u(x)\,d\hat{x}$$

and then repeat the argument leading to (32) in Lemma 5.1. □

For $s\in(0,1)$, we define the operator $\mathcal{K}_\mu(\delta)\colon L_p(\Omega)^n\to W_p^1(\mathbb{R}^d;\tilde{W}_p^1(Q)^n)$ by

$$\mathcal{K}_\mu(\delta)=ND_1\mathcal{J}_\delta\mathcal{E}(\mathcal{A}_\mu^0)^{-1}; \tag{34}$$

we agree to set $\mathcal{K}_\mu(\delta)=\mathcal{K}_\mu$ for $s=1$. It follows from the assumptions (23) and (26), together with Lemma 5.1, that

$$\begin{aligned}\delta^{1-s}\|D_1D_2\mathcal{K}_\mu(\delta)f\|_{p,\mathbb{R}^d\times Q}&+\delta^{1-s}\|D_1\mathcal{K}_\mu(\delta)f\|_{1,p,\mathbb{R}^d\times Q}\\&+\|D_2\mathcal{K}_\mu(\delta)f\|_{p,\mathbb{R}^d\times Q}+\|\mathcal{K}_\mu(\delta)f\|_{p,\mathbb{R}^d\times Q}\lesssim\|f\|_{p,\Omega}\end{aligned} \tag{35}$$

for any $s\in(0,1]$ uniformly in $\delta>0$. Applying the Sobolev embedding theorem, we then see that, for any $s\in(0,1]$ and $q\in[p,p^*]$,

$$\delta^{1-s}\|\mathcal{K}_\mu(\delta)f\|_{q,\mathbb{R}^d\times Q}\lesssim\|f\|_{p,\Omega} \tag{36}$$

uniformly in $\delta\in(0,1]$.

Since we do not impose any extra assumptions on the coefficients of $\mathcal{A}^\varepsilon$, the function $\tau^\varepsilon N$ may fail to be measurable, and therefore the classical corrector $\tau^\varepsilon\mathcal{K}_\mu$ – and even the mollified one, $\tau^\varepsilon\mathcal{K}_\mu(\delta)$, – may not map $L_p(\Omega)^n$ into $L_0(\Omega)^n$. We use the Steklov smoothing operator to further regularize $\mathcal{K}_\mu(\delta)$.

5.1. **Smoothing.** Let $\mathcal{T}^\varepsilon\colon L_0(\mathbb{R}^d\times Q)\to L_0(\mathbb{R}^d\times Q;L_0(Q))$ be the translation operator

$$\mathcal{T}^\varepsilon u(x,y,z)=u(x+\varepsilon z,y), \tag{37}$$

where $(x,y)\in\mathbb{R}^d\times Q$ and $z\in Q$. Obviously, $\mathcal{T}^\varepsilon(u+v)=\mathcal{T}^\varepsilon u+\mathcal{T}^\varepsilon v$ and $\mathcal{T}^\varepsilon uv=\mathcal{T}^\varepsilon u\cdot\mathcal{T}^\varepsilon v$, so $\mathcal{T}^\varepsilon$ is an algebra homomorphism. Next, the formal adjoint of $\mathcal{T}^\varepsilon$ with respect to the L_2-pairing is given by the formula

$$(\mathcal{T}^\varepsilon)^*u(x,y)=\int_Q u(x-\varepsilon z,y,z)\,dz.$$

Then the Steklov smoothing operator $\mathcal{S}^\varepsilon$ is the restriction of $(\mathcal{T}^\varepsilon)^*$ to $L_1(\mathbb{R}^d\times Q)+L_\infty(\mathbb{R}^d\times Q)$; in other words,

$$\mathcal{S}^\varepsilon u(x,y)=\int_Q\mathcal{T}^\varepsilon u(x,y,z)\,dz. \tag{38}$$

The operator $\mathcal{S}^\varepsilon$ thus defined is formally self-adjoint.

Here we collect some well-known facts about $\mathcal{T}^\varepsilon$ and $\mathcal{S}^\varepsilon$, cf. [ZhP16, Subsection 2.1].

Lemma 5.3. *For any $q\in[1,\infty)$ and $\varepsilon>0$, $\tau^\varepsilon\mathcal{T}^\varepsilon$ is an isometry of $\tilde{L}_q(\mathbb{R}^d\times Q)$ into $L_q(\mathbb{R}^d;L_q(Q))$.*

Proof. By change of variable,

$$\|\tau^\varepsilon\mathcal{T}^\varepsilon u\|_{q,\mathbb{R}^d\times Q}^q=\int_{\mathbb{R}^d}\int_Q|u(x+\varepsilon z,\varepsilon^{-1}x)|^q\,dx\,dz=\int_{\mathbb{R}^d}\int_Q|u(x,\varepsilon^{-1}x-z)|^q\,dx\,dz.$$

But since u is periodic in the second variable, this equals $\|u\|_{q,\mathbb{R}^d\times Q}^q$. □

A related result for $\mathcal{S}^\varepsilon$ is immediate from Hölder's inequality and Lemma 5.3.

Lemma 5.4. *For any $q \in [1,\infty)$ and $\varepsilon > 0$, $\tau^\varepsilon \mathcal{S}^\varepsilon$ is a bounded operator from $\tilde{L}_q(\mathbb{R}^d \times Q)$ to $L_q(\mathbb{R}^d)$ of norm 1.*

Both $\mathcal{T}^\varepsilon$ and $\mathcal{S}^\varepsilon$ converge to the identity operator in uniform operator topologies, where the domain is "smoother" that the codomain.

Lemma 5.5. *Let Σ be a domain in $\mathbb{R}^d$, and let $r \in (0,1]$ and $q \in [1,\infty)$. Then for any $\varepsilon > 0$ and $u \in C_c^\infty(\mathbb{R}^d \times Q)$ we have*

$$\|(\mathcal{T}^\varepsilon - \mathcal{I})u\|_{q,\Sigma\times Q\times Q} \lesssim \varepsilon^r \|D_1^{r,q} u\|_{q,\Sigma_\varepsilon \times Q}. \tag{39}$$

Proof. For $r < 1$, the inequality (39) follows just by scaling. For $r = 1$, we write

$$u(x+\varepsilon z, y) - u(x,y) = \varepsilon i \int_0^1 \langle D_1 u(x+\varepsilon t z, y), z\rangle \, dt.$$

Hence,

$$\|(\mathcal{T}^\varepsilon - \mathcal{I})u(\,\cdot\,, y, z)\|_{q,\Sigma} \le \varepsilon r_Q \|D_1 u(\,\cdot\,, y)\|_{q,\Sigma_\varepsilon}.$$

Raising both sides to the qth power and integrating then yields (39). □

The next lemma comes from the previous one, together with Hölder's inequality.

Lemma 5.6. *Let Σ be a domain in $\mathbb{R}^d$, and let $r \in (0,1]$ and $q \in [1,\infty)$. Then for any $\varepsilon > 0$ and $u \in C_c^\infty(\mathbb{R}^d \times Q)$ we have*

$$\|(\mathcal{S}^\varepsilon - \mathcal{I})u\|_{q,\Sigma\times Q} \lesssim \varepsilon^r \|D_1^{r,q} u\|_{q,\Sigma_\varepsilon \times Q}. \tag{40}$$

5.2. **Corrector.** We define the corrector $\mathcal{K}_\mu^\varepsilon \colon L_p(\Omega)^n \to W_p^1(\Omega)^n$ by

$$\mathcal{K}_\mu^\varepsilon = \tau^\varepsilon \mathcal{S}^\varepsilon \mathcal{K}_\mu(\varepsilon). \tag{41}$$

Thanks to the smoothing $\mathcal{S}^\varepsilon$, it is bounded with

$$\varepsilon\|D\mathcal{K}_\mu^\varepsilon f\|_{p,\Omega} + \|\mathcal{K}_\mu^\varepsilon f\|_{p,\Omega} \lesssim \|f\|_{p,\Omega}. \tag{42}$$

Indeed, taking into account that $\varepsilon D\tau^\varepsilon\mathcal{S}^\varepsilon = \varepsilon\tau^\varepsilon\mathcal{S}^\varepsilon D_1 + \tau^\varepsilon\mathcal{S}^\varepsilon D_2$ and using Lemma 5.4, we see that

$$\begin{aligned}\varepsilon\|D\mathcal{K}_\mu^\varepsilon f\|_{p,\Omega} + \|\mathcal{K}_\mu^\varepsilon f\|_{p,\Omega} &\lesssim \varepsilon\|D_1\mathcal{K}_\mu(\varepsilon) f\|_{p,\Omega_\varepsilon\times Q} \\ &\quad + \|D_2\mathcal{K}_\mu(\varepsilon) f\|_{p,\Omega_\varepsilon\times Q} + \|\mathcal{K}_\mu(\varepsilon) f\|_{p,\Omega_\varepsilon\times Q}.\end{aligned}$$

The estimate (42) then follows from (35). We also notice that (36) implies the bound

$$\varepsilon^{1-s}\|\mathcal{K}_\mu^\varepsilon f\|_{q,\Omega} \lesssim \|f\|_{p,\Omega} \tag{43}$$

for the same range of q as in (36).

Remark 5.7. The operator $\mathcal{K}_\mu^\varepsilon$ may be written explicitly as

$$\mathcal{K}_\mu^\varepsilon f(x) = \int_Q N(x+\varepsilon z, \varepsilon^{-1}x)\,\mathcal{J}_\varepsilon \mathcal{E} D(\mathcal{A}_\mu^0)^{-1} f(x+\varepsilon z)\, dz.$$

It first appeared for $s = 1$ (in which case $\mathcal{J}_\varepsilon$ is dropped from $\mathcal{K}_\mu^\varepsilon$) in the paper [PT07].

6. Main results

Now we formulate the main results of the paper. The first one deals with approximation under minimal assumptions on the initial problem.

Theorem 6.1. *If* (14), (23) *and* (26) *hold, then for any* $\varepsilon \in \mathscr{E}_\mu$ *and* $f \in L_p(\Omega)^n$ *we have*

$$\|(\mathcal{A}_\mu^\varepsilon)^{-1} f - (\mathcal{A}_\mu^0)^{-1} f\|_{q,\Omega} \lesssim \varepsilon^{s/p} \|f\|_{p,\Omega}, \tag{44}$$

$$\|D(\mathcal{A}_\mu^\varepsilon)^{-1} f - D(\mathcal{A}_\mu^0)^{-1} f - \varepsilon D\mathcal{K}_\mu^\varepsilon f\|_{p,\Omega} \lesssim \varepsilon^{s/p} \|f\|_{p,\Omega}, \tag{45}$$

where $q \in [p, p^*]$. *The constants depend only on the parameters* d, s, p, q, n, μ, *the domain* Ω, *the* $C^{0,1}$*-norms of* A *and* N *and the constants in the bounds* (14) *and* (26).

Notice that the inverse of $\mathcal{A}_\mu^\varepsilon$ actually does converge in the operator norm from L_p to W_p^r with $r < 1$, yet the rate may be not as good.

Corollary 6.2. *Under the hypotheses of Theorem* 6.1, *for any* $r \in (0, 1)$, $\varepsilon \in \mathscr{E}_\mu$ *and* $f \in L_p(\Omega)^n$ *it holds that*

$$\|D^{r,p}((\mathcal{A}_\mu^\varepsilon)^{-1} f - (\mathcal{A}_\mu^0)^{-1} f)\|_{p,\Omega} \lesssim \varepsilon^{s/p \wedge (1-r)} \|f\|_{p,\Omega}. \tag{46}$$

The constant depends only on the parameters d, r, s, p, n, μ, *the domain* Ω, *the* $C^{0,1}$*-norms of* A *and* N *and the constants in the bounds* (14) *and* (26).

We can improve the estimate (44) for $q = p$ provided that the adjoint problem enjoys the same regularity properties as the initial one.

Theorem 6.3. *Suppose that* (14), (23), (26) *and* (23$^+$), (26$^+$) *hold. Then for any* $\varepsilon \in \mathscr{E}_\mu$ *and* $f \in L_p(\Omega)^n$ *we have*

$$\|(\mathcal{A}_\mu^\varepsilon)^{-1} f - (\mathcal{A}_\mu^0)^{-1} f\|_{p,\Omega} \lesssim \varepsilon^{s} \|f\|_{p,\Omega}. \tag{47}$$

The constant depends only on the parameters d, s, p, n, μ, *the domain* Ω, *the* $C^{0,1}$*-norms of* A, N *and* N^+ *and the constants in the bounds* (14), (26) *and* (26$^+$).

The other estimate in Theorem 6.1 can be improved as well, but only if restricted to an interior of Ω.

Theorem 6.4. *Suppose that* (14), (23), (26) *and* (23$^+$), (26$^+$) *hold. Suppose further that for a given* $\chi \in C^{0,1}(\bar{\Omega})$ *with* $\operatorname{supp}\chi \subset \Omega$ *there is* $\chi' \in C^{0,1}(\bar{\Omega})$ *with* $\operatorname{supp}\chi \subset \operatorname{supp}\chi' \subset \Omega$ *such that for all* $\varepsilon \in \mathscr{E}_\mu$ *the interior energy estimate*

$$\|D\chi u\|_{p,\Omega} \lesssim \|u\|_{p,\Omega} + |||\chi' \mathcal{A}_\mu^\varepsilon u|||_{-1,p,\Omega}, \qquad u \in \mathscr{W}_p^1(\Omega; \mathbb{C}^n), \tag{48}$$

holds. Then for any $\varepsilon \in \mathscr{E}_\mu$ *and* $f \in L_p(\Omega)^n$

$$\|D\chi((\mathcal{A}_\mu^\varepsilon)^{-1} f - (\mathcal{A}_\mu^0)^{-1} f - \varepsilon \mathcal{K}_\mu^\varepsilon f)\|_{p,\Omega} \lesssim \varepsilon^{s} \|f\|_{p,\Omega}. \tag{49}$$

The constant depends only on the parameters d, s, p, n, μ, *the domain* Ω, *the* $C^{0,1}$*-norms of* A, N, N^+ *and* χ' *and the constants in the bounds* (14), (26), (26$^+$) *and* (48).

As an immediate corollary we have:

Corollary 6.5. *Let hypotheses be as in Theorem 6.4 with χ having the property that χ^{-1} is uniformly bounded on a domain Σ with $\bar{\Sigma} \subset \Omega$. Then, for any $\varepsilon \in \mathscr{E}_\mu$ and $f \in L_p(\Omega)^n$,*

$$\|D(\mathcal{A}_\mu^\varepsilon)^{-1} f - D(\mathcal{A}_\mu^0)^{-1} f - \varepsilon D\mathcal{K}_\mu^\varepsilon f\|_{p,\Sigma} \lesssim \varepsilon^s \|f\|_{p,\Omega}. \tag{50}$$

The constant depends only on the parameters d, s, p, n, μ, the domain Ω, the $C^{0,1}$-norms of A, N, N^+ and χ', the L_∞-norms of $D\chi$ and $\chi^{-1}|_\Sigma$ and the constants in the bounds (14), (26), (26$^+$) *and* (48).

The next result follows from Corollary 6.5 in the same manner as Corollary 6.2 comes from Theorem 6.1.

Corollary 6.6. *Let hypotheses be as in Theorem 6.4 with χ having the property that χ^{-1} is uniformly bounded on a domain Σ with $\bar{\Sigma} \subset \Omega$. Then, for any $r \in (0,1)$, $\varepsilon \in \mathscr{E}_\mu$ and $f \in L_p(\Omega)^n$,*

$$\|D^{r,p}((\mathcal{A}_\mu^\varepsilon)^{-1} f - (\mathcal{A}_\mu^0)^{-1} f)\|_{p,\Sigma} \lesssim \varepsilon^{s\wedge(1-r)} \|f\|_{p,\Omega}. \tag{51}$$

The constant depends only on the parameters d, r, s, p, n, μ, the domain Ω, the $C^{0,1}$-norms of A, N, N^+ and χ', the L_∞-norms of $D\chi$ and $\chi^{-1}|_\Sigma$ and the constants in the bounds (14), (26), (26$^+$) *and* (48).

Remark 6.7. The corrector $\varepsilon\mathcal{K}_\mu^\varepsilon$ is usually involved in an approximation for $(\mathcal{A}_\mu^\varepsilon)^{-1}$ in the "energy" norm. If we want to approximate $D(\mathcal{A}_\mu^\varepsilon)^{-1}$ only, we may use the operator $\tau^\varepsilon\mathcal{S}^\varepsilon D_2\mathcal{K}_\mu(\varepsilon)$ instead, because

$$\varepsilon D\mathcal{K}_\mu^\varepsilon = \varepsilon\tau^\varepsilon\mathcal{S}^\varepsilon D_1\mathcal{K}_\mu(\varepsilon) + \tau^\varepsilon\mathcal{S}^\varepsilon D_2\mathcal{K}_\mu(\varepsilon),$$

where

$$\|\tau^\varepsilon\mathcal{S}^\varepsilon D_1\mathcal{K}_\mu(\varepsilon) f\|_{p,\Omega} \lesssim \|f\|_{p,\Omega}$$

by Lemma 5.4 and the estimate (35).

Remark 6.8. The results of Theorem 6.4 and Corollaries 6.5 and 6.6 rely on an a priory bound (48). In view of Lemma 3.5, for a compactly supported function χ this can be reduced to a similar bound with a smaller exponent $q \geq 1$, provided that (14) holds also for q in place of p.

Remark 6.9. A glance at (45) and (50) suggests that the rate of approximation for $D(\mathcal{A}_\mu^\varepsilon)^{-1}$ becomes worse only near the boundary of Ω. In fact, one can introduce a boundary-layer correction term $\mathcal{B}_\mu^\varepsilon$ so that for any $\varepsilon \in \mathscr{E}_\mu$ and $f \in L_p(\Omega)^n$

$$\|(\mathcal{A}_\mu^\varepsilon)^{-1} f - (\mathcal{A}_\mu^0)^{-1} f - \varepsilon\mathcal{K}_\mu^\varepsilon f - \mathcal{B}_\mu^\varepsilon f\|_{1,p,\Omega} \lesssim \varepsilon^s \|f\|_{p,\Omega}. \tag{52}$$

For $s = 1$ and $p = 2$, such a result was the starting point of the approach suggested in [ZhP05] (see also [PT07], [PSu12], [Su13$_1$] and [Su13$_2$]). However, the construction of $\mathcal{B}_\mu^\varepsilon$ is no simpler than the original problem and actually amounts to finding the inverse of $\mathcal{A}_\mu^\varepsilon$. Thus, that approach required further analysis of the boundary-layer correction term to obtain bounds on its norms.

We also note that if $\Omega = \mathbb{R}^d$ (or, more generally, Ω is a flat manifold without boundary, such as, e.g., $\mathbb{T}^d$), then $\mathcal{B}_\mu^\varepsilon = 0$. This enables one to improve the rates in (44)–(45) to ε^s, which, at least for $s = 1$, is known to be sharp.

Remark 6.10. By inspection of the proofs, one can see that the estimates in Theorem 6.1–Corollary 6.6 follow from inequalities with $\|(\mathcal{A}_\mu^0)^{-1} f\|_{1+s,p,\Omega}$ in place of $\|f\|_{p,\Omega}$ on the right. Thus, if (26) fails to hold, but $(\mathcal{A}_\mu^0)^{-1} f \in \mathscr{W}_p^1(\Omega;\mathbb{C}^n) \cap$

$W_p^{1+s}(\Omega)^n$ for some $f \in L_p(\Omega)^n$, then for fixed such f we still have, e.g., results similar to Theorem 6.1 and Corollary 6.2.

7. Examples

In the examples below we assume that Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with $C^{1,1}$ boundary and $\mathscr{W}_q^1(\Omega;\mathbb{C}^n)$ is either $\mathring{H}^1(\Omega)^n$ for all q or $H^1(\Omega)^n$ for all q, in which case $\mathscr{W}_q^1(\Omega;\mathbb{C}^n) \subset \mathscr{W}_p^1(\Omega;\mathbb{C}^n)$ whenever $q \geq p$.

7.1. **Strongly elliptic operators.** Let $p = 2$, and let $\mathscr{H}^1(\Omega;\mathbb{C}^n) = \mathscr{W}_2^1(\Omega;\mathbb{C}^n)$. Suppose that the operator $\mathcal{A}^\varepsilon$ is weakly coercive uniformly in ε for ε sufficiently small, that is, there are $\varepsilon_0 \in (0,1]$ and $c_A > 0$ and $C_A < \infty$ so that for all $\varepsilon \in \mathscr{E}_0 = (0,\varepsilon_0]$

$$\operatorname{Re}(\mathcal{A}^\varepsilon u, u)_\Omega + C_A\|u\|_{2,\Omega}^2 \geq c_A\|Du\|_{2,\Omega}^2, \qquad u \in \mathscr{H}^1(\Omega;\mathbb{C}^n). \tag{53}$$

With this assumption, $\mathcal{A}^\varepsilon$ becomes strongly elliptic, which means that the function A satisfies the Legendre–Hadamard condition

$$\operatorname{Re}\langle A(\,\cdot\,)\xi\otimes\eta, \xi\otimes\eta\rangle \geq c_A|\xi|^2|\eta|^2, \qquad \xi \in \mathbb{R}^d, \eta \in \mathbb{C}^n \tag{54}$$

(see Lemma 7.2). What is more, a simple calculation based on boundedness and coercivity of A^ε shows that if $\varepsilon \in \mathscr{E}_0$, then $\mathcal{A}^\varepsilon$ is an m-sectorial operator with sector

$$\mathscr{S} = \left\{z \in \mathbb{C}\colon |\operatorname{Im} z| \leq c_A^{-1}\|A\|_{L_\infty}(\operatorname{Re} z + C_A)\right\}$$

independent of ε, and therefore (14) holds for $p = 2$ and any $\varepsilon \in \mathscr{E}_0$ provided that $\mu \notin \mathscr{S}$. As we have seen in Remark 3.4, the estimate (14) is then valid for any $p \in (p_\mu^+, p_\mu)$, with $p_\mu > 2$ depending only on d, μ, Ω and the ellipticity constants c_A, C_A and $\|A\|_C$.

Let $p_0 = \sup_{\mu\notin\mathscr{S}} p_\mu$, and set $\mathscr{P}_0 = (p_0^+, p_0)$. We show that (14) holds, in fact, for any $p \in \mathscr{P}_0$ and $\mu \notin \mathscr{S}$ uniformly in $\varepsilon \in \mathscr{E}_0$. Indeed, suppose that $\mu, \nu \notin \mathscr{S}$ and choose $p \in (2, p_\nu)$. From the Sobolev embedding theorem, we know that $L_2(\Omega)^n$ is continuously embedded in $W_{q^+}^1(\Omega)^*$ for $q \in [2, 2^*]$ and in particular in $\mathscr{W}_{2^*\wedge p}^{-1}(\Omega;\mathbb{C}^n)$ (see (11)). Therefore, the first resolvent identity

$$(\mathcal{A}_\mu^\varepsilon)^{-1} = (\mathcal{A}_\nu^\varepsilon)^{-1} + (\mu - \nu)(\mathcal{A}_\nu^\varepsilon)^{-1}(\mathcal{A}_\mu^\varepsilon)^{-1}$$

yields that $(\mathcal{A}_\mu^\varepsilon)^{-1}$ is bounded from $\mathscr{W}_{2^*\wedge p}^{-1}(\Omega;\mathbb{C}^n)$ to $\mathscr{W}_{2^*\wedge p}^{1}(\Omega;\mathbb{C}^n)$. Repeating this procedure finitely many times, if need be, we conclude that the operator $(\mathcal{A}_\mu^\varepsilon)^{-1}$ is bounded from $\mathscr{W}_p^{-1}(\Omega;\mathbb{C}^n)$ to $\mathscr{W}_p^{1}(\Omega;\mathbb{C}^n)$.

Remark 7.1. No necessary and sufficient algebraic condition for A to assure (53) is known. A simpler condition not involving ε and still implying the weak coercivity on $\mathring{H}^1(\Omega)^n$ is that for some $c > 0$ and all $x \in \Omega$

$$\operatorname{Re}(A(x,\,\cdot\,)Du, Du)_{\mathbb{R}^d} \geq c\|Du\|_{2,\mathbb{R}^d}^2, \qquad u \in H^1(\mathbb{R}^d)^n. \tag{55}$$

That this hypothesis suffices can be seen by noticing that (55) is invariant under dilation and therefore remains true with $A(x,\varepsilon^{-1}y)$ in place of $A(x,y)$. Since A is uniformly continuous in the first variable, a localization argument then leads to (53), with $c_A < c$, $C_A > 0$ and $\mathscr{H}^1(\Omega;\mathbb{C}^n) = \mathring{H}^1(\Omega)^n$.

To give an example of A satisfying the strong coercivity condition on $\mathring{H}^1(\Omega)^n$ (i.e., with $C_A = 0$), take a matrix first-order differential operator $b(D)$ with symbol

$$\xi \mapsto b(\xi) = \sum_{k=1}^{d} b_k \xi_k,$$

where $b_k \in \mathbb{C}^{m\times n}$. Suppose that the symbol has the property that $\operatorname{rank} b(\xi) = n$ for any $\xi \in \mathbb{R}^d \setminus \{0\}$, or, equivalently, that

$$b(\xi)^* b(\xi) \ge c_b |\xi|^2, \qquad \xi \in \mathbb{R}^d.$$

Extending $u \in \mathring{H}^1(\Omega)^n$ by zero outside Ω and applying the Fourier transform, we see that the operator $b(D)^* b(D)$ is strongly coercive on $\mathring{H}^1(\Omega)^n$:

$$\|b(D)u\|_{2,\Omega}^2 \ge c_b \|Du\|_{2,\Omega}^2, \qquad u \in \mathring{H}^1(\Omega)^n. \tag{56}$$

Let $g \in C^{0,1}(\bar{\Omega}; \tilde{L}_\infty(Q))^{m\times m}$ with $\operatorname{Re} g$ uniformly positive definite and let $A_{kl} = b_k^* g b_l$. Then, by (56),

$$\begin{aligned}\operatorname{Re}(A^\varepsilon Du, Du)_\Omega &= \operatorname{Re}(g^\varepsilon b(D)u, b(D)u)_\Omega \\ &\ge c_b \|(\operatorname{Re} g)^{-1}\|_{L_\infty}^{-1} \|Du\|_{2,\Omega}^2\end{aligned}$$

for all $u \in \mathring{H}^1(\Omega)^n$. Purely periodic operators of this type were studied, e.g., in [PSu12] and [Su13$_1$].

For coercivity on $H^1(\Omega)^n$, we require a stronger condition on the symbol, namely, that $\operatorname{rank} b(\xi) = n$ for any $\xi \in \mathbb{C}^d \setminus \{0\}$, not just $\xi \in \mathbb{R}^d \setminus \{0\}$, which implies that

$$\|b(D)u\|_{2,\Omega}^2 \ge c_b \|Du\|_{2,\Omega}^2 - C_b \|u\|_{2,\Omega}^2, \qquad u \in H^1(\Omega)^n, \tag{57}$$

see [Ne12, Section 3.7, Theorem 7.8]. Then, obviously, for any $u \in H^1(\Omega)^n$

$$\begin{aligned}\operatorname{Re}(A^\varepsilon Du, Du)_\Omega &= \operatorname{Re}(g^\varepsilon b(D)u, b(D)u)_\Omega \\ &\ge \|(\operatorname{Re} g)^{-1}\|_{L_\infty}^{-1} \big(c_b \|Du\|_{2,\Omega}^2 - C_b \|u\|_{2,\Omega}^2\big),\end{aligned}$$

where A and g are as above. Such operators in the purely periodic setting appeared in [Su13$_2$].

Now we turn to the cell problem and the effective operator. The first thing that we need to check is that the cell problem (22) has a unique solution, for which (23) holds. Lemma 4.1 contains a sufficient condition to conclude these, and we will see in a moment that the operator $\mathcal{A}(x)$ does indeed meet the hypothesis of that lemma.

Lemma 7.2. *Assume that* (53) *holds. Then for any* $x \in \Omega$

$$\operatorname{Re}(\mathcal{A}(x)u, u)_Q \ge c_A \|Du\|_{2,Q}^2, \qquad u \in \tilde{H}^1(Q)^n. \tag{58}$$

Proof. Fix $u^{(\varepsilon)} = \varepsilon u^\varepsilon \varphi$ with $u \in \tilde{C}^1(Q)^n$ and $\varphi \in C_c^\infty(\Omega)$. We substitute $u^{(\varepsilon)}$ into (53) and let ε tend to 0. Then, because $u^{(\varepsilon)}$ and $Du^{(\varepsilon)} - (Du)^\varepsilon \varphi$ converge in L_2 to 0,

$$\lim_{\varepsilon\to 0} \operatorname{Re} \int_\Omega \langle A^\varepsilon(x)(Du)^\varepsilon(x), (Du)^\varepsilon(x)\rangle |\varphi(x)|^2\, dx \ge \lim_{\varepsilon\to 0} c_A \int_\Omega |(Du)^\varepsilon(x)|^2 |\varphi(x)|^2\, dx.$$

It is well known that if $f \in C_c(\mathbb{R}^d; \tilde{L}_\infty(Q))$, then

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} f^\varepsilon(x)\, dx = \int_{\mathbb{R}^d} \int_Q f(x,y)\, dx\, dy$$

(see, e.g., [A92, Lemmas 5.5 and 5.6]). As a result,

$$\operatorname{Re}\int_\Omega\int_Q \langle A(x,y)\,Du(y),Du(y)\rangle|\varphi(x)|^2\,dx\,dy \geq c_A\int_\Omega\int_Q |Du(y)|^2|\varphi(x)|^2\,dx\,dy.$$

But φ is an arbitrary function in $C_c^\infty(\Omega)$ and A is uniformly continuous in the first variable, so

$$\operatorname{Re}\int_Q \langle A(x,y)\,Du(y),Du(y)\rangle\,dy \geq c_A\int_Q |Du(y)|^2\,dy$$

for all $x\in\Omega$, as required. □

We see that, for any $x\in\Omega$, the operator $\mathcal{A}(x)$ is an isomorphism of $\tilde{H}^1(Q)^n/\mathbb{C}$ onto $\tilde{H}^{-1}(Q)^n$ and that the ellipticity constants of $\mathcal{A}(x)$ are better that those of $\mathcal{A}^\varepsilon$ (cf. (53) with (58)). Then the same arguments as in the proof of Lemma 3.3 and in Remark 3.4 show that $\mathcal{A}(x)$ is an isomorphism of $\tilde{W}_p^1(Q)^n/\mathbb{C}$ onto $\tilde{W}_p^{-1}(Q)^n$ for any $p\in\mathscr{P}_0$. Thus, the hypothesis of Lemma 4.1 is verified.

As for the effective operator, one can prove that, for any $\mu\notin\mathscr{S}$, the inverse for $\mathcal{A}_\mu^\varepsilon$ converges in the weak operator topology and then the limit is an isomorphism of $\mathscr{H}^{-1}(\Omega)^n$ onto $\mathscr{H}^1(\Omega)^n$, which is, in fact, the inverse for $\mathcal{A}_\mu^0$, see [Tar10, Lemma 6.2]. Now that we know that $\mathcal{A}_\mu^0$ is an isomorphism whenever $\mu\notin\mathscr{S}$ and that the function A^0 is Lipschitz, the assumption (26) follows for $s=1$ and actually any $p\in(1,\infty)$ by elliptic regularity, see, e.g., [McL00, Chapter 4].

Of course, all these results are true for the dual counterparts with the same range of p, because $c_{A^+}=c_A$, $C_{A^+}=C_A$ and $\|A^+\|_{L_\infty}=\|A\|_{L_\infty}$.

It remains to discuss the interior energy estimate (48). Let $p\in[2,p_0)$. Applying the functional $\mathcal{A}_\mu^\varepsilon u$ to $|\chi|^2u$, where $\chi\in C_c^{0,1}(\Omega)$, and using (53), we arrive at the well-known Caccioppoli inequality:

$$\|\chi Du\|_{2,\operatorname{supp}\chi}\lesssim\|u\|_{2,\operatorname{supp}\chi}+\|\chi\mathcal{A}_\mu^\varepsilon u\|_{-1,2,\Omega}^*,\qquad u\in\mathscr{H}^1(\Omega;\mathbb{C}^n),$$

Therefore, Lemma 3.5, for $q=2$, yields (48).

To summarize, if the coercivity condition (53) holds true, then the global results (see Theorem 6.1–Theorem 6.3) are valid with $s=1$ and $p\in\mathscr{P}_0$ and the local results (see Theorem 6.4–Corollary 6.6) are valid with $s=1$ and $p\in\mathscr{P}_0\cap[2,\infty)$.

Remark 7.3. The constants p_μ, and hence p_0, can be expressed explicitly. We note that generally one would not expect p_0 to be too large. In fact, it must tend to 2 as the ellipticity of the family $\mathcal{A}^\varepsilon$ becomes "bad" (that is, the ratio $c_A^{-1}\|A\|_{L_\infty}$ grows), see [Mey63]. In the next subsection we provide an example where p may be chosen arbitrary large.

7.2. **Strongly elliptic operators with VMO-coefficients.** Let $\mathcal{A}^\varepsilon$ be as in the previous subsection. Assume further that $A\in L_\infty(\Omega;\mathrm{VMO}(\mathbb{R}^d))$, meaning that $\sup_{x\in\Omega}\eta_{A(x,\cdot)}(r)\to0$ as $r\to0$.

Using the reflection technique, we extend A to be a function belonging to both $C_c^{0,1}(\mathbb{R}^d;\tilde{L}_\infty(Q))$ and $L_\infty(\mathbb{R}^d;\mathrm{VMO}(\mathbb{R}^d))$. Notice that A^ε is then a VMO-function. Indeed, A^ε obviously belongs to the space $\mathrm{BMO}(\mathbb{R}^d)$, with $\|A^\varepsilon\|_{\mathrm{BMO}}\leq2\|A\|_{L_\infty}$. Next, after dilation, we may suppose that $\varepsilon=1$. Given an $\epsilon>0$ small, there is

$r>0$ such that $\omega_A(r)<\epsilon/3$ and $\eta_{A(x,\,\cdot\,)}(r)<\epsilon/3$. Then, since

$$\fint_{B_R(x_0)}|A^1(x)-m_{B_R(x_0)}(A^1)|\,dx\le\fint_{B_R(x_0)}|A(x_0,x)-m_{B_R(x_0)}(A(x_0,\,\cdot\,))|\,dx+2\fint_{B_R(x_0)}|A(x,x)-A(x_0,x)|\,dx,$$

we have

$$\eta_{A^1}(r)\le\sup_{x_0\in\mathbb{R}^d}\eta_{A(x_0,\,\cdot\,)}(r)+2\omega_A(r)<\epsilon,$$

and the claim follows.

As a result, if $\varepsilon\in\mathscr{E}_0$ is fixed and $\mu\notin\mathcal{S}$, the inverse of $\mathcal{A}^\varepsilon_\mu$ is a continuous map from $\mathscr{W}^{-1}_p(\Omega;\mathbb{C}^n)$ to $\mathscr{W}^1_p(\Omega;\mathbb{C}^n)$ for each $p\in(1,\infty)$, see [She18]. Hence, in order to prove (14), we need only show that its norm is uniformly bounded in ε. We do this by treating $\mathcal{A}^\varepsilon$ as a local perturbation of a purely periodic operator and then applying results for purely periodic operators with rapidly oscillating coefficients.

First observe that if B_R is a ball with center in $\bar\Omega$ and radius R, then, by (53),

$$\operatorname{Re}(A^\varepsilon(x_0,\,\cdot\,)Dv,Dv)_{B_R\cap\Omega}\ge(c_A-\omega_A(R))\|Dv\|^2_{2,B_R\cap\Omega}-C_A\|v\|^2_{2,B_R\cap\Omega}$$

for all v in $\mathscr{H}^1(B_R\cap\Omega;\mathbb{C}^n)$, the space of functions whose zero extensions to Ω belong to $\mathscr{H}^1(\Omega;\mathbb{C}^n)$. It follows that for R small enough, the operator $D^*A^\varepsilon(x_0,\,\cdot\,)D$ from $\mathscr{H}^1(B_R\cap\Omega;\mathbb{C}^n)$ to the dual space $\mathscr{H}^{-1}(B_R\cap\Omega;\mathbb{C}^n)$ is m-sectorial, with sector

$$\mathcal{S}_R=\bigl\{z\in\mathbb{C}\colon|\operatorname{Im}z|\le(c_A-\omega_A(R))^{-1}\|A\|_{L_\infty}(\operatorname{Re}z+C_A)\bigr\}$$

converging pointwise to $\mathcal{S}$ as $R\to0$, that is, $\operatorname{dist}(z,\mathcal{S}_R)\to\operatorname{dist}(z,\mathcal{S})$ for $z\in\mathbb{C}$.

Now, fix $\mu\notin\mathcal{S}$ and find $R_0>0$ such that $\omega_A(R)\le c_A/2$ and $\mu\notin\mathcal{S}_R$ as long as $R\le R_0$. Let $F\in C^\infty_c(\Omega)^{dn}$ and $u_\varepsilon=(\mathcal{A}^\varepsilon_\mu)^{-1}D^*F$. Take $\chi\in C^\infty_c(B_R)$, $R\le R_0$, with the properties that $0\le\chi(x)\le1$ and $\chi=1$ on $1/2B_R$. Then $v_\varepsilon=\chi u_\varepsilon$ obviously satisfies

$$D^*A^\varepsilon(x_0,\,\cdot\,)Dv_\varepsilon-\mu v_\varepsilon=D^*(A^\varepsilon(x_0,\,\cdot\,)-A^\varepsilon)Dv_\varepsilon+g$$

in the sense of functionals on $\mathscr{H}^1(B_{R_0}\cap\Omega;\mathbb{C}^n)$, where $g=\chi D^*F+D^*(A^\varepsilon D\chi\cdot u_\varepsilon)-(D\chi)^*\cdot A^\varepsilon Du_\varepsilon$. This is a purely periodic problem, for which we know that the operator $D^*A^\varepsilon(x_0,\,\cdot\,)D-\mu$ is an isomorphism of $\mathscr{W}^1_q(B_{R_0}\cap\Omega;\mathbb{C}^n)$ onto $\mathscr{W}^{-1}_q(B_{R_0}\cap\Omega;\mathbb{C}^n)$ for any $q\in(1,\infty)$, with uniformly bounded inverse, see [She18]. Assuming that $p>2$ (the other case will follow by duality), we immediately find that

$$\|Dv_\varepsilon\|_{2^*\wedge p,B_{R_0}\cap\Omega}\lesssim\omega_A(R)\|Dv_\varepsilon\|_{2^*\wedge p,B_{R_0}\cap\Omega}+\|g\|^*_{-1,2^*\wedge p,B_{R_0}\cap\Omega},$$

the constant not depending on R. Choosing R sufficiently small, we may absorb the first term on the right into the left-hand side. Since

$$\|g\|^*_{-1,2^*\wedge p,B_{R_0}\cap\Omega}\lesssim\|F\|_{2^*\wedge p,B_R\cap\Omega}+\|u_\varepsilon\|_{1,2,B_R\cap\Omega}$$

(we have used the Sobolev embedding theorem to estimate the $L_{2^*\wedge p}$-norm of u_ε and the $\mathscr{W}^{-1}_{2^*\wedge p}$-norm of Du_ε), it follows that

$$\|Du_\varepsilon\|_{2^*\wedge p,1/2B_R\cap\Omega}\lesssim\|F\|_{2^*\wedge p,B_R\cap\Omega}+\|u_\varepsilon\|_{1,2,B_R\cap\Omega}.$$

Now, cover Ω with balls of radius R to obtain

$$\|Du_\varepsilon\|_{2^*\wedge p,\Omega}\lesssim\|F\|_{2^*\wedge p,\Omega}+\|u_\varepsilon\|_{1,2,\Omega}\lesssim\|F\|_{2^*\wedge p,\Omega}.$$

After a finite number of repetitions, if need be, we get

$$\|Du_\varepsilon\|_{p,\Omega}\lesssim\|F\|_{p,\Omega}.$$

Next, the hypothesis of Lemma 4.1 is satisfied, because of Lemma 7.2 and the fact that $A(x,\cdot) \in \mathrm{VMO}(\mathbb{R}^d)$ with VMO-modulus bounded uniformly in x. Finally, (26) and (48) hold for, respectively, $s = 1$ and any $p \in (1,\infty)$ and any $p \in [2,\infty)$, as indicated previously.

Summarizing, if $A \in L_\infty(\Omega; \mathrm{VMO}(\mathbb{R}^d))$ satisfies the coercivity condition (53), then the global results (see Theorem 6.1–Theorem 6.3) are valid with $s = 1$ and $p \in (1,\infty)$ and the local results (see Theorem 6.4–Corollary 6.6) are valid with $s = 1$ and $p \in [2,\infty)$.

8. Proof of the main results

We start with a "resolvent" identity involving $(\mathcal{A}^\varepsilon_\mu)^{-1}$, $(\mathcal{A}^0_\mu)^{-1}$ and $\mathcal{K}^\varepsilon_\mu$, which is a central part of the proof.

Fix $f \in L_p(\Omega)^n$ and $g \in (W^1_p(\Omega)^n)^*$. For $\delta = \varepsilon$, we set $u_0 = (\mathcal{A}^0_\mu)^{-1} f$, $u_{0,\delta} = \mathcal{J}_\delta \mathcal{E} (\mathcal{A}^0_\mu)^{-1} f$, $U = \mathcal{K}_\mu f$, $U_\delta = \mathcal{K}_\mu(\delta) f$, $U_{\varepsilon,\delta} = \tau^\varepsilon \mathcal{S}^\varepsilon U_\delta = \mathcal{K}^\varepsilon_\mu f$ and $u^+_\varepsilon = ((\mathcal{A}^\varepsilon_\mu)^+)^{-1} g$. We then have

$$((\mathcal{A}^\varepsilon_\mu)^{-1} f - (\mathcal{A}^0_\mu)^{-1} f - \varepsilon \mathcal{K}^\varepsilon_\mu f, g)_\Omega = (f, u^+_\varepsilon)_\Omega - (u_0, g)_\Omega - \varepsilon (U_{\varepsilon,\delta}, g)_\Omega.$$

By definition of u_0 and u^+_ε,

$$(f, u^+_\varepsilon)_\Omega - (u_0, g)_\Omega = (A^0 D u_0, D u^+_\varepsilon)_\Omega - (A^\varepsilon D u_0, D u^+_\varepsilon)_\Omega.$$

Choose a function $\rho_\varepsilon \in C^{0,1}(\bar{\Omega})$ with support in the closure of $(\partial\Omega)_{3\varepsilon} \cap \Omega$ and values in $[0,1]$ such that $\rho_\varepsilon|_{(\partial\Omega)_{2\varepsilon}\cap\Omega} = 1$ and $\|D\rho_\varepsilon\|_{\infty,\Omega} \lesssim \varepsilon^{-1}$. For example, we may set $\rho_\varepsilon(x) = 3 - \operatorname{dist}(x, \partial\Omega)/r_Q\varepsilon$ for $x \in \Omega \cap (\partial\Omega)_{3\varepsilon} \setminus (\partial\Omega)_{2\varepsilon}$. If $\chi_\varepsilon = 1 - \rho_\varepsilon$, then $\chi_\varepsilon U_{\varepsilon,\delta} \in \mathscr{W}^1_p(\Omega; \mathbb{C}^n)$, and we immediately conclude that

$$(\chi_\varepsilon U_{\varepsilon,\delta}, g)_\Omega = (A^\varepsilon D \chi_\varepsilon U_{\varepsilon,\delta}, D u^+_\varepsilon)_\Omega - \mu (\chi_\varepsilon U_{\varepsilon,\delta}, u^+_\varepsilon)_\Omega.$$

As a result,

(59)
$$\begin{aligned}
&((\mathcal{A}^\varepsilon_\mu)^{-1} f - (\mathcal{A}^0_\mu)^{-1} f - \varepsilon \mathcal{K}^\varepsilon_\mu f, g)_\Omega \\
&\quad = (\chi_\varepsilon A^0 D u_0, D u^+_\varepsilon)_\Omega - (\chi_\varepsilon A^\varepsilon D (u_0 + \varepsilon U_{\varepsilon,\delta}), D u^+_\varepsilon)_\Omega + \varepsilon\mu (\chi_\varepsilon U_{\varepsilon,\delta}, u^+_\varepsilon)_\Omega \\
&\qquad + (\rho_\varepsilon (A^0 - A^\varepsilon) D u_0, D u^+_\varepsilon)_\Omega + \varepsilon (A^\varepsilon D\rho_\varepsilon \cdot U_{\varepsilon,\delta}, D u^+_\varepsilon)_\Omega - \varepsilon (\rho_\varepsilon U_{\varepsilon,\delta}, g)_\Omega.
\end{aligned}$$

Let us focus on the first two terms on the right-hand side. The first one can be written, using (25), as

$$\begin{aligned}
(\chi_\varepsilon A^0 D u_0, D u^+_\varepsilon)_\Omega &= (\chi_\varepsilon A (D_1 u_0 + D_2 U), D_1 u^+_\varepsilon)_{\Omega\times Q} \\
&= (\chi_\varepsilon A (D_1 u_{0,\delta} + D_2 U_\delta), D_1 u^+_\varepsilon)_{\Omega\times Q} \\
&\quad + (\chi_\varepsilon A (D_1 (u_0 - u_{0,\delta}) + D_2 (U - U_\delta)), D_1 u^+_\varepsilon)_{\Omega\times Q}.
\end{aligned} \tag{60}$$

As for the second, notice that $\varepsilon D U_{\varepsilon,\delta} = \varepsilon \tau^\varepsilon \mathcal{S}^\varepsilon D_1 U_\delta + \tau^\varepsilon \mathcal{S}^\varepsilon D_2 U_\delta$, and hence

$$\begin{aligned}
(\chi_\varepsilon A^\varepsilon D (u_0 + \varepsilon U_{\varepsilon,\delta}), D u^+_\varepsilon)_\Omega &= (\tau^\varepsilon \chi_\varepsilon A \mathcal{T}^\varepsilon (D_1 u_{0,\delta} + D_2 U_\delta), D_1 u^+_\varepsilon)_{\Omega\times Q} \\
&\quad + (\tau^\varepsilon \chi_\varepsilon A \mathcal{T}^\varepsilon D_1 (u_0 - u_{0,\delta}), D_1 u^+_\varepsilon)_{\Omega\times Q} \\
&\quad + \varepsilon (\tau^\varepsilon \chi_\varepsilon A \mathcal{T}^\varepsilon D_1 U_\delta, D_1 u^+_\varepsilon)_{\Omega\times Q} \\
&\quad + (\chi_\varepsilon A^\varepsilon (\mathcal{I} - \mathcal{S}^\varepsilon) D u_0, D u^+_\varepsilon)_\Omega.
\end{aligned} \tag{61}$$

We commute $\mathcal{T}^\varepsilon$ past $\chi_\varepsilon A$ in the first term on the right,

(62)
$$\begin{aligned}(\tau^\varepsilon \chi_\varepsilon A\mathcal{T}^\varepsilon(D_1u_{0,\delta}+D_2U_\delta), D_1u_\varepsilon^+)_{\Omega\times Q} &= (\tau^\varepsilon\mathcal{T}^\varepsilon\chi_\varepsilon A(D_1u_{0,\delta}+D_2U_\delta), D_1u_\varepsilon^+)_{\Omega\times Q}\\ &\quad + (\tau^\varepsilon\chi_\varepsilon[A,\mathcal{T}^\varepsilon](D_1u_{0,\delta}+D_2U_\delta), D_1u_\varepsilon^+)_{\Omega\times Q}\\ &\quad - (\tau^\varepsilon[\rho_\varepsilon,\mathcal{T}^\varepsilon]A(D_1u_{0,\delta}+D_2U_\delta), D_1u_\varepsilon^+)_{\Omega\times Q},\end{aligned}$$

and then examine the difference

(63) $$(\chi_\varepsilon A(D_1u_{0,\delta}+D_2U_\delta), D_1u_\varepsilon^+)_{\Omega\times Q} - (\tau^\varepsilon\mathcal{T}^\varepsilon\chi_\varepsilon A(D_1u_{0,\delta}+D_2U_\delta), D_1u_\varepsilon^+)_{\Omega\times Q}.$$

Using Lemma 5.3 and noticing that χ_ε vanishes near the boundary and, moreover, so does $\mathcal{T}^\varepsilon\chi_\varepsilon$, we obtain

(64)
$$\begin{aligned}(\chi_\varepsilon A(D_1u_{0,\delta}+D_2U_\delta), D_1u_\varepsilon^+)_{\Omega\times Q} &= (D_1^*\chi_\varepsilon A(D_1u_{0,\delta}+D_2U_\delta), u_\varepsilon^+)_{\Omega\times Q}\\ &= (\tau^\varepsilon\mathcal{T}^\varepsilon D_1^*\chi_\varepsilon A(D_1u_{0,\delta}+D_2U_\delta), \mathcal{T}^\varepsilon u_\varepsilon^+)_{\Omega\times Q}.\end{aligned}$$

A similar result for the other term in (63) requires a technical lemma.

Lemma 8.1. *Fix $\varepsilon>0$. Let $F\in C^{0,1}(\bar\Omega;\tilde L_p(Q))^d$ be such that $F(x,\cdot)=0$ for $x\in(\partial\Omega)_\varepsilon$ and $D_2^*F(x,\cdot)=0$ as a functional in $\tilde W_p^{-1}(Q)$ for each $x\in\Omega$. Then $D_1^*\tau^\varepsilon\mathcal{T}^\varepsilon F=\tau^\varepsilon\mathcal{T}^\varepsilon D_1^*F$ on $C_c^1(\Omega)$, viewed as a subspace of $C_c(\Omega\times Q)$.*

Proof. Let φ be a function in $C_c^1(\Omega)^n$, extended by zero to all of $\mathbb{R}^d$. After a change of variables, we must show that

(65)
$$\begin{aligned}&\int_\Omega\int_Q\langle F(x,x/\varepsilon+y), D_1\varphi(x+\varepsilon y)\rangle\,dx\,dy\\ &\qquad=\int_\Omega\int_Q\langle D_1^*F(x,x/\varepsilon+y), \varphi(x+\varepsilon y)\rangle\,dx\,dy.\end{aligned}$$

Were $F(x,\cdot)$ smooth, this would be nothing but the usual integration by parts formula. But we can find a sequence of smooth functions F_K with $D_2^*F_K=0$ that converges, in a suitable sense, to the function F, and that will complete the proof.

If $e_k(y)=e^{2\pi i\langle y,k\rangle}$, where $k\in\mathbb{Z}^d$, then we let $F_K(x,\cdot)$ denote the square partial sum of the Fourier series for $F(x,\cdot)$:

$$F_K(x,\cdot)=\sum_{|k_j|\le K}\hat F_k(x)e_k.$$

By hypothesis, $D_2^*F(x,\cdot)=0$ on $\tilde W^1_{p^+}(Q)^n$, so

$$\langle\hat F_k(x),k\rangle=(2\pi)^{-1}\int_Q\langle F(x,y), De_k(y)\rangle\,dy=0$$

for each $k\in\mathbb{Z}^d$. Also notice that $D^*\hat F_k(x)$ are the Fourier coefficients of $D_1^*F(x,\cdot)$. An integration by parts then gives

(66)
$$\begin{aligned}&\int_\Omega\int_Q\langle F_K(x,x/\varepsilon+y), D_1\varphi(x+\varepsilon y)\rangle\,dx\,dy\\ &\qquad=\int_\Omega\int_Q\langle (D_1^*F)_K(x,x/\varepsilon+y), \varphi(x+\varepsilon y)\rangle\,dx\,dy.\end{aligned}$$

Here $(D_1^*F)_K(x,\cdot)$ is the square partial sum of the Fourier series for $D_1^*F(x,\cdot)$.

We now show that (66) implies (65). Let G be a function in $L_\infty(\mathbb{R}^d;\tilde L_p(Q))$, and let $G_K(x,\cdot)$ be the square partial sum of the Fourier series for $G(x,\cdot)$. We claim that $G_K \to G$ in the weak-$*$ topology on $C_c(\mathbb{R}^d \times Q)^*$ as $K \to \infty$. Indeed, given any $\psi \in C_c(\mathbb{R}^d \times Q)$, the sequence of functions $x \mapsto (G_K(x,\cdot), \psi(x,\cdot))_Q$ converges pointwise to the function $x \mapsto (G(x,\cdot), \psi(x,\cdot))_Q$, because $G_K(x,\cdot) \to G(x,\cdot)$ in $L_p(Q)$ (see [Gra14$_1$, Theorem 4.1.8]). In addition, all the functions in the sequence are supported in a single compact set and are uniformly bounded, since

$$|(G_K(x,\cdot), \psi(x,\cdot))_Q| \lesssim \|G(x,\cdot)\|_{p,Q} \|\psi(x,\cdot)\|_{p^+,Q} \le \|G\|_{L_\infty(\mathbb{R}^d; L_p(Q))} \|\psi\|_C,$$

where we have used the fact that $\sup_{K\in\mathbb{N}} \|G_K(x,\cdot)\|_{p,Q} \lesssim \|G(x,\cdot)\|_{p,Q}$ (see [Gra14$_1$, Corollary 4.1.3]). Then $(G_K, \psi)_{\mathbb{R}^d\times Q} \to (G,\psi)_{\mathbb{R}^d\times Q}$ by the Lebesgue dominated convergence theorem, and the claim follows. Applying this to the functions $(x,y) \mapsto \chi_\Omega(x) F(x, x/\varepsilon + y)$ and $(x,y) \mapsto \chi_\Omega(x) D_1^* F(x, x/\varepsilon + y)$ (χ_Ω is the characteristic function of Ω), which obviously belong to $L_\infty(\mathbb{R}^d; \tilde{L}_p(Q))$, we immediately obtain (65). □

Choose a cutoff function $\eta_\varepsilon \in C^{0,1}(\bar{\Omega})$ satisfying $\eta_\varepsilon|_{(\operatorname{supp}\chi_\varepsilon)_\varepsilon} = 1$. By definition of U_δ, the second term in (63) is

$$(\tau^\varepsilon \mathcal{T}^\varepsilon \chi_\varepsilon A(I + D_2N) D_1 u_{0,\delta}, D_1 \eta_\varepsilon u_\varepsilon^+)_{\Omega\times Q}.$$

Assume for the moment that $\eta_\varepsilon u_\varepsilon^+ \in C_c^1(\Omega)^n$ and recall from (22) that, for each fixed $x \in \Omega$, $D_2^* A(x,\cdot)(I + D_2 N(x,\cdot)) D u_{0,\delta}(x) = 0$ on $\tilde{W}_{p^+}^1(Q)^n$. Then Lemma 8.1 tells us that

$$(\tau^\varepsilon \mathcal{T}^\varepsilon \chi_\varepsilon A(I+D_2N) D_1 u_{0,\delta}, D_1 \eta_\varepsilon u_\varepsilon^+)_{\Omega\times Q} = (\tau^\varepsilon \mathcal{T}^\varepsilon D_1^* \chi_\varepsilon A(I+D_2N) D_1 u_{0,\delta}, \eta_\varepsilon u_\varepsilon^+)_{\Omega\times Q}.$$

But the form

$$\eta_\varepsilon u_\varepsilon^+ \mapsto (\tau^\varepsilon \mathcal{T}^\varepsilon \chi_\varepsilon A(I + D_2N) D_1 u_{0,\delta}, D_1 \eta_\varepsilon u_\varepsilon^+)_{\Omega\times Q}$$

is continuous on $\mathring{W}_{p^+}^1(\Omega)^n$ and the form

$$\eta_\varepsilon u_\varepsilon^+ \mapsto (\tau^\varepsilon \mathcal{T}^\varepsilon D_1^* \chi_\varepsilon A(I + D_2N) D_1 u_{0,\delta}, \eta_\varepsilon u_\varepsilon^+)_{\Omega\times Q}$$

is continuous on $L_{p^+}(\Omega)^n$ (by Lemma 5.3 and the hypothesis (23)), so the equality

$$(\tau^\varepsilon \mathcal{T}^\varepsilon \chi_\varepsilon A(D_1 u_{0,\delta} + D_2 U_\delta), D_1 \eta_\varepsilon u_\varepsilon^+)_{\Omega\times Q} = (\tau^\varepsilon \mathcal{T}^\varepsilon D_1^* \chi_\varepsilon A(D_1 u_{0,\delta} + D_2 U_\delta), \eta_\varepsilon u_\varepsilon^+)_{\Omega\times Q} \tag{67}$$

holds, in fact, for any $u_\varepsilon^+ \in W_{p^+}^1(\Omega)^n$. Recalling that $\eta_\varepsilon = 1$ on $(\operatorname{supp}\chi_\varepsilon)_\varepsilon$ and combining (64) with (67), we see that

$$\begin{aligned}(\chi_\varepsilon A(D_1 u_{0,\delta} + D_2 U_\delta), D_1 u_\varepsilon^+)_{\Omega\times Q} &- (\tau^\varepsilon \mathcal{T}^\varepsilon \chi_\varepsilon A(D_1 u_{0,\delta} + D_2 U_\delta), D_1 u_\varepsilon^+)_{\Omega\times Q} \\ &= (\tau^\varepsilon \mathcal{T}^\varepsilon D_1^* \chi_\varepsilon A(D_1 u_{0,\delta} + D_2 U_\delta), (\mathcal{T}^\varepsilon - \mathcal{I}) u_\varepsilon^+)_{\Omega\times Q}.\end{aligned} \tag{68}$$

Putting together (59)–(62) and (68), we arrive at the operator identity

$$(\mathcal{A}_\mu^\varepsilon)^{-1} - (\mathcal{A}_\mu^0)^{-1} - \varepsilon \mathcal{K}_\mu^\varepsilon|_{L_p(\Omega)^n} = \mathcal{I}_\mu^\varepsilon + \mathcal{D}_\mu^\varepsilon + \mathcal{B}_\mu^\varepsilon \tag{69}$$

that effectively splits the problem into the interior parts, given by

$$\begin{aligned}(\mathcal{I}_\mu^\varepsilon f, g)_\Omega = {} & (\tau^\varepsilon \mathcal{T}^\varepsilon \chi_\varepsilon D_1^* A(D_1 u_{0,\delta} + D_2 U_\delta), (\mathcal{T}^\varepsilon - \mathcal{I}) u_\varepsilon^+)_{\Omega\times Q} \\ & - (\tau^\varepsilon \chi_\varepsilon [A, \mathcal{T}^\varepsilon](D_1 u_{0,\delta} + D_2 U_\delta), D_1 u_\varepsilon^+)_{\Omega\times Q} \\ & - \varepsilon(\tau^\varepsilon \chi_\varepsilon A \mathcal{T}^\varepsilon D_1 U_\delta, D_1 u_\varepsilon^+)_{\Omega\times Q} \\ & - (\chi_\varepsilon A^\varepsilon(\mathcal{I} - \mathcal{S}^\varepsilon) D u_0, D u_\varepsilon^+)_\Omega \\ & + \varepsilon\mu(\chi_\varepsilon U_{\varepsilon,\delta}, u_\varepsilon^+)_\Omega\end{aligned} \tag{70}$$

and

$$(\mathcal{D}^\varepsilon_\mu f, g)_\Omega = (\chi_\varepsilon A(D_1(u_0 - u_{0,\delta}) + D_2(U - U_\delta)), D_1 u^+_\varepsilon)_{\Omega\times Q} \\ - (\tau^\varepsilon \chi_\varepsilon A \mathcal{T}^\varepsilon D_1(u_0 - u_{0,\delta}), D_1 u^+_\varepsilon)_{\Omega\times Q}, \tag{71}$$

and the boundary part, given by

$$\begin{aligned}(\mathcal{B}^\varepsilon_\mu f, g)_\Omega = {} & ((D_1\rho_\varepsilon)^* \cdot \tau^\varepsilon \mathcal{T}^\varepsilon A(D_1 u_{0,\delta} + D_2 U_\delta), (\mathcal{T}^\varepsilon - \mathcal{I})u^+_\varepsilon)_{\Omega\times Q} \\ & + (\tau^\varepsilon[\rho_\varepsilon, \mathcal{T}^\varepsilon] A(D_1 u_{0,\delta} + D_2 U_\delta), D_1 u^+_\varepsilon)_{\Omega\times Q} \\ & + (\rho_\varepsilon(A^0 - A^\varepsilon) D u_0, D u^+_\varepsilon)_\Omega \\ & + \varepsilon(A^\varepsilon D\rho_\varepsilon \cdot U_{\varepsilon,\delta}, D u^+_\varepsilon)_\Omega \\ & - \varepsilon(\rho_\varepsilon U_{\varepsilon,\delta}, g)_\Omega.\end{aligned} \tag{72}$$

We are finally in a position to prove Theorem 6.1.

Proof of Theorem 6.1. We estimate each term in (70), (71) and (72), bearing in mind that χ_ε vanishes on $(\partial\Omega)_{2\varepsilon}$ and ρ_ε is supported in $(\partial\Omega)_{3\varepsilon}$.

We begin with the "interior" operator $\mathcal{I}^\varepsilon_\mu$. By Lemmas 5.3 and 5.5,

$$\begin{aligned}&\bigl|(\tau^\varepsilon \mathcal{T}^\varepsilon \chi_\varepsilon D_1^* A(D_1 u_{0,\delta} + D_2 U_\delta), (\mathcal{T}^\varepsilon - \mathcal{I})u^+_\varepsilon)_{\Omega\times Q}\bigr| \\ &\qquad \le \|\tau^\varepsilon \mathcal{T}^\varepsilon \chi_\varepsilon D_1^* A(D_1 u_{0,\delta} + D_2 U_\delta)\|_{p,(\operatorname{supp}\chi_\varepsilon)_\varepsilon\times Q} \|(\mathcal{T}^\varepsilon - \mathcal{I})u^+_\varepsilon\|_{p^+,(\operatorname{supp}\chi_\varepsilon)_\varepsilon\times Q} \\ &\qquad \lesssim \varepsilon\bigl(\|Du_{0,\delta}\|_{1,p,\Omega} + \|D_1 D_2 U_\delta\|_{p,\Omega\times Q} + \|D_2 U_\delta\|_{p,\Omega\times Q}\bigr)\|Du^+_\varepsilon\|_{p^+,\Omega}.\end{aligned}$$

For the second term, observe that

$$\tau^\varepsilon[A, \mathcal{T}^\varepsilon] = \tau^\varepsilon(\mathcal{I} - \mathcal{T}^\varepsilon)A \cdot \tau^\varepsilon \mathcal{T}^\varepsilon.$$

This, together with Lemma 5.3, implies that

$$\begin{aligned}&\bigl|(\tau^\varepsilon \chi_\varepsilon [A, \mathcal{T}^\varepsilon](D_1 u_{0,\delta} + D_2 U_\delta), D_1 u^+_\varepsilon)_{\Omega\times Q}\bigr| \\ &\qquad \le \|(\mathcal{I} - \mathcal{T}^\varepsilon)A\|_{L_\infty} \|\tau^\varepsilon \mathcal{T}^\varepsilon(D_1 u_{0,\delta} + D_2 U_\delta)\|_{p,\operatorname{supp}\chi_\varepsilon\times Q} \|D_1 u^+_\varepsilon\|_{p^+,\Omega\times Q} \\ &\qquad \lesssim \varepsilon\bigl(\|Du_{0,\delta}\|_{p,\Omega} + \|D_2 U_\delta\|_{p,\Omega\times Q}\bigr)\|Du^+_\varepsilon\|_{p^+,\Omega}.\end{aligned}$$

By Lemma 5.3 again, we see that

$$\begin{aligned}\varepsilon\bigl|(\tau^\varepsilon \chi_\varepsilon A \mathcal{T}^\varepsilon D_1 U_\delta, D_1 u^+_\varepsilon)_{\Omega\times Q}\bigr| &\le \varepsilon\|A\|_{L_\infty} \|\tau^\varepsilon \mathcal{T}^\varepsilon D_1 U_\delta\|_{p,\operatorname{supp}\chi_\varepsilon\times Q} \|D_1 u^+_\varepsilon\|_{p^+,\Omega\times Q} \\ &\lesssim \varepsilon\|D_1 U_\delta\|_{p,\Omega\times Q}\|Du^+_\varepsilon\|_{p^+,\Omega},\end{aligned}$$

while Lemmas 5.6 and 5.4 show that, respectively,

$$\begin{aligned}\bigl|(\chi_\varepsilon A^\varepsilon(\mathcal{I} - \mathcal{S}^\varepsilon) Du_0, Du^+_\varepsilon)_\Omega\bigr| &\le \|A\|_{L_\infty} \|(\mathcal{I} - \mathcal{S}^\varepsilon)Du_0\|_{p,\operatorname{supp}\chi_\varepsilon} \|Du^+_\varepsilon\|_{p^+,\Omega} \\ &\lesssim \varepsilon^s \|D^{s,p} Du_0\|_{p,\Omega} \|Du^+_\varepsilon\|_{p^+,\Omega}\end{aligned}$$

and

$$\varepsilon\bigl|(\chi_\varepsilon U_{\varepsilon,\delta}, u^+_\varepsilon)_\Omega\bigr| \le \varepsilon\|U_{\varepsilon,\delta}\|_{p,\operatorname{supp}\chi_\varepsilon} \|u^+_\varepsilon\|_{p^+,\Omega} \lesssim \varepsilon\|U_\delta\|_{p,\Omega\times Q}\|u^+_\varepsilon\|_{p^+,\Omega}.$$

We have found that

$$\begin{aligned}\bigl|(\mathcal{I}^\varepsilon_\mu f, g)_\Omega\bigr| \lesssim \varepsilon^s\bigl(&\|D^{s,p} Du_0\|_{p,\Omega} + \varepsilon^{1-s}\|Du_{0,\delta}\|_{1,p,\Omega} \\ &+ \varepsilon^{1-s}\|D_1 D_2 U_\delta\|_{p,\Omega\times Q} + \varepsilon^{1-s}\|D_1 U_\delta\|_{p,\Omega\times Q} \\ &+ \|D_2 U_\delta\|_{p,\Omega\times Q} + \|U_\delta\|_{p,\Omega\times Q}\bigr)\|u^+_\varepsilon\|_{1,p^+,\Omega}.\end{aligned} \tag{73}$$

Turning to the "interior" operator $\mathcal{D}^\varepsilon_\mu$, we see that

$$\begin{aligned}\big|(\chi_\varepsilon A(D_1(u_0-u_{0,\delta})+D_2(U-U_\delta)),D_1u_\varepsilon^+)_{\Omega\times Q}\big| &\le\\ \le \|A\|_{L_\infty}\big(\|D_1(u_0-u_{0,\delta})\|_{p,\Omega\times Q}+\|D_2(U-U_\delta)\|_{p,\Omega\times Q}\big)\|D_1u_\varepsilon^+\|_{p^+,\Omega\times Q}\\ \lesssim \big(\|D(u_0-u_{0,\delta})\|_{p,\Omega}+\|D_2(U-U_\delta)\|_{p,\Omega\times Q}\big)\|Du_\varepsilon^+\|_{p^+,\Omega}\end{aligned}$$

and

$$\begin{aligned}&\big|(\tau^\varepsilon\chi_\varepsilon A\mathcal{T}^\varepsilon D_1(u_0-u_{0,\delta}),D_1u_\varepsilon^+)_{\Omega\times Q}\big|\\ &\quad\le \|A\|_{L_\infty}\|\tau^\varepsilon\mathcal{T}^\varepsilon D_1(u_0-u_{0,\delta})\|_{p,\operatorname{supp}\chi_\varepsilon\times Q}\|D_1u_\varepsilon^+\|_{p^+,\Omega\times Q}\\ &\quad\lesssim \|D(u_0-u_{0,\delta})\|_{p,\Omega}\|Du_\varepsilon^+\|_{p^+,\Omega}\end{aligned}$$

(according to Lemma 5.3). Hence

$$\big|(\mathcal{D}^\varepsilon_\mu f,g)_\Omega\big|\lesssim\big(\|D(u_0-u_{0,\delta})\|_{p,\Omega}+\|D_2(U-U_\delta)\|_{p,\Omega\times Q}\big)\|Du_\varepsilon^+\|_{p^+,\Omega}. \tag{74}$$

It remains to estimate the "boundary" operator $\mathcal{B}^\varepsilon_\mu$. Arguing as above and then applying Lemma A.1 and the bound (14^+), we easily find that

$$\begin{aligned}&\big|(\mathcal{T}^\varepsilon(D_1\rho_\varepsilon)^*\cdot\tau^\varepsilon\mathcal{T}^\varepsilon A(D_1u_{0,\delta}+D_2U_\delta),(\mathcal{T}^\varepsilon-\mathcal{I})u_\varepsilon^+)_{\Omega\times Q}\big|\\ &\quad\lesssim\big(\|Du_{0,\delta}\|_{p,\operatorname{supp}D\rho_\varepsilon}+\|D_2U_\delta\|_{p,\operatorname{supp}D\rho_\varepsilon\times Q}\big)\|Du_\varepsilon^+\|_{p^+,(\operatorname{supp}D\rho_\varepsilon)_{2\varepsilon}}\\ &\quad\lesssim\varepsilon^{s/p}\big(\|Du_{0,\delta}\|_{s,p,\Omega}+\|D_1^{s,p}D_2U_\delta\|_{p,\Omega\times Q}+\|D_2U_\delta\|_{p,\Omega\times Q}\big)\|g\|^*_{-1,p^+,\Omega}\end{aligned}\tag{75}$$

and

$$\begin{aligned}&\big|(\tau^\varepsilon[\rho_\varepsilon,\mathcal{T}^\varepsilon]A(D_1u_{0,\delta}+D_2U_\delta),D_1u_\varepsilon^+)_{\Omega\times Q}\big|\\ &\quad\lesssim\big(\|Du_{0,\delta}\|_{p,(\operatorname{supp}D\rho_\varepsilon)_{2\varepsilon}}+\|D_2U_\delta\|_{p,(\operatorname{supp}D\rho_\varepsilon)_{2\varepsilon}\times Q}\big)\|Du_\varepsilon^+\|_{p^+,(\operatorname{supp}D\rho_\varepsilon)_\varepsilon}\\ &\quad\lesssim\varepsilon^{s/p}\big(\|Du_{0,\delta}\|_{s,p,\Omega}+\|D_1^{s,p}D_2U_\delta\|_{p,\Omega\times Q}+\|D_2U_\delta\|_{p,\Omega\times Q}\big)\|g\|^*_{-1,p^+,\Omega}.\end{aligned}\tag{76}$$

Likewise,

$$\begin{aligned}\big|(\rho_\varepsilon(A^0-A^\varepsilon)Du_0,Du_\varepsilon^+)_\Omega\big|&\lesssim\|Du_0\|_{p,\operatorname{supp}\rho_\varepsilon}\|Du_\varepsilon^+\|_{p^+,\operatorname{supp}\rho_\varepsilon}\\ &\lesssim\varepsilon^{s/p}\|Du_0\|_{s,p,\Omega}\|g\|^*_{-1,p^+,\Omega}.\end{aligned}\tag{77}$$

As for the last two terms in (72),

$$\begin{aligned}\varepsilon\big|(A^\varepsilon D\rho_\varepsilon\cdot U_{\varepsilon,\delta},Du_\varepsilon^+)_\Omega\big|&\lesssim\|U_{\varepsilon,\delta}\|_{p,\operatorname{supp}D\rho_\varepsilon}\|Du_\varepsilon^+\|_{p^+,\operatorname{supp}D\rho_\varepsilon}\\ &\lesssim\varepsilon^{s/p}\big(\|D_1^{s,p}U_\delta\|_{p,\Omega\times Q}+\|U_\delta\|_{p,\Omega\times Q}\big)\|g\|^*_{-1,p^+,\Omega}\end{aligned}\tag{78}$$

and

$$\begin{aligned}\varepsilon\big|(\rho_\varepsilon U_{\varepsilon,\delta},g)_\Omega\big|&\lesssim\big(\varepsilon\|DU_{\varepsilon,\delta}\|_{p,\operatorname{supp}\rho_\varepsilon}+\|U_{\varepsilon,\delta}\|_{p,\operatorname{supp}D\rho_\varepsilon}\big)\|g\|^*_{-1,p^+,\Omega}\\ &\lesssim\varepsilon^{s/p}\big(\|D_1^{s,p}D_2U_\delta\|_{p,\Omega_1\times Q}+\|D_1^{s,p}U_\delta\|_{p,\Omega\times Q}\\ &\qquad+\varepsilon^{1-s}\|D_1U_\delta\|_{p,\Omega_1\times Q}+\|D_2U_\delta\|_{p,\Omega_1\times Q}+\|U_\delta\|_{p,\Omega\times Q}\big)\|g\|^*_{-1,p^+,\Omega},\end{aligned}\tag{79}$$

where we have used the estimates

$$\begin{aligned}\varepsilon\|DU_{\varepsilon,\delta}\|_{p,\operatorname{supp}\rho_\varepsilon}\lesssim\varepsilon^{s/p}\big(&\|D_1^{s,p}D_2U_\delta\|_{p,\Omega_1\times Q}\\ &+\varepsilon^{1-s}\|D_1U_\delta\|_{p,\Omega_1\times Q}+\|D_2U_\delta\|_{p,\Omega_1\times Q}\big)\end{aligned}\tag{80}$$

and

$$\|U_{\varepsilon,\delta}\|_{p,\operatorname{supp}D\rho_\varepsilon}\le\varepsilon^{s/p}\big(\|D_1^{s,p}U_\delta\|_{p,\Omega\times Q}+\|U_\delta\|_{p,\Omega\times Q}\big)\tag{81}$$

(recall that U_δ is extended to all of $\mathbb{R}^d$ and hence is well-defined on Ω_1). To verify the first one, we substitute $\varepsilon DU_{\varepsilon,\delta} = \varepsilon\tau^\varepsilon\mathcal{S}^\varepsilon D_1U_\delta + \tau^\varepsilon\mathcal{S}^\varepsilon D_2U_\delta$ to obtain, via Lemma 5.4,

$$\begin{aligned}\varepsilon\|DU_{\varepsilon,\delta}\|_{p,\operatorname{supp}\rho_\varepsilon} &\le \varepsilon\|\tau^\varepsilon\mathcal{S}^\varepsilon D_1U_\delta\|_{p,\Omega} + \|\tau^\varepsilon\mathcal{S}^\varepsilon D_2U_\delta\|_{p,\operatorname{supp}\rho_\varepsilon}\\ &\lesssim \varepsilon\|D_1U_\delta\|_{p,\Omega_1\times Q} + \|D_2U_\delta\|_{p,(\operatorname{supp}\rho_\varepsilon)_\varepsilon\times Q}.\end{aligned}$$

Since $(\operatorname{supp}\rho_\varepsilon)_\varepsilon$ is the union of $(\operatorname{supp}\rho_\varepsilon)_\varepsilon\cap\Omega$ and $(\operatorname{supp}\rho_\varepsilon)_\varepsilon\setminus\Omega$, we may apply Lemma A.1, with $\Sigma=\Omega$ and $\Sigma=\Omega_1\setminus\bar{\Omega}$, to get (80). The other inequality is checked in a similar fashion. Summarizing,

(82)
$$\begin{aligned}\big|(\mathcal{B}^\varepsilon_\mu f,g)_\Omega\big| \lesssim \varepsilon^{s/p}\big(&\|Du_0\|_{s,p,\Omega} + \|Du_{0,\delta}\|_{s,p,\Omega} + \|D_1^{s,p}D_2U_\delta\|_{p,\Omega_1\times Q}\\ &+ \|D_1^{s,p}U_\delta\|_{p,\Omega\times Q} + \varepsilon^{1-s}\|D_1U_\delta\|_{p,\Omega_1\times Q}\\ &+ \|D_2U_\delta\|_{p,\Omega_1\times Q} + \|U_\delta\|_{p,\Omega\times Q}\big)\|g\|^*_{-1,p^+,\Omega}.\end{aligned}$$

Now from (73), (74) and (82), together with Lemmas 5.1 and 5.2 and the estimates (26), (28) and (14^+), we obtain

$$\|(\mathcal{A}^\varepsilon_\mu)^{-1}f - (\mathcal{A}^0_\mu)^{-1}f - \varepsilon\mathcal{K}^\varepsilon_\mu f\|_{1,p,\Omega} \lesssim \varepsilon^{s/p}\|f\|_{p,\Omega}, \tag{83}$$

which immediately implies (45). The L_q-bound (44) comes from (83) as well, since, according to the Sobolev embedding theorem,

$$\|(\mathcal{A}^\varepsilon_\mu)^{-1}f - (\mathcal{A}^0_\mu)^{-1}f\|_{q,\Omega} \lesssim \|(\mathcal{A}^\varepsilon_\mu)^{-1}f - (\mathcal{A}^0_\mu)^{-1}f - \varepsilon\mathcal{K}^\varepsilon_\mu f\|_{1,p,\Omega} + \varepsilon\|\mathcal{K}^\varepsilon_\mu f\|_{q,\Omega}$$

and the terms on the right are estimated by using (43) and (83). □

Proof of Corollary 6.2. From (83) and the fact that $W^1_p(\Omega)^n$ is continuously embedded in $W^r_p(\Omega)^n$, we conclude that

$$\|D^{r,p}((\mathcal{A}^\varepsilon_\mu)^{-1}f - (\mathcal{A}^0_\mu)^{-1}f - \varepsilon\mathcal{K}^\varepsilon_\mu f)\|_{p,\Omega} \lesssim \varepsilon^{s/p}\|f\|_{p,\Omega}.$$

On the other hand, interpolation between the W^1_p- and L_p-bounds in (42) gives

$$\varepsilon^r\|D^{r,p}\mathcal{K}^\varepsilon_\mu f\|_{p,\Omega} \lesssim \|f\|_{p,\Omega},$$

and (46) follows. □

Proof of Theorem 6.3. Knowing that $(\mathcal{A}^\varepsilon_\mu)^+$ satisfies the hypotheses of Theorem 6.1^+ and $g\in L_{p^+}(\Omega)^n$, we can get a better estimate on $\mathcal{B}^\varepsilon_\mu$ than (82). Indeed, if $U^+_\delta = \mathcal{K}^+_\mu(\delta)g$ and $U^+_{\varepsilon,\delta} = \tau^\varepsilon\mathcal{S}^\varepsilon U^+_\delta = (\mathcal{K}^\varepsilon_\mu)^+g$, then ($45^+$) implies that

$$\|Du^+_\varepsilon\|_{p^+,(\partial\Omega)_{5\varepsilon}\cap\Omega} \lesssim \|Du^+_0\|_{p^+,(\partial\Omega)_{5\varepsilon}\cap\Omega} + \varepsilon\|DU^+_{\varepsilon,\delta}\|_{p^+,(\partial\Omega)_{5\varepsilon}\cap\Omega} + \varepsilon^{s/p^+}\|g\|_{p^+,\Omega},$$

and, therefore, by Lemma A.1 and the estimate (80^+) with $(\partial\Omega)_{5\varepsilon}$ in place of $\operatorname{supp}\rho_\varepsilon$,

(84)
$$\begin{aligned}\|Du^+_\varepsilon\|_{p^+,(\partial\Omega)_{5\varepsilon}\cap\Omega} \lesssim \varepsilon^{s/p^+}\big(&\|Du^+_0\|_{s,p^+,\Omega} + \|D_1^{s,p^+}D_2U^+_\delta\|_{p^+,\Omega_1\times Q}\\ &+ \varepsilon^{1-s}\|D_1U^+_\delta\|_{p^+,\Omega_1\times Q} + \|D_2U^+_\delta\|_{p^+,\Omega_1\times Q} + \|g\|_{p^+,\Omega}\big).\end{aligned}$$

Using this to bound the norm of Du_ε^+ in (75)–(78), as well as Lemma 5.4 to handle the last term in (72), yields

(85)
$$\begin{aligned}\big|(\mathcal{B}_\mu^\varepsilon f, g)_\Omega\big| \lesssim \varepsilon^s\big(&\|Du_0\|_{s,p,\Omega} + \|Du_{0,\delta}\|_{s,p,\Omega} + \|D_1^{s,p} D_2 U_\delta\|_{p,\Omega_1\times Q} + \|D_1^{s,p} U_\delta\|_{p,\Omega\times Q}\\ &+ \varepsilon^{1-s}\|D_1 U_\delta\|_{p,\Omega_1\times Q} + \|D_2 U_\delta\|_{p,\Omega_1\times Q} + \|U_\delta\|_{p,\Omega\times Q}\big)\\ &\times\big(\|Du_0^+\|_{s,p^+,\Omega} + \|D_1^{s,p^+} D_2 U_\delta^+\|_{p^+,\Omega_1\times Q} + \varepsilon^{1-s}\|D_1 U_\delta^+\|_{p^+,\Omega_1\times Q}\\ &+ \|D_2 U_\delta^+\|_{p^+,\Omega_1\times Q} + \|g\|_{p^+,\Omega}\big).\end{aligned}$$

Combining (73), (74) and (85) with Lemmas 5.1 and 5.2 and the estimates (26), (28) and (14^+), (26^+), (28^+), we obtain

$$\big|((\mathcal{A}_\mu^\varepsilon)^{-1} f - (\mathcal{A}_\mu^0)^{-1} f, g)_\Omega\big| \lesssim \varepsilon^s \|f\|_{p,\Omega}\|g\|_{p^+,\Omega},$$

and this is what we wanted to prove. □

As we have seen in the proof of Theorem 6.1, the interior terms in (69) are of order ε^s. To go further, we establish an "interior" operator identity, which is similar to (69) but involves no boundary terms.

So let $\chi' \in C^{0,1}(\bar\Omega)$ with $\chi' = 0$ in $(\partial\Omega)_\sigma$ for some $\sigma > 0$. Define the linear operator $\mathcal{P}^\varepsilon\colon W_p^1(\Omega)^n \to (C_c^\infty(\Omega)^n)^*$ associated with the form $(u, v) \mapsto (A^\varepsilon Du, Dv)_\Omega$ and set $\mathcal{P}_\mu^\varepsilon = \mathcal{P}^\varepsilon - \mu$. If $u_\varepsilon = (\mathcal{A}_\mu^\varepsilon)^{-1} f$, then we have

$$(\chi'\mathcal{P}_\mu^\varepsilon u_\varepsilon, u_\varepsilon^+)_\Omega = (f, \chi' u_\varepsilon^+)_\Omega = (\mathcal{A}_\mu^0 u_0, \chi' u_\varepsilon^+)_\Omega.$$

Thus,

$$\begin{aligned}(\chi'\mathcal{P}_\mu^\varepsilon(u_\varepsilon - u_0 - \varepsilon U_{\varepsilon,\delta}), u_\varepsilon^+)_\Omega = (A^0 Du_0, D\chi' u_\varepsilon^+)_\Omega - (A^\varepsilon D(u_0 + \varepsilon U_{\varepsilon,\delta}), D\chi' u_\varepsilon^+)_\Omega\\ + \varepsilon\mu(U_{\varepsilon,\delta}, \chi' u_\varepsilon^+)_\Omega.\end{aligned}$$

The first two terms on the right-hand side are similar to those in (59), with u_ε^+ replaced by $\chi' u_\varepsilon^+$, in which case $\chi_\varepsilon|_{\operatorname{supp}\chi'} = 1$ for $5\varepsilon \le \sigma$, so the previous calculations go over without change to yield, for such ε,

$$(\mathcal{A}_\mu^\varepsilon)^{-1}\chi'\mathcal{P}_\mu^\varepsilon((\mathcal{A}_\mu^\varepsilon)^{-1} - (\mathcal{A}_\mu^0)^{-1} - \varepsilon\mathcal{K}_\mu^\varepsilon)|_{L_p(\Omega)^n} = \mathring{\mathcal{I}}_\mu^\varepsilon + \mathring{\mathcal{D}}_\mu^\varepsilon, \tag{86}$$

where

$$\begin{aligned}(\mathring{\mathcal{I}}_\mu^\varepsilon f, g)_\Omega = {}&(\tau^\varepsilon\mathcal{T}^\varepsilon D_1^* A(D_1 u_{0,\delta} + D_2 U_\delta), (\mathcal{T}^\varepsilon - \mathcal{I})\chi' u_\varepsilon^+)_{\Omega\times Q}\\ &- (\tau^\varepsilon[A, \mathcal{T}^\varepsilon](D_1 u_{0,\delta} + D_2 U_\delta), D_1\chi' u_\varepsilon^+)_{\Omega\times Q}\\ &- \varepsilon(\tau^\varepsilon A\mathcal{T}^\varepsilon D_1 U_\delta, D_1\chi' u_\varepsilon^+)_{\Omega\times Q}\\ &- (A^\varepsilon(\mathcal{I} - \mathcal{S}^\varepsilon)Du_0, D\chi' u_\varepsilon^+)_\Omega\\ &+ \varepsilon\mu(U_{\varepsilon,\delta}, \chi' u_\varepsilon^+)_\Omega\end{aligned}$$

and

$$\begin{aligned}(\mathring{\mathcal{D}}_\mu^\varepsilon f, g)_\Omega = {}&(A(D_1(u_0 - u_{0,\delta}) + D_2(U - U_\delta)), D_1\chi' u_\varepsilon^+)_{\Omega\times Q}\\ &- (\tau^\varepsilon A\mathcal{T}^\varepsilon D_1(u_0 - u_{0,\delta}), D_1\chi' u_\varepsilon^+)_{\Omega\times Q}.\end{aligned}$$

This is the interior operator identity that we seek.

Proof of Theorem 6.4. Set $v_\varepsilon = u_\varepsilon - u_0 - \varepsilon U_{\varepsilon,\delta}$ and $f_\varepsilon = \chi'\mathcal{P}_\mu^\varepsilon v_\varepsilon$. If η is a smooth cutoff function which is supported in Ω and is identically 1 on $\operatorname{supp}\chi'$, then ηv_ε

belongs to $\mathscr{W}_p^1(\Omega;\mathbb{C}^n)$ and therefore $f_\varepsilon = \chi'\mathcal{A}_\mu^\varepsilon \eta v_\varepsilon$ belongs to $\mathscr{W}_p^{-1}(\Omega;\mathbb{C}^n)$. To estimate the norm of f_ε, we use the identity (86):

$$\begin{aligned}\bigl|(f_\varepsilon, u_\varepsilon^+)_\Omega\bigr| \lesssim \varepsilon^s \bigl(&\|D^{s,p}Du_0\|_{p,\Omega} + \varepsilon^{1-s}\|Du_{0,\delta}\|_{1,p,\Omega} + \varepsilon^{1-s}\|D_1D_2U_\delta\|_{p,\Omega\times Q}\\ &+ \varepsilon^{1-s}\|D_1U_\delta\|_{p,\Omega\times Q} + \|D_2U_\delta\|_{p,\Omega\times Q} + \|U_\delta\|_{p,\Omega\times Q}\\ &+ \varepsilon^{-s}\|D(u_0-u_{0,\delta})\|_{p,\Omega} + \varepsilon^{-s}\|D_2(U-U_\delta)\|_{p,\Omega\times Q}\bigr)\|u_\varepsilon^+\|_{1,p^+,\Omega}\end{aligned}$$

(cf. (73) and (74) in the proof of Theorem 6.1); taking the supremum over all $g \in (W_p^1(\Omega)^n)^*$, or, equivalently, over all $u_\varepsilon^+ \in \mathscr{W}_{p^+}^1(\Omega;\mathbb{C}^n)$ (recall that the quotient map (10^+) is an epimorphism), and applying Lemmas 5.1 and 5.2 and the inequalities (26) and (28) shows that

$$|||f_\varepsilon|||_{-1,p,\Omega} \lesssim \varepsilon^s\|f\|_{p,\Omega}. \tag{87}$$

On the other hand, according to (48),

$$\|D\chi v_\varepsilon\|_{p,\Omega} \lesssim \|v_\varepsilon\|_{p,\Omega} + |||f_\varepsilon|||_{-1,p,\Omega},$$

because $\chi'\mathcal{A}_\mu^\varepsilon\eta v_\varepsilon = f_\varepsilon$ and $\eta = 1$ on $\operatorname{supp}\chi'$. The result now follows from (42), (87) and Theorem 6.3. □

Appendix A. An estimate for integrals over a neighborhood of the boundary

The following lemma is a slight modification of [PSu12, Lemma 5.1].

Lemma A.1. *Let Σ be a uniformly weakly Lipschitz domain in $\mathbb{R}^d$. Then for each fixed $r \in (0,1]$ and $q \in [1,\infty)$ and any $\varepsilon > 0$*

$$\|u\|_{q,(\partial\Sigma)_\varepsilon\cap\Sigma} \lesssim \varepsilon^{r/q}\|u\|_{r,q,\Sigma}, \qquad u \in C_c^\infty(\bar\Sigma). \tag{88}$$

The constant in the inequality depends only on r, q, d and Σ.

Proof. We show that

$$\|u\|_{q,(\partial\Sigma)_\varepsilon\cap\Sigma} \lesssim \varepsilon^{1/q}\|u\|_{1,q,\Sigma}^{1/q}\|u\|_{q,\Sigma}^{1-1/q}, \tag{89}$$

which, via interpolation, clearly implies (88).

Recall that B denotes the open unit ball centered at the origin and B_+ denotes the open unit half-ball with $x_d \in (0,1)$. Let S_t be the cross-section of B at $x_d = t$ and P_t be the piece of B_+ with $x_d \in (0,t)$. If (W_k,ω_k) are local boundary coordinate patches, then $\omega_k(W_k\cap\Sigma) = B_+$ and $\omega_k(W_k\cap\partial\Sigma) = S_0$, and for any $y \in \omega_k(W_k\cap\Sigma)$

$$\operatorname{dist}(y,S_0) \le L_\Sigma \operatorname{dist}(x, W_k\cap\partial\Sigma),$$

where $x = \omega_k^{-1}(y)$ and $L_\Sigma = \sup_k[\omega_k]_{C^{0,1}}$. It follows that $\omega_k(W_k\cap(\partial\Sigma)_\varepsilon\cap\Sigma) \subset P_{\varepsilon/\varepsilon_1}$ with $\varepsilon_1 r_Q = L_\Sigma^{-1}$. On the other hand, we know that the cover is sufficiently tight in the sense that the union of $\omega_k^{-1}(B_+)$ contains $(\partial\Sigma)_\delta\cap\Sigma$ for some $\delta > 0$. Therefore, taking $\varepsilon_0 = \varepsilon_1\wedge\delta$, we can insure that $(\partial\Sigma)_\varepsilon\cap\Sigma$ is covered by $\{W_k\}$ for any $\varepsilon \le \varepsilon_0$.

Now, using a partition of unity $\{\varphi_k\}$ subordinate to $\{W_k\}$ (see Section 2) and making a change of variables to flatten out the boundary, we reduce (89) to proving that, for any $\varepsilon \le \varepsilon_0$ and any smooth function u on B_+ vanishing near the boundary of B, it holds that

$$\|u\|_{q,P_{\varepsilon/\varepsilon_0}} \lesssim \varepsilon^{1/q}\|u\|_{1,q,B_+}^{1/q}\|u\|_{q,B_+}^{1-1/q}. \tag{90}$$

By the divergence theorem, for any $t \in (0,1)$ we have

$$\int_{S_t} |u(x',t)|^q \, dx' = -\int_{B_+ \setminus P_t} \partial_{x_d} |u(x)|^q \, dx,$$

and hence

$$\begin{aligned}\int_{S_t} |u(x',t)|^q \, dx' &\le q \int_{B_+ \setminus P_t} |\partial_{x_d} u(x)| |u(x)|^{q-1} \, dx \\ &\le q \Big(\int_{B_+} |\partial_{x_d} u(x)|^q \, dx \Big)^{1/q} \Big(\int_{B_+} |u(x)|^q \, dx \Big)^{1-1/q}.\end{aligned}$$

Integrating in t from 0 to $\varepsilon/\varepsilon_0$ now gives (90). □

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