Homogenization of periodic and locally periodic operators

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Homogenization theory studies the asymptotic behavior of solutions to differential equations with rapidly oscillating coefficients. Such equations appear naturally in applications, e. g., related to physical processes in highly heterogeneous media. From a mathematical point of view, it is convenient to consider a parametrized family of problems, where the parameter measures the heterogeneity of the medium. Often, the more heterogeneous the medium, the more similar the process appears to one in an "effective" homogeneous medium. This corresponds to the fact that the solutions of the equations in the family converge, in a suitable sense, to the solution of an equation with slowly varying (or even constant) coefficients.

In addition to proving the convergence, it is certainly a matter of considerable interest to find the rate as well. So-called operator error estimates make it possible to achieve both of these ends: they yield convergence in the strongest type of operator topology and, at the same time, provide the rate.

Such results have been receiving a great deal of attention since the pioneering 2001 paper of Birman and Suslina. The problems, where the coefficients are periodic in each variable, are now well developed. On the other hand, more general periodic and locally periodic problems are not nearly extensively studied as one would like. It was our goal to fill this gap.

Let $\mathcal{A}^\varepsilon$ be the matrix differential operator given by the formula

$$\mathcal{A}^\varepsilon = -\operatorname{div} A^\varepsilon \nabla \tag{1}$$

and acting from the complex Sobolev space $H^1(\mathbb{R}^d)^n$ to its dual $H^{-1}(\mathbb{R}^d)^n$. The coefficient A^ε depends on $\varepsilon > 0$, which plays the role of parameter. It is assumed that $\mathcal{A}^\varepsilon$ is bounded and weakly coercive uniformly in ε for any ε in a neighborhood $\mathscr{E}$ of zero, that is, for all $\varepsilon \in \mathscr{E}$ and $u \in H^1(\mathbb{R}^d)^n$ it holds that

$$\|\mathcal{A}^\varepsilon u\|_{H^{-1}(\mathbb{R}^d)^n} \leqslant C\|u\|_{H^1(\mathbb{R}^d)^n} \tag{2}$$

and

$$\operatorname{Re}(\mathcal{A}^\varepsilon u, u)_{\mathbb{R}^d} \geqslant c_A\|\nabla u\|^2_{L_2(\mathbb{R}^d)^n} - C_A\|u\|^2_{L_2(\mathbb{R}^d)^n}. \tag{3}$$

In this case, $\mathcal{A}^\varepsilon$ is m-sectorial with sector independent of $\varepsilon \in \mathscr{E}$. If μ is outside the sector and f belongs to $L_2(\mathbb{R}^d)^n$, the strongly elliptic system $\mathcal{A}^\varepsilon u_\varepsilon - \mu u_\varepsilon = f$ has a unique solution u_ε. The problem is then to study the behavior of u_ε as $\varepsilon \to 0$.

The first attempt to extend the Birman–Suslina approach to cover general periodic problems was made shortly after their first paper on the subject. It turned out that the approach is not really well suited for these, but nevertheless yields operator error estimates in certain cases. The simplest example due to Suslina is the scalar self-adjoint operator $\mathcal{A}^\varepsilon$ on the plane, where the matrix $A^\varepsilon(x) = A(x_1, x_2/\varepsilon)$ is diagonal and periodic along the x_2-axis. In [1], the operator $\mathcal{A}^\varepsilon$ may also involve lower-order terms with unbounded coefficients. Another result proved in [1] is related to the problem on $\mathbb{R} \times (0,1)$ with Dirichlet or Neumann boundary conditions. Still, the requirement that $A^\varepsilon(x)$ be diagonal cannot be dropped within the framework used there.

The starting point of our analysis was the paper [3] (see also the note [2]), where a new method was proposed, allowing to prove various operator approximations for the resolvent $(\mathcal{A}^\varepsilon - \mu)^{-1}$ without severe restrictions on the structure of $A^\varepsilon(x)$. The paper dealt with a periodic (possibly non-self-adjoint) operator $\mathcal{A}^\varepsilon$ with the same principal part as in (1) and lower-order terms. Now $A^\varepsilon(x)$ was of the form $A(x_1, x_2/\varepsilon)$, where $x = x_1 \oplus x_2 \in \mathbb{R}^d$ and A is a bounded function that is Lipschitz in the first variable and periodic in the second, and the coefficients in the lower-order terms were multipliers between appropriate Sobolev spaces. We showed that $(\mathcal{A}^\varepsilon - \mu)^{-1}$ converges in the uniform operator topology and found the first two terms in the corresponding approximation:

$$\|(\mathcal{A}^\varepsilon - \mu)^{-1} f - (\mathcal{A}^0 - \mu)^{-1} f\|_{L_2(\mathbb{R}^d)^n} \leqslant C\varepsilon \|f\|_{L_2(\mathbb{R}^d)^n}, \tag{4}$$

$$\|(\mathcal{A}^\varepsilon - \mu)^{-1} f - (\mathcal{A}^0 - \mu)^{-1} f - \varepsilon \mathcal{C}_\mu^\varepsilon f\|_{L_2(\mathbb{R}^d)^n} \leqslant C\varepsilon^2 \|f\|_{L_2(\mathbb{R}^d)^n}. \tag{5}$$

We also obtained an approximation for $\nabla(\mathcal{A}^\varepsilon - \mu)^{-1}$:

$$\|\nabla(\mathcal{A}^\varepsilon - \mu)^{-1} f - \nabla(\mathcal{A}^0 - \mu)^{-1} f - \varepsilon \nabla \mathcal{K}_\mu^\varepsilon f\|_{L_2(\mathbb{R}^d)^n} \leqslant C\varepsilon \|f\|_{L_2(\mathbb{R}^d)^n}. \tag{6}$$

The rates are sharp with respect to order and generally cannot be improved. The estimates (4) and (6) are typical for homogenization and were known in particular cases. The most subtle result is the one involving the corrector $\mathcal{C}_\mu^\varepsilon$, which has no analogue in the classical theory. For purely periodic operators, such a result is due to Birman and Suslina. They employed a spectral method, and this is the point requiring the periodicity in each variable.

Instead, we proposed a non-spectral approach based on a version of the resolvent identity. Although it was still convenient to use the Floquet–Bloch theory as in the works of Birman–Suslina, it did not play a crucial role here. Later [4] this enables us to transfer the estimates (4)–(6) to locally periodic (possibly non-self-adjoint) operators. The coefficient A^ε is now given by $A^\varepsilon(x) = A(x, x/\varepsilon)$, where A is bounded and Lipschitz in the first variable and periodic in the second. Because A^ε is no longer periodic even in an approximate sense, this case is technically significantly different and more challenging than the periodic, yet similar in spirit. Thus, resolvent identities are the heart of the proof, as before.

Of course, the Lipschitz continuity assumption on $A(\cdot, y)$ is not required in order to just formulate the problem; for instance, mere uniform continuity would be enough. The generalization to locally periodic operators when the function A is Hölder continuous, with exponent $s \in [0,1)$, in the first argument was announced in [5]; a detailed treatment is part of the PhD thesis [6]. It turned out that the rates of approximation depend on the smoothness of A. For $s > 0$, the rates in (4) and (5) are equal to, respectively, ε^s and $\varepsilon^{2s/(2-s)}$:

$$\|(\mathcal{A}^\varepsilon - \mu)^{-1} f - (\mathcal{A}^0 - \mu)^{-1} f\|_{L_2(\mathbb{R}^d)^n} \leqslant C\varepsilon^s \|f\|_{L_2(\mathbb{R}^d)^n}, \tag{7}$$

$$\|(\mathcal{A}^\varepsilon - \mu)^{-1} f - (\mathcal{A}^0 - \mu)^{-1} f - \varepsilon \mathcal{C}_\mu^\varepsilon(s) f\|_{L_2(\mathbb{R}^d)^n} \leqslant C\varepsilon^{2s/(2-s)} \|f\|_{L_2(\mathbb{R}^d)^n}. \tag{8}$$

We note that the corrector in (8) depends on the exponent s, and even in the Lipschitz case it has more complicated structure than that of the "periodic" corrector in (5). An estimate of the form (6) with the same $\mathcal{K}_\mu^\varepsilon$ does not hold, however. Roughly speaking, the regularity of the image of $\mathcal{K}_\mu^\varepsilon$ in the scale of Sobolev–Slobodetskii spaces follows the regularity of A in the scale of Hölder spaces, and hence for $s<1$ the operator $\nabla\mathcal{K}_\mu^\varepsilon$ does not map $L_2(\mathbb{R}^d)^n$ into itself. On the other hand, if we replace the differentiation ∇ by the fractional derivative $\nabla^s=(-\Delta)^{s/2}$ and assume that the function

$$\nabla_x^s A(x,y)=\left(\int_{\mathbb{R}^d}|h|^{-d-2s}|A(x+h,y)-A(x,y)|^2\,dh\right)^{1/2}$$

(which is an analogue of $\nabla_x A(x,y)$ for the non-Lipschitz case) is uniformly bounded, then the error of the corresponding approximation will become of order ε^s:

$$\|\nabla^s(\mathcal{A}^\varepsilon-\mu)^{-1}f-\nabla^s(\mathcal{A}^0-\mu)^{-1}f-\varepsilon\nabla^s\mathcal{K}_\mu^\varepsilon f\|_{L_2(\mathbb{R}^d)^n}\leqslant C\varepsilon^s\|f\|_{L_2(\mathbb{R}^d)^n}. \tag{9}$$

As for the case $s=0$, we showed that $(\mathcal{A}^\varepsilon-\mu)^{-1}$ and $\nabla^r(\mathcal{A}^\varepsilon-\mu)^{-1}$ with $r\in(0,1)$ converge to $(\mathcal{A}^0-\mu)^{-1}$ and $\nabla^r(\mathcal{A}^0-\mu)^{-1}$.

In the further analysis, we focused on more difficult locally periodic problems – namely, those on a bounded domain $\Omega\subset\mathbb{R}^d$. Any "reasonable" type of boundary conditions was allowed, provided that the domain of the operator $\mathcal{A}^\varepsilon$ lies between the Sobolev spaces $\mathring{H}^1(\Omega)^n$ and $H^1(\Omega)^n$; moreover, in the vector case (i. e., for $n>1$) different components may satisfy different boundary conditions. It was assumed that, first, the estimates (2) and (3) hold for all u belonging to the operator domain and, secondly, the effective problem has the elliptic regularity property that $(\mathcal{A}^0-\mu)^{-1}L_2(\Omega)^n\subset H^{1+s}(\Omega)^n$ for some $s\in(0,1]$. The basic examples are the Dirichlet problem or the Neumann problem for strongly elliptic systems. The former was treated in the note [7]. The general case is contained in [8], where we also proved that

$$\|(\mathcal{A}^\varepsilon-\mu)^{-1}f-(\mathcal{A}^0-\mu)^{-1}f\|_{L_p(\Omega)^n}\leqslant C\varepsilon^s\|f\|_{L_p(\Omega)^n}, \tag{10}$$

$$\|\nabla(\mathcal{A}^\varepsilon-\mu)^{-1}f-\nabla(\mathcal{A}^0-\mu)^{-1}f-\varepsilon\nabla\mathcal{K}_\mu^\varepsilon f\|_{L_p(\Omega)^n}\leqslant C\varepsilon^{s/p}\|f\|_{L_p(\Omega)^n} \tag{11}$$

for any p in a neighborhood of 2. If, in addition, A is a VMO-function in the second argument, then these estimates are valid for every $p\in(1,\infty)$.

The study of such problems away from the boundary resembles the study of problems on the whole of $\mathbb{R}^d$. The influence of the boundary comes into play in a very small neighborhood of the boundary. It is because of boundary correction terms that the rate in (11) is worse than the rate in (10), and there seems to be no analogue of the approximation (5) for a "non-trivial" boundary.

In [7] we also studied the homogenization problem for the parabolic semigroup of the operator $\mathcal{A}^\varepsilon$ with Dirichlet boundary conditions and C_A equal to zero. Under these assumptions, the spectrum of $\mathcal{A}^\varepsilon$ lies to the right of the line $\{\operatorname{Re}z=2\gamma\}$, $\gamma>0$, and for any $t\geqslant\varepsilon^2$

$$\|e^{-t\mathcal{A}^\varepsilon}\varphi-e^{-t\mathcal{A}^0}\varphi\|_{L_2(\Omega)^n}\leqslant C\varepsilon e^{-\gamma t}t^{-1/2}\|\varphi\|_{L_2(\Omega)^n}, \tag{12}$$

$$\|\nabla e^{-t\mathcal{A}^\varepsilon}\varphi-\nabla e^{-t\mathcal{A}^0}\varphi-\varepsilon\nabla\mathcal{K}^\varepsilon(t)\varphi\|_{L_2(\Omega)^n}\leqslant C\varepsilon^{1/2}e^{-\gamma t}t^{-3/4}\|\varphi\|_{L_2(\Omega)^n}. \tag{13}$$

The proof is based on the link between the parabolic semigroup and the resolvent via the Laplace transform, which was previously exploited in the purely periodic settings by Suslina and Meshkova.

References

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