Examples of measures with trivial left and non-trivial right random walk tail boundary

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May 25, 2021

Abstract

In early 80's Vadim Kaimanovich presented a construction of a non-degenerate measure, on the standard lamplighter group, that has a trivial left and non-trivial right random walk tail boundary. We show that examples of such kind are possible precisely for amenable groups that have non-trivial factors with ICC property.

1 Introduction

Let G be a countable group and ν be a probability measure on G. A measure on G is called *non-degenerate* if its support generates G as a semigroup. The ν-random walk on G is defined in the following way. First let $(X_i)_{i=1}^{\infty}$ be the i.i.d. process with distribution ν. We set $Z_i = X_1 \cdot \ldots \cdot X_i$. Process (Z_i) is called the right ν-random walk on G. Similarly, we can define the left random walk by setting $Z'_i = X_i \cdot \ldots \cdot X_1$. By default, *random walk* will mean right random walk. We will restrict ourselves to non-degenerate measures on groups. If ν is a measure on a countable group G, we may define an opposite measure ν^{-1} by $\nu^{-1}(g) = \nu(g^{-1})$. It is trivial to see that instead of left random walks, we may consider right random walks with opposite measures. The *tail boundary* or the *tail subalgebra* of random walk (Z_i) is defined as the intersection $\bigcap_j \sigma(Z_j, Z_{j+1}, \ldots)$, where $\sigma(Z_j, Z_{j+1}, \ldots)$ denotes the minimal σ-algebra under which all variables $Z_j, Z_{j+1}, \ldots$ are measurable. Pair (G, ν) (or, abusing notation, measure ν itself) is called Liouville if the tail boundary of ν-random walk on G is trivial. One of the fundamental questions of asymptotic theory of random walks is whether a measure on a group is Liouville. Another notion of boundary is that of the *Poisson boundary*, it is defined as the invariant-set subalgebra of the process $(Z_i)_{i\in\mathbb{N}}$ under the time-shift action; in the setting of the random walk on group with non-degenerate measure, the Poisson Boundary coincides with the tail boundary (see [KaVe83], [Ka92]), so we will will use these notions interchangeably. Due to the Kaimanovich-Vershik entropy

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criterion for boundary triviality [KaVe83], we have that if a measure ν on G has finite Shannon entropy (defined by $H(\nu) = -\sum_{g\in G} \nu(g)\log\nu(g)$, assuming $0\log 0 = 0$), then left and right ν-random walks have trivial tail boundaries simultaneously. Surprisingly, this is not the case if the finite entropy assumption is waived: in [Ka83] Kaimanovich constructed an example of a measure on the standard lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ such that the left random walk has trivial tail boundary, while the right random walk has non-trivial. The purpose of the present note is to explore which countable groups admit examples akin to that of Kaimanovich. Our main result is the following:

Theorem 1. *Let G be a countable group. There is a non-degenerate probability measure ν on G with trivial left and non-trivial right random walk tail boundaries iff G is amenable and has a non-trivial ICC factor-group.*

We remind that a group is called an ICC (short for infinite conjugacy classes) if conjugacy class of each nontrivial element of the group is non-trivial. Note that a finitely-generated group lacks an ICC factor exactly when it is virtually-nilpotent (=has polynomial growth, due to the famous Gromov theorem), see [DuM56], [M56].

We note that using more subtle techniques from [ErKa19], one can prove that the boundary is not only non-trivial, but the action of any ICC factor-group on the corresponding factor-boundary could be made to be essentially free. In this note we only show that the boundary is non-trivial.

It is well known that amenable groups and only them admit non-degenerate Liouville measures, see Theorems 4.2 and 4.3 from [KaVe83]). It is also well known that all measures on groups without ICC factors are Liouville, see [Ja], a self-contained proof could be also found in the second preprint version of [Feta19]. Thus examples of Kaimanovich type are possible only for amenable groups with non-trivial ICC factors. In the sequel we will show that for every such group there is a measure of full support with non-trivial left and trivial right random walk boundary. Our construction is based on that of the breakthrough paper [Feta19] of Frish, Hartman, Tamuz and Vahidi Ferdowsi, where a non-Liouville measure was constructed for every group with an ICC factor, combined with the classic construction of a Liouville measure for every amenable group by Kaimanovich and Vershik [KaVe83] and Rosenblatt [Ro81], although in the proof of non-triviality of boundary we employ the approach similar to that of Ershler and Kaimanovich [ErKa19].

Acknowledgements. I'm thankful to Vadim Kaimanovich for helpful discussions. Prof A. M. Vershik constantly pushed me to study asymptotic behaviour of Markov chains, for which I'm grateful. This research project was started during my postdoc in the Einstein Institute of Mathematics at the Hebrew University of Jerusalem, supported by the Israel Science Foundation grant 1702/17, and finished in the Euler Mathematical Institute at Saint-Petersburg State University.

2 A process with heavy tail

Let K be a random variable such that $P(K = k) = (1/c)k^{-5/4}$, for $k \in \mathbb{N}$. Consider an i.i.d. process $(K_i)_{i\in\mathbb{N}}$ (each K_i has the same distribution as K). A number $i \in \mathbb{N}$ is a record-time if $K_i \geq K_j$ for all $j < i$, and the value (K_i) is a record-value; we will call pair (i, K_i) a record, and usually denote it, abusing notation a bit, as K_i. A record is simple if $K_i > K_j$ for all $j < i$.

The following lemma could be found in [Feta19, Lemma 2.6] and [ErKa19, Sections 2.B and 2.C].

Lemma 1. *For almost every realization of the random process (K_i), there is i_0 such that*

1. *for all $i \geq i_0$ we have $\max\{K_1, \dots K_i\} > i$;*
2. *all record-times starting from i_0 are simple.*

We have a random variable K, let us construct coupled random variable Y. If $K = k_0$, we set $Y =$ '*red*' with probability 2^{-k_0} and $Y =$ '*blue*' with probability $1 - 2^{-k_0}$.

Now consider the process $(K_i, Y_i)_{i=1}^{\infty}$ such that pairs (K_i, Y_i) form an i.i.d.

Consider a trajectory of the random process $(K_i, Y_i)_{i\in\mathbb{N}}$. We will say that this trajectory *stabilizes* if there is i_0 such that

1. for all $i \geq i_0$ we have $\max\{K_1, \dots K_i\} > i$;
2. all record-times i starting from i_0 are simple and $Y_i =' blue'$ for these record-times.

We will call the smallest such i_0 (if it exists) the *stabilization time*. Now it is easy to extend the previous lemma in the following way using the Borel-Cantelli lemma:

Lemma 2. *Almost every realization of the random process $(K_i, Y_i)_{i\in\mathbb{N}}$ stabilizes.*

3 Construction

Let G be a group, and A be a subset of G. We will say that a finite subset F of G is (A, δ)-invariant if $|aF \setminus F| < \delta|F|$ for all $a \in A$.

Let H be a group. Let A be a finite subset of H. We will say that an element b is an A-lock if for any a_1', a_2', a_1'', a_2'' from A, equality $a_1'ba_2' = a_1''ba_2''$ implies $a_1' = a_1''$ and $a_2' = a_2''$, and sets A and AbA are disjoint.

The proof of the following for amenable groups could be found in [Feta19, Proposition 2.5] and in the general case in [ErKa19, Poposition 4.25].

Lemma 3. *If Γ is an ICC group, then for every finite subset A of Γ there is an A-lock.*

Let G be a group, and let φ be a canonical epimorphism onto an ICC group Γ. Let (c_i) be any sequence enumerating all the elements of G.

We will construct the measure ν for the main theorem as a distribution of a certain random variable X coupled with (K, Y).

We will construct the variable in an iterative manner, together with sets A_i, F_i, D_i and a sequence b_i for each $i \in \mathbb{N}$.

Let $A_1 = \{e\}$. For each $i \geq 1$ we choose F_i to be $((A_i \cup \{c_i\} \cup \{c_i^{-1}\})^{i+1}, 1/i)$ - invariant. We denote $D_i = F_i^{-1} \cup F_i \cup A_i \cup \{c_i\} \cup \{c_i^{-1}\}$, for $i \in \mathbb{N}$. For each $i \geq 1$ we choose b_i to be such that $\varphi(b_i)$ is a $\varphi(D_i^{10i+10})$-lock. For each $i \in \mathbb{N}$ we set $A_{i+1} = D_i \cup b_i F_i^{-1} \cup F_i b_i^{-1}$.

We are ready to construct a random variable X that is coupled to (K, Y). Assume $K = i$. If $Y =$ "*red*", we set $X = c_i$. Otherwise let X be uniformly distributed in $b_i F_I^{-1}$.

So let ν be the distribution of X. It is trivial that the support of ν is G. The following proposition appears as a part of Theorem 4.2 from [KaVe83]:

Proposition 1. *Let ν be a non-degenerate measure on a countable group G. The Poisson boundary of ν-random walk on G is trivial iff for every $g \in G$ we have $\|g * \nu^{*n} - \nu^{*n}\| \to 0$.*

Lemma 4. *ν^{-1} - random walk on G has trivial Poisson boundary.*

Proof. Let g be fixed. Assuming that n is big enough, the sequence $K_1, \ldots, K_n$ with probability close to 1 has unique maximal value, and the corresponding $Y_i =$ "*blue*"; this is a trivial consequence of Lemma 2. So we have that $(\nu^{-1})^{*n}$ could be decomposed as

$$(\nu^{-1})^{*n} = \sum_{q', q'', m} p_{q', q'', m} \cdot q' * \lambda_{F_m} b_n^{-1} q'' + \eta_n,$$

where $q', q'' \in A_n^n$, $m > n$, $p_{q', q'', m} \geq 0$, λ_{F_m} is the uniform measure on F_m, and $\|\eta_n\| \to 0$ as $n \to \infty$. From this we readily conclude that $\|(\nu^{-1})^{*n} - g * (\nu^{-1})^{*n}\| \leq 4/n + 4\|\eta_n\|$, as soon as $g \in A_n$, so the assumption of Lemma 1 is fulfilled. □

Now we will show that the tail boundary is nontrivial. For this we will construct a tail-measurable function and show that its image is nontrivial.

Denote $W_n = \varphi(A_n^n b_n F_n^{-1} A_n^n)$ for all $n \in \mathbb{N}$. Let $p : \bigcup_n W_n \to \Gamma$ be a function defined by the formula $p(q'\varphi(b_n f) q'') = q'$, where $q', q'' \in A_n^n$, $f \in \varphi(F_n^{-1})$. Note that p is defined properly since by construction $\varphi(b_n)$ is a $\varphi(D_i^{10i+10})$-lock. Note that for any $w \in W_n$ if $p(w)$ belongs to $\bigcup_n W_n$, then $p(w) \in W_m$ for some $m < n$, since $p(w) \in A_n^n$ and W_m is disjoint from A_n^n for any $m \geq n$ by the construction of b_n. For any $w \in \Gamma$ we define $t(w)$ as the (possibly empty) set of all $p(w), p(p(w)), p(p(p(w)))$ that lie in $\gamma \in \bigcup_i W_i$. Let $(K_i, Y_i, X_i)_{i \in \mathbb{N}}$ be the process described above. We can make the following simple observation: if $i < j$ are bigger than the stabilization time, then $p(\varphi(Z_i)) \subset p(\varphi(Z_j))$, This is easy to prove for i and $i+1$ (either $i+1$ is a new record-time , and then $p(\varphi(Z_{i+1})) = \varphi(Z_i)$, or it is not a new record-time, and then $p(\varphi(Z_{i+1})) = p(\varphi(Z_i))$; either way we get $t(\varphi(Z_i)) \subset t(\varphi(Z_{i+1}))$. Also, if i_1 is at least the second record-time after the stabilization time, then $\varphi(Z_{i-1}) \in t(\varphi(Z_i))$. We conclude that for almost every realization of the process $(K_i, Y_i, X_i)_{i\in\mathbb{N}}$, the limit $\lim_{i\to\infty} t(\varphi(Z_i))$ exists and is equal to $\bigcup_{i\geq i_0} t(\varphi(Z_i))$. We define this limit $\tau(\omega)$. It is trivial that τ is a tail-measurable random variable. Let us collect our observations concerning τ.

Lemma 5. *1. τ is tail-measurable;*

2. $\tau \subset \bigcup_n W_n$;

3. $\tau \cap W_n$ has at most one element for any $n \in \mathbb{N}$;

4. if i_1 is at least the second record-time after the stabilization time, then $\varphi(Z_{i-1}) \in \tau(\varphi(Z_i))$;

5. if the trajectory of the process stabilizes, then there is n_0, such that $\tau \cap \bigcup_{n\geq n_0} W_n$ contains exactly elements of the form $\varphi(Z_{i-1})$, where i runs through all the record-time bigger than the stabilization time, except for the first one.

The purpose of ours is now to prove that the distribution of the random variable τ is not concentrated on one point. Denote $\Omega = (\mathbb{N}\times\{`red', `blue'\}\times G)^{\mathbb{N}}$ the space of trajectories of the random process $(K_i, Y_i, X_i)_{i'in\mathbb{N}}$, and

$$\Xi = (\mathbb{N} \times \{`red', `blue'\} \times G)^{\mathbb{N}} \times \mathbb{N},$$

the space of trajectories augmented by values of the stabilization times. Both Ω and Ξ are endowed with probability measures and are naturally isomorphic.

Take any point ξ_0 from the support of the measure on Ξ and such that the statement of Lemma 2 holds for the corresponding realization of the random process (K_i, Y_i, Z_i). For big enough m there are (at least) two γ_1, γ_2 such that $P(\varphi(X) = \gamma_1 | K < m) > 0$ and $P(\varphi(X) = \gamma_2 | K < m) > 0$. We fix the realization ω_0 of the random process that corresponds to ξ_0. Let i_0 be the stabilization time for that realization. Let i_1 be a record-time that is bigger than the stabilization time and such that the corresponding record-value k_{i_1} is bigger than m; let i_2 be the next record-time. By the previous lemma, $\varphi(Z_{i_2-1}) \in \tau$. Consider the neighbourhood of ξ_0 defined by constraints that $X_i = x_i, K_i = k_i, Y_i = y_i$ for all $i = 1 \dots i_2 - 1$ and that the stabilization time is not bigger than i_1. Denote S the projection of this neighbourhood into Ω. Note that S has positive measure. By construction of m, there are k_1', y_1', x_1' such that $\varphi(x_1') \neq \varphi(x_1)$, $k_1 < m$ and that $P(K = k_1', Y = y_1', X = x_1') > 0$. We define a map $T : A \to \Omega$ that changes the first triple (k_1, y_1, x_1) to (k_1', y_1', x_1'):

$$T(k_1, y_1, x_1, k_2, y_2, x_2, k_3, y_3, x_3, \dots) = (k_1', y_1', x_1', k_2, y_2, x_2, k_3, y_3, x_3, \dots).$$

This map preserves measure up to a positive multiplicative constant, so $T(S)$ has positive measure. Also, for every $\omega \in T(S)$ we have that the stabilization time is at most i_1. We also note that for every $\omega \in S$, $\tau(\omega) \cap W_{i_1} =$

$\{\varphi(x_1(\omega_0)x_2(\omega_0)\dots x_{i_2-1}(\omega_0))\}$, and for every $\omega \in T(S)$, we have $\tau(\omega) \cap W_{i_1} = \{\varphi(x_1' x_2(\omega_0)\dots x_{i_2-1}(\omega_0))\}$, so the distribution of τ is not concentrated on one point, since sets of values $\tau(S)$ and $\tau(T(S))$ are disjoint.

References

[DuM56] A.M. Duguid and D.H. McLain,*FC-nilpotent and FC-soluble groups*, Mathematical proceedings of the Cambridge philosophical society, 1956, pp. 391–398.

[ErKa19] A. Erschler and V. Kaimanovich, *Arboreal structures on groups and the associated boundaries*, arXiv preprint arXiv:1903.02095 (2019).

[Feta19] Frisch, Joshua, Yair Hartman, Omer Tamuz, and Pooya Vahidi Ferdowsi. *Choquet-Deny groups and the infinite conjugacy class property*, Annals of Mathematics 190, no. 1 (2019): 307–320.

[Ja] Wojciech Jaworski, *Countable amenable identity excluding groups*, Canadian Mathematical Bulletin 47 (2004), no. 2, 215–228.

[Ka83] V.A. Kaimanovich *Examples of non-abelian discrete groups with non-trivial exit boundary*, Zap. Nauchn. Sem. LOMI, 1983, Volume 123, 167–184.

[Ka92] V.A. Kaimanovich, *Measure-theoretic boundaries of Markov chains, 0–2 laws and entropy*, Harmonic analysis and discrete potential theory. Springer, Boston, MA, 1992. 145–180.

[KaVe83] V.A. Kaimanovich, and A. M. Vershik, *Random walks on discrete groups: boundary and entropy*, The annals of probability (1983): 457–490.

[M56] D.H. McLain, *Remarks on the upper central series of a group*, Glasgow Mathematical Journal 3 (1956), no. 1, 38–44.

[Ro81] J. Rosenblatt, *Ergodic and mixing random walks on locally compact groups*, Mathematische Annalen 257 (1981), no. 1, 31–42.