PREDICTABILITY, TOPOLOGICAL ENTROPY AND INVARIANT RANDOM ORDERS

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Abstract. We prove that a topologically predictable action of a countable amenable group has zero topological entropy, as conjectured by Hochman. On route, we investigate invariant random orders and formulate a unified Kieffer-Pinsker formula for the Kolmogorov-Sinai entropy of measure preserving actions of amenable groups. We also present a proof due to Weiss for the fact that topologically prime actions of sofic groups have non-positive topological sofic entropy.

The aim of this note is to prove the following:

Theorem 0.1. *Let Γ be a countable amenable group that acts on a compact metric space X by homeomorphism, and let $S \subset \Gamma$ be a subsemigroup of Γ with $1_\Gamma \notin S$. If every continuous function $f \in C(X)$ is contained in the closed algebra generated by $\{f \circ s : \ s \in S\}$ and the constant functions, then the action $\Gamma \curvearrowright X$ has zero topological entropy.*

Theorem 0.1 was initially proved in the case $\Gamma = \mathbb{Z}$ and $S = \mathbb{Z}_+$ by Kamiński, Siemaszko and Szymański in [17] (see also [18] and [11]). In [14] Hochman gave another proof and also generalized this to the case $\Gamma = \mathbb{Z}^d$. In the same work Hochman conjectured Theorem 0.1. Later, Huang, Jin and Ye in [15] proved Theorem 0.1 under the additional assumption that Γ is torsion-free and locally nilpotent. It turns out that the proof of Theorem 0.1 does not involve any considerable new ideas or tools beyond those developed by Hochman in the same paper where the question had been posed. However, as suggested to us by L. Bowen, without paying too great a price, we are able to obtain more general results about predictability in the presence of *invariant random orders* on groups. We will deduce Theorem 0.1 above as a particular case of the slightly more general Theorem 0.2 below.

For a countable group Γ we denote by $Ord(\Gamma) \subset \{0,1\}^{\Gamma\times\Gamma}$ the space of all orders on Γ. The space $Ord(\Gamma)$ is metrizable and compact, and admits a natural Γ-action. (see Section 2). Let ν be a Γ-invariant measure on $Ord(\Gamma)$. We will say that action $\Gamma \curvearrowright X$ is *ν-topologically predictable relative to a topological factor map $\pi : X \to Y$* if the following holds: For any $f \in C(X)$, and ν-a.e. $\prec \in Ord(\Gamma)$ the function f is contained in the closed algebra generated by $\{f \circ g : g \prec 1_\Gamma\}$ together with the image of $C(Y)$ in $C(X)$ under the map $\pi_* : C(Y) \to C(X)$.

Theorem 0.2. *Let Γ be a countable amenable group that acts on two compact metric spaces X, Y by homeomorphisms with a continuous Γ-equivariant map $\pi : X \to Y$. If there exists a Γ-invariant probability measure ν on the space $Ord(\Gamma)$*

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such that $\Gamma \curvearrowright X$ is ν-topologically predictable relative to $\pi : X \to Y$, then the topological entropy of the action $\Gamma \curvearrowright X$ is equal to the topological entropy of the action $\Gamma \curvearrowright Y$.

Theorem 0.1 is a special case of Theorem 0.2, obtained by taking $\Gamma \curvearrowright Y$ to be the trivial action on a singleton, and taking ν to be the delta measure concentrated on the order $\prec_S$ given by

$$g_1 \prec_S g_2 \;\Leftrightarrow\; g_1 g_2^{-1} \in S.$$

In the last decade entropy theory has emerged for actions of non-amenable groups. Entropy for measure preserving actions of sofic group was developed starting with the seminal paper [4]. Sofic topological entropy was introduced by Kerr and Li in [19]. The reader may find more details in [5], [6] and [20]. It is natural to ask if the above results extend to the non-amenable setting.

In Section 5 we include a short proof that was communicated to us by Benjy Weiss for the fact that topologically prime actions have zero entropy. Weiss's proof uses some similar techniques as the result about predictable systems. The result about prime systems also applies to actions of sofic groups, with appropriate adjustments.

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1. Preliminaries

In this paper $\Gamma \curvearrowright X$ will denote a left action of a countable group Γ by homeomorphisms on a compact metric space X. We will denote by $Prob(X)$ the (compact, convex) space of Borel probability measures on X, and by $Prob_\Gamma(X)$ the subset of Γ-invariant Borel probability measures. We will use the notation $\Gamma \curvearrowright (X, \mu)$ to indicate that $\mu \in Prob_\Gamma(X)$, and in this case we will say that the Γ action on (X, μ) is measure preserving. For two partitions α and β denote their join by $\alpha \vee \beta$. Similarly, for two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ denote by $\mathcal{A} \vee \mathcal{B}$ the smallest σ-algebra containing both. We will say that a partition β is finer than a partition α if each element of α is a union of elements from β.

Let α be a countable measurable partition of a standard probability space X. We denote $H_\mu(\alpha)$ its Shannon entropy; for two partitions α and β we denote $H_\mu(\alpha|\beta)$ the Shannon entropy of partition α relative to partition β. For a partition α and a σ-subalgebra $\mathcal{A}$ of a standard probability space, the Shannon entropy of α relative to $\mathcal{A}$ is given by

$$H_\mu(\alpha|\mathcal{A}) = -\int \log \mu\left(\alpha \mid \mathcal{A}\right) d\mu.$$

If α is a partition of finite Shannon entropy and $\mathcal{A}$ is a σ-subalgebra, then the following holds:

$$H_\mu(\alpha|\mathcal{A}) = \inf\{H_\mu(\alpha|\beta),\ \text{where } \beta \subseteq \mathcal{A}, H_\mu(\beta) < +\infty\};$$

Moreover, if $(\beta_i)_{i=1}^{\infty}$ is a sequence of partitions having finite Shannon entropy and $\mathcal{A}$ is the smallest σ-subalgebra that contains each β_i then

$$H_\mu(\alpha|\mathcal{A}) = \lim_{i\to\infty} H_\mu(\alpha|\bigvee_{j\le i}\beta_j).$$

We recall that countable group Γ is *amenable* if it has a Følner sequence, namely a sequence $(F_i)_{i=1}^{\infty}$ of finite subsets such that for any $g \in \Gamma$ the following holds:

$$\lim_{i\to\infty} \frac{|gF_i \setminus F_i|}{|F_i|} = 0.$$

It is a well known fact that a countable group Γ is amenable if and only if any action of Γ by continuous affine transformations on a compact convex subset of a separable locally convex topological vector space has a fixed point (for this and many other equivalent conditions for amenability, see for instance [28]).

For a countable partition $\alpha = \{B_1, B_2, \ldots\}$ and $g \in \Gamma$ we denote

$$\alpha^g = \{g^{-1}(B_1), g^{-1}(B_2), \ldots\}.$$

If $F \subset \Gamma$ is finite let $\alpha^F = \bigvee_{g\in F}\alpha^g$. This is again a countable partition. If $F \subset \Gamma$ is infinite, we let α^F denote the smallest σ-subalgebra containing all α^g for $g \in F$.

Now suppose that $(F_i)_{i=1}^{\infty}$ is a Følner sequence for the group Γ. We will denote the *Kolmogorov-Sinai entropy* of the partition α relative to a Γ-invariant σ-subalgebra $\mathcal{A} \subset Borel(X)$ by $h_\Gamma(\alpha, X, \mu|\mathcal{A})$. This is defined by the formula

$$h_\Gamma(\alpha, X, \mu|\mathcal{A}) = \lim_{i\to\infty} \frac{H_\mu(\alpha^{F_i}|\mathcal{A})}{|F_i|}.$$

It is known that $h_\Gamma(\alpha, X, \mu|\mathcal{A})$ does not depend on the choice of the Følner sequence. The *Kolmogorov-Sinai entropy* of the action relative to a σ-subalgebra $\mathcal{A}$ will be denoted by $h_\Gamma(X, \mu|\mathcal{A})$. This is given by

$$h_\Gamma(X, \mu|\mathcal{A}) = \sup\{h_\Gamma(\alpha, X, \mu|\mathcal{A}) : \ H_\mu(\alpha) < \infty\}.$$

We denote the Kolmogorov-Sinai entropy of the factor corresponding to $\mathcal{A}$ by

$$h_\Gamma(\mathcal{A}, X, \mu) = \sup\{h_\Gamma(\alpha, X, \mu) : \ \alpha \subset \mathcal{A}, H_\mu(\alpha) < \infty\}.$$

For an invariant σ-subalgebra $\mathcal{B}$ we denote $h_\Gamma(\mathcal{B}, X, \mu|\mathcal{A})$ the entropy of the factor corresponding to $\mathcal{B}$ relative to $\mathcal{A}$.

The Kolmogorov-Sinai theorem asserts that

$$h_\Gamma(\alpha^\Gamma \vee \mathcal{A}, X, \mu|\mathcal{A}) = h_\Gamma(\alpha, X, \mu|\mathcal{A}).$$

The topological entropy of an action $\Gamma \curvearrowright X$ of a countable amenable group Γ on a compact space X will be denoted by $h_\Gamma(X)$.

2. Invariant random orders and invariant random pasts

In the sequel we will employ some rudimentary theory of random orders on groups. Random orders were successfully used in [24] and [25] to prove results concerning deterministic orders on amenable groups. See the book [8] for background and much more. Particular invariant random orders have been applied to entropy theory, notably Keiffer's paper [21] about actions of amenable groups, and [1, 3, 31] for actions of countable but not necessarily amenable groups.

Consider the set $2^{\Gamma\times\Gamma}$ of binary relations on a countable group Γ, endowed with the product topology. This topology makes $2^{\Gamma\times\Gamma}$ a compact metrizable space. We will consider the left action $\Gamma \curvearrowright 2^{\Gamma\times\Gamma}$ given by

$$x(g\cdot R)y \Leftrightarrow (xg)R(yg) \text{ for } x,y,g\in\Gamma \text{ and } R\in 2^{\Gamma\times\Gamma}.$$

Recall that a relation $\prec\in 2^{\Gamma\times\Gamma}$ on Γ is called a *partial order* if the following requirements hold:

(1) It is antisymmetric, which means that if $x\prec y$ then $y\not\prec x$.
(2) It is transitive, which means that if $x\prec y$ and $y\prec z$ then $x\prec z$.

A partial order $\prec$ is called *total* if for any $x,y\in\Gamma$ either $x\prec y$, $y\prec x$ or $x=y$. Let $\mathit{Ord}(\Gamma)$ denote the set of all partial orders on Γ. Denote the set of all total orders on Γ by $\mathit{TotalOrd}(\Gamma)$. It is not hard to see that both $\mathit{TotalOrd}(\Gamma)$ and $\mathit{Ord}(\Gamma)$ are closed subsets of $2^{\Gamma\times\Gamma}$. It is easy to see that $\mathit{Ord}(\Gamma)$ and $\mathit{TotalOrd}(\Gamma)$ are Γ-invariant subsets.

To a partial order $\prec$ we associate the *past* (at the identity):

$$\Phi_\prec = \{\gamma\in\Gamma \;:\; 1_\Gamma\prec\gamma\}.$$

A fixed point for the action $\Gamma\curvearrowright \mathit{Ord}(\Gamma)$ is called a (deterministic) invariant order on Γ. If $\prec$ is an invariant order, then it is straightforward to check that the associated past $\Phi_\prec\subset\Gamma$ is a semigroup that does not contain the identity 1_Γ. If $\prec\in \mathit{Ord}(\Gamma)$ is a (deterministic) invariant *total* order, then the associated past $\Phi_\prec\subset\Gamma$ is an *algebraic past* for Γ, namely it is a semigroup with the property that

$$\Gamma = \Phi_\prec \uplus \Phi_\prec^{-1} \uplus \{1_\Gamma\}.$$

A group Γ that admits a deterministic invariant total order (or equivalently, admits an algebraic past) is called *left-orderable*. An *invariant random order* on Γ is a Γ-invariant Borel probability measure on $\mathit{Ord}(\Gamma)$. An *invariant random total order* is a Γ-invariant Borel probability measure on $\mathit{TotalOrd}(\Gamma)$. Equivalently, it is an invariant random order that is supported on the set of total orders. Thus, consistently with our notation the space of invariant random orders will be denoted by

$$\mathit{Prob}_\Gamma(\mathit{Ord}(\Gamma)),$$

and the space of invariant random total orders will be denoted by

$$\mathit{Prob}_\Gamma(\mathit{TotalOrd}(\Gamma)).$$

In the sequel we will use the probabilistic convention and write "$\prec$ is an invariant random total order with law ν" to mean that $\prec$ is an $\mathit{Ord}(\Gamma)$-valued random variable with distribution ν, where

$$\nu\in \mathit{Prob}_\Gamma(\mathit{Ord}(\Gamma)).$$

In this case for $F\in L^1(\nu)$ we will use the notation

$$\mathbb{E}_\prec F(\prec) = \int F(\prec)d\nu(\prec).$$

In contrast to deterministic invariant orders, every countable group Γ admits at least one invariant random total order. Namely, consider the random process $(\xi_\gamma)_{\gamma\in\Gamma}$ of independent random variables such that each ξ_γ has uniform distribution on $[0,1]$. Then each realization of this process induces an order on Γ.

We now define what it means for one invariant random total order to extend another: Let $\nu,\tilde\nu\in \mathit{Prob}_\Gamma(\mathit{Ord}(\Gamma))$ be invariant random orders on Γ. Recall that a *joining* of ν and $\tilde{\nu}$ is a probability measure $\lambda \in Prob(Ord(\Gamma) \times Ord(\Gamma))$ that is invariant under the Γ-action on the product space and has the property that push-forward of the projection onto the first coordinate coincides with ν and the push-forward of the projection of λ onto the second coordinate coincides with $\tilde{\nu}$. We say that an invariant random order $\tilde{\nu}$ extends ν if there exists a joining λ of ν and $\tilde{\nu}$ with the property that

$$\lambda\left(\{(\prec, \tilde{\prec}) \in Ord(\Gamma) \times Ord(\Gamma) \ : \ x \prec y \Rightarrow x \tilde{\prec} y\}\right) = 1 \tag{1}$$

Lemma 2.1. *Let Γ be an amenable group. Then any invariant random order on Γ can be extended to an invariant random total order.*

Proof. Let $\nu \in Prob_\Gamma(Ord(\Gamma))$ be an invariant random order. Consider the set $J(\nu)$ that consists of Borel probability measures $\lambda \in Prob(Ord(\Gamma) \times TotalOrd(\Gamma))$ whose push-forward via the projection onto the first coordinate is equal to ν and have the property that (1) holds. Then $J(\nu)$ is a non-empty, compact convex set. Because ν is a Γ-invariant probability measure, the set $J(\nu)$ is also invariant under the natural action of Γ. By amenability of Γ, the action $\Gamma \curvearrowright J(\nu)$ by affine transformations admits a fixed point $\lambda \in J(\nu)$. It follows that any such fixed point is a joining of ν with some invariant random total order $\tilde{\nu}$ that extends ν. $\square$

For torsion free locally nilpotent groups, the Rhemtulla-Formanek Theorem [9, 29] asserts that any *deterministic* invariant order extends to a *deterministic* invariant total order. Equivalently, for this class of groups any sub-semigroup that does not contain the identity extends to an algebraic past. The Rhemtulla-Formanek theorem was used in [15] to prove Theorem 0.1 for the class of torsion free locally nilpotent groups. Examples provided in the same paper show that the conclusion of the Rhemtulla-Formanek Theorem fails for more general groups, including some left-orderable amenable ones. Lemma 2.1 can be viewed as an easy "random substitute" for the Rhemtulla-Formanek theorem that applies to all amenable groups

Question 2.2. Does the statement of Lemma 2.1 hold without the amenability assumption on the group?

Let $\nu \in Prob_\Gamma(Ord(\Gamma))$ be an invariant random order. Recall that the action $\Gamma \curvearrowright X$ is *topologically ν-predictable* relative to $\pi : X \to Y$ if for any $f \in C(X)$, and ν-a.e. $\prec \in Ord(\Gamma)$ the function f is contained in the closed algebra generated by $\{f \circ g \ : \ g \prec 1_\Gamma\}$ together with the image of $C(Y)$ in $C(X)$ under the map $\pi_* : C(Y) \to C(X)$.

Now suppose $\mu \in Prob_\Gamma(X)$ is a Γ-invariant probability measure for the action $\Gamma \curvearrowright (X, \mu)$ and $\nu \in Prob_\Gamma(Ord(\Gamma))$ is an invariant random order. We say that the measure preserving action $\Gamma \curvearrowright (X, \mu)$ is *measure-theoretically ν-predictable relative to* $\pi : X \to Y$ if for every countable Borel partition α with $H_\mu(\alpha) < \infty$ we have that for ν-a.e $\prec \in Ord(\Gamma)$, the partition α is measurable with respect to the μ-completion of $\alpha^{\Phi_\prec} \vee \pi^{-1}(Borel(Y))$.

Let us introduce the following random generalization for the notion of an algebraic past. An *invariant random past* on Γ is a random function $\tilde{\Phi} : \Gamma \to 2^\Gamma$, or equivalently a Borel probability measure on $(2^\Gamma)^\Gamma$, with the following properties:

(i) For almost every instance of $\tilde{\Phi} : \Gamma \to 2^\Gamma$ and for all $g \in \Gamma$ the condition $g \notin \tilde{\Phi}(g)$ holds.

(ii) For almost every instance of $\tilde{\Phi} : \Gamma \to 2^\Gamma$, for all $g, h \in \Gamma$, if $g \in \tilde{\Phi}(h)$ then $\tilde{\Phi}(g) \subset \tilde{\Phi}(h)$.
(iii) If $g \neq h$ then either $g \in \tilde{\Phi}(h)$ or $h \in \tilde{\Phi}(g)$.
(iv) For all $g \in \Gamma$ the random subsets $\tilde{\Phi}(g)$ and $\tilde{\Phi}(1_\Gamma)g$ have the same distribution.

It follows directly from the definitions that if $\prec$ is an invariant random total order, then the random function given by $g \mapsto \{h \in \Gamma : \ h \prec g\}$ defines an invariant random past. If $\tilde{\Phi}$ is a random past on Γ with law $\tilde{\nu} \in \mathit{Prob}((2^\Gamma)^\Gamma)$ and $F \in L^1(\tilde{\nu})$ we use the following probabilistic notation:

$$\mathbb{E}_{\tilde{\Phi}} F(\tilde{\Phi}) = \int F(\tilde{\Phi}) d\tilde{\nu}(\tilde{\Phi}).$$

3. The Kieffer-Pinsker formula

In this section we state and prove a simultaneous but rather straightforward generalization of Kieffer's entropy formula [21] and of Pinsker's entropy formula for actions of amenable groups [16, Theorem 3.1]. The earliest and most basic case of this formula for the group $\Gamma = \mathbb{Z}$ with the usual order goes back to Kolmogorov's very first paper [23] on entropy.

Theorem 3.1 (The Kieffer-Pinsker formula)**.** *Let $\Gamma \curvearrowright (X, \mu)$ be a probability measure preserving action of a countable amenable group Γ, let $\tilde{\Phi} : \Gamma \to 2^\Gamma$ be an invariant random past on Γ with $\Phi = \tilde{\Phi}(1_\Gamma) \subset \Gamma$. Suppose that α is a Borel partition with $H_\mu(\alpha) < +\infty$ and that $\mathcal{A}$ is a Γ-invariant σ-algebra on X. Then the following holds:*

$$h_\Gamma(\alpha, X, \mu|\mathcal{A}) = \mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^\Phi \vee \mathcal{A}),$$

Before the proof we will establish a couple of auxiliary lemmata. We assume that $\tilde{\Phi} : \Gamma \to 2^\Gamma$, $\Phi = \tilde{\Phi}(1_\Gamma)$, α and $\mathcal{A}$ are as in the statement of Theorem 3.1.

Lemma 3.2. *For any $\varepsilon > 0$ there is such a finite subset D of Γ so that for any $D' \supset D$*

$$\mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^{D'\cap\Phi} \vee \mathcal{A}) \leq \mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^\Phi \vee \mathcal{A}) + \varepsilon. \tag{2}$$

Proof. Let us consider an arbitrary subset $\Phi \subset \Gamma$. Let $(D_i)_{i=1}^\infty$ be an increasing sequence of finite subsets of Γ such that $\bigcup D_i = \Gamma$. A classical argument using the Martingale convergence theorem and Chung's Lemma (as in [10, Theorem 14.28]) implies that

$$\lim_{i\to\infty} H_\mu(\alpha|\alpha^{D_i\cap\Phi} \vee \mathcal{A}) = H_\mu(\alpha|\alpha^\Phi \vee \mathcal{A}).$$

Using the monotone convergence theorem it follows that for sufficiently big $i \in \mathbb{N}$

$$\mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^{D_i\cap\Phi} \vee \mathcal{A}) \leq \mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^\Phi \vee \mathcal{A}) + \varepsilon. \tag{3}$$

Choose $i \in \mathbb{N}$ that satisfies (3), and let $D = D_i$. By monotonicity of conditional entropy (as in [10, Proposition 14.18]), for any superset $D' \supset D$ we have

$$\mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^{D'\cap\Phi} \vee \mathcal{A}) \leq \mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^{D\cap\Phi} \vee \mathcal{A}).$$

Thus (2) holds for any $D' \supset D$.

□

Lemma 3.3. *For every finite $F \subset \Gamma$ we have*

$$H_\mu\left(\alpha^F \mid \mathcal{A}\right) = \sum_{g \in F} \mathbb{E}_{\tilde{\Phi}} H_\mu\left(\alpha \mid \alpha^{Fg^{-1} \cap \Phi} \vee \mathcal{A}\right). \tag{4}$$

Proof. Fix an instance of $\tilde{\Phi} : \Gamma \to 2^\Gamma$ that satisfies properties (i)–(iii) of an invariant random past. Write F as $F = \{g_1, \ldots, g_{|F|}\}$ ordered so that $g_j \in \tilde{\Phi}(g_i)$ iff $i < j$. Applying the chain rule for entropy (as in [10, Proposition 14.18]) we have:

$$H_\mu\left(\alpha^F \mid \mathcal{A}\right) = \sum_{i=1}^{|F|} H_\mu\left(\alpha^{g_i} \mid \bigvee_{j<i} \alpha^{g_j} \vee \mathcal{A}\right)$$

This can be rewritten as :

$$H_\mu(\alpha^F|\mathcal{A}) = \sum_{g \in F} H_\mu(\alpha^g|\alpha^{F \cap (\tilde{\Phi}(g))} \vee \mathcal{A}).$$

Taking the expectation over $\tilde{\Phi}$, using property (iv) of an invariant random past and linearity of expectation we get (4). □

Proof of Theorem 3.1. Let $(F_i)_{i=1}^\infty$ be a Følner sequence in Γ.

By Lemma 3.3 we have:

$$H_\mu(\alpha^{F_i}|\mathcal{A}) = \sum_{g \in F_i} \mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^{F_i g^{-1} \cap \Phi} \vee \mathcal{A}). \tag{5}$$

Choose any $\varepsilon > 0$, and let $D \subset \Gamma$ as in Lemma 3.2. Then for any $D' \supset D$ we have that

$$\mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^\Phi \vee \mathcal{A}) \leq \mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^{D' \cap \Phi} \vee \mathcal{A}) \leq \mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^\Phi \vee \mathcal{A}) + \varepsilon$$

Let $\partial_D F_i$ denote the set of $g \in F_i$ such that $D \not\subset F_i g^{-1}$. From the definition of a Følner sequence we can derive that

$$\lim_{i \to \infty} \frac{|\partial_D F_i|}{|F_i|} = 0.$$

For any $g \in F_i \setminus \partial_D F_i$ we will have

$$\mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^\Phi \vee \mathcal{A}) \leq \mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^{F_i g^{-1} \cap \Phi} \vee \mathcal{A}) \leq \mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^\Phi \vee \mathcal{A}) + \varepsilon$$

The latter together with equation (5) implies that

$$\mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^\Phi \vee \mathcal{A}) \leq \lim_{i \to \infty} \frac{H_\mu(\alpha^{F_i}|\mathcal{A})}{|F_i|} \leq \mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^\Phi \vee \mathcal{A}) + \varepsilon.$$

This finishes the proof since $\varepsilon > 0$ is arbitrary. □

Corollary 3.4. *Let Γ be a countable amenable group. Then a probability measure preserving action $\Gamma \curvearrowright (X, \mu)$ has zero Kolmogorov-Sinai entropy relative to a factor map $\pi : X \to Y$ if and only if it is measure-theoretically ν-predictable relative to $\pi : X \to Y$ with respect to some (hence any) invariant random total order.*

Proof. An action $\Gamma \curvearrowright (X, \mu)$ has zero Kolmogorov-Sinai entropy if and only if for any finite measurable partition α we have

$$h_\Gamma(\alpha, X, \mu|\mathcal{A}) = 0.$$

By the Kieffer-Pinkser formula, for any invariant random past this is equivalent to

$$\mathbb{E}_{\tilde{\Phi}} H_\mu(\alpha|\alpha^\Phi \vee \mathcal{A}) = 0,$$

which is equivalent to having $H_\mu(\alpha|\alpha^\Phi \vee \mathcal{A}) = 0$ for a.e. realization of $\tilde{\Phi}$. This is equivalent to the statement that α is measurable with respect to the μ-completion of $\alpha^\Phi \vee \mathcal{A}$. $\square$

Furthermore, predictability of a relative generator with respect to an invariant random partial order implies zero relative entropy:

Proposition 3.5. *Let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving action of a countable amenable group Γ and let $\mathcal{A}$ be a Γ-invariant σ-subalgebra. Suppose that $\prec$ is an invariant random partial order on Γ. Let α be a finite Shannon entropy partition of X such that $\alpha \subset \mathcal{A} \vee \alpha^{\Phi_\prec}$ for almost every instance of $\prec$ and*

$$\alpha^\Gamma \vee \mathcal{A} = \mathit{Borel}(X) \mod \mu. \tag{6}$$

Then $h_\Gamma(X, \mu|\mathcal{A}) = 0$.

Proof. Denote the law of the invariant random partial order $\prec$ by $\nu \in \mathit{Prob}_\Gamma(\mathit{Ord}(\Gamma))$. By Lemma 2.1, we can find an invariant random total order $\tilde{\nu} \in \mathit{Prob}_\Gamma(\mathit{Ord}(\Gamma))$ that extends ν. Because $\tilde{\nu}$ extends ν it follows that $\alpha \subset \mathcal{A} \vee \alpha^{\Phi_\prec}$ for $\tilde{\nu}$ -a.e instance of $\prec$. This implies by Theorem 3.1 that $h_\Gamma(X, \mu|\mathcal{A}) = 0$. $\square$

Seward [30] defined the *relative Rokhlin entropy* for measure-preserving actions of countable groups. For an ergodic measure-preserving action $\Gamma \curvearrowright (X, \mu)$ of a countable group Γ and Γ-invariant σ-subalgebra, it is given by

$$h_\Gamma^{Rok}(X, \mu \mid \mathcal{A}) = \inf_\alpha H_\mu(\alpha \mid \mathcal{A}),$$

where α ranges over all generating partitions (countable partitions α that satisfy (6)). For a free action of an amenable group, Rokhlin entropy coincides with Kolmogorov-Sinai entropy. Seward proved the following far-reaching extension of Sinai's theorem: Any free ergodic measure-preserving action $\Gamma \curvearrowright (X, \mu)$ with positive Rokhlin entropy admits a Bernoulli factor, or equivalently it admits a (non-trivial) partition whose iterates under Γ are jointly independent [32]. As an immediate corollary of Seward's Bernoulli factor theorem we have:

Proposition 3.6. *Let Γ be a countable group, let ν be an invariant random partial order on Γ. A free ergodic action $\Gamma \curvearrowright (X, \mu)$ that is measure-theoretically predictable with respect to some invariant random partial order has zero Rokhlin entropy.*

Proof. If a free ergodic action $\Gamma \curvearrowright (X, \mu)$ has positive Rokhlin entropy then Seward's Bernoulli factor theorem says that it admits a non-trivial finite partition α with independent Γ-iterates. Such a partition α is not measurable with respect to the μ-completion of $\alpha^{\Phi_\prec}$, for any order $\prec$ on Γ. Hence, a free ergodic action with positive Rokhlin entropy is not measure-theoretically predictable with respect to any invariant random order. $\square$

Question 3.7. Is there a direct proof of Proposition 3.6 that does not use Seward's Bernoulli factor theorem?

4. From topological predictability to measure-theoretical predictability via μ-continuous partitions

In this section we complete the proof of Theorem 0.2. The steps are essentially identical to Hochman's proof in [14], where only the lexicographic past on the group $\mathbb{Z}^d$ was considered, without the "relative" version (the image of the factor map $\pi : X \to Y$ was the trivial one-point space). Just as in Hochman's proof, we will rely on the variational principle for topological entropy.

Theorem 4.1 (The Variational principle [34, 26, 27]). *Let $\Gamma \curvearrowright X$ be an action of a countable amenable group by homeomorphisms on a compact metrizable space X. Then the topological entropy $h_\Gamma(X)$ is given by*

$$h_\Gamma(X) = \sup_{\mu \in Prob_\Gamma(X)} h_\Gamma(X, \mu).$$

Note that Kerr and Li proved a more general variational principal for sofic entropy [19].

As in the previous sections, let X be a compact metric space and let μ be a Borel probability measure on it. The Rokhlin distance $d_\mu(\alpha, \beta)$ between two partitions α and β of finite Shannon entropy is defined by the formula

$$d_\mu(\alpha, \beta) = H_\mu(\alpha|\beta) + H_\mu(\beta|\alpha).$$

It is well known that for a measure preserving action of an amenable group the Kolmogorov-Sinai entropy is an 1-Lipschitz function with respect to the Rokhlin metric on the space of partition with finite Shannon entropy.

A partition α of (X, μ) is said to be a *μ-continuous partition* if there is a continuous function $f : X \to \mathbb{R}$ such that pieces of α are equal to the level sets of f up to μ-null subset. This definition is due to Hochman [14]; he proved the following (in the more general setup where X is a normal topological space and μ is a regular Borel probability measure):

Proposition 4.2 (Hochman [14]). *For any Borel probability measure $\mu \in Prob(X)$, the μ-continuous partitions are dense with respect to the Rokhlin metric.*

Proof of Theorem 0.2. Let $\Gamma \curvearrowright X, \Gamma \curvearrowright Y$ and $\pi : X \to Y$ be as in the statement of Theorem 0.2. Let

$$\mathcal{A} = \pi^{-1}(Borel(Y)) \subset Borel(X).$$

Let μ be a Γ-invariant measure on X. We note that $h_\Gamma(Y, \pi(\mu)) = h_\Gamma(\mathcal{A}, X, \mu)$. Take any μ-continuous partition α of (X, μ) with $H(\alpha) < \infty$. Then topological predictability of the action relative to $\pi : X \to Y$ implies that α is measurable with respect to the completion of $\alpha^{\Phi_\prec} \vee \mathcal{A}$ for almost every instance of the invariant random total order $\prec$. By Corollary 3.4 of Theorem 3.1 it follows that

$$h_\Gamma(\alpha, X, \mu \mid \mathcal{A}) = 0.$$

Since this holds for a set of partitions that is dense with respect to the Rokhlin metric, and the function $\alpha \mapsto h_\Gamma(\alpha, X, \mu \mid \mathcal{A})$ is continuous (in fact 1-Lipschitz with respect to the Rokhlin-metric), it follows that

$$h_\Gamma(X, \mu \mid \mathcal{A}) = 0.$$

The Abramov-Rokhlin entropy addition formula for amenable group actions [35] asserts that

$$h_\Gamma(X, \mu) = h_\Gamma(\mathcal{A}, X, \mu) + h_\Gamma(X, \mu \mid \mathcal{A}).$$

Thus for any $\mu \in Prob_\Gamma(X)$:

$$h_\Gamma(X,\mu) = h_\Gamma(\mathcal{A}, X, \mu) = h_\Gamma(Y, \pi(\mu)).$$

By the variational principle (Theorem 4.1) it follows that

$$h_\Gamma(X) \le h_\Gamma(Y).$$

Since the topological entropy for actions of amenable groups is factor-monotone, we have

$$h_\Gamma(X) = h_\Gamma(Y).$$

□

5. Prime actions have zero topological entropy

An action $\Gamma \curvearrowright X$ is called *topologically prime* if every factor map is either an isomorphism or it maps onto the trivial action on the one-point space. More generally, if $\pi : X \to Y$ is a topological factor map between $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$, we say that $\pi : X \to Y$ is a *topologically prime extension* if $\Gamma \curvearrowright X$ has no intermediate factors. Equivalently, $C(X)$ has no strict Γ-invariant C^*-subalgebras that strictly contain $\pi_*(C(Y))$. King constructed and example of a homeomorphism on the Cantor set that is topologically prime [22], because it has "topological minimal self-joinings" in the sense of del Junco [7]. The later property, called "doubly minimal" by Weiss [36], means that any pair of points $x, y \in X$ that are not in the same orbit have a dense orbit in $X \times X$. More generally, any free and ergodic measure-preserving $\mathbb{Z}$-action with zero entropy admits a uniquely ergodic doubly minimal model [36]. In particular there is topologically prime model for any free, ergodic $\mathbb{Z}$-action with zero entropy [11, Theorem 13.1].

The following result was communicated to us by Benjy Weiss. With his kind permission we reproduce his proof.

Theorem 5.1. *Suppose that Γ is a countable amenable group that acts on X and Y, and that $\pi : X \to Y$ is a topologically prime extension. Then $h_\Gamma(X) = h_\Gamma(Y)$.*

For $\Gamma = \mathbb{Z}$ and π equal to the trivial factor, a proof of Theorem 5.1 appears in [33]. See also [11, Section 13].

Proof. Let $\pi : X \to Y$ be a topologically prime extension. Choose any Γ-invariant measure $\mu \in Prob_\Gamma(X)$, and let $\nu \in Prob_\gamma(Y)$ be the push-forward of μ via π. We will prove that

$$h_\Gamma(X,\mu) = h_\Gamma(Y,\nu). \tag{7}$$

We first prove this under the additional assumption that μ satisfies the following property:

$$\inf\left\{\mu(A) :\ A \text{ is not contained in the } \mu\text{-completion of } \pi^{-1}(Borel(Y))\right\} = 0. \tag{8}$$

Since μ-continuous partitions are dense with respect to the Rokhlin metric, we can find for any $\epsilon > 0$ a μ-continuous partition α such that

$$0 < H_\mu(\alpha \mid \pi^{-1} Borel(Y)) < \epsilon. \tag{9}$$

It follows that

$$h_\mu(\alpha, X, \mu \mid \pi^{-1} Borel(Y)) < \epsilon.$$

Since $\pi : X \to Y$ is a topologically prime extension, for any $f \in C(X) \setminus \pi_*(C(Y))$, $C(X)$ is contained in the Γ-invariant C^*-algebra generated by f and $\pi_*(C(Y))$. By the left inequality in (9), α is not contained in the μ-completion of $\pi^{-1}(Borel(Y))$. It follows that

$$(\alpha \vee \pi^{-1} Borel(Y))^{\Gamma} = Borel(X) \mod \mu.$$

By Kolmogorov-Sinai theorem,

$$h_{\Gamma}(X, \mu \mid \pi^{-1} Borel(Y)) < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this shows that

$$h_{\Gamma}(X, \mu \mid \pi^{-1} Borel(Y)) = 0.$$

By the Abramov-Rokhlin entropy addition formula, this implies (7).

It remains to prove that (7) holds even when $\mu \in Prob_{\Gamma}(X)$ does not satisfy (8). If (8) does not hold, then there exists $\epsilon > 0$ so that every $A \in Borel(X)$ with $\mu(A) < \epsilon$ is also measurable with respect to the μ-completion of $\pi^{-1}Borel(Y)$. In this case the factor map $\pi : X \to Y$ is of the following very degenerate form: Let $\mu = p \cdot \mu_c + (1-p) \cdot \mu_a$ be the representation of μ as a convex combination of a purely continuous measure $\mu_c \in Prob_{\Gamma}(X)$ and purely atomic measure $\mu_a \in Prob_{\Gamma}(X)$. Then $\pi : X \to Y$ gives a measure-preserving isomorphism between $\Gamma \curvearrowright (X, \mu_c)$ and $\Gamma \curvearrowright (Y, \pi(\mu_c))$. In particular, the factor map $\pi : X \to Y$ is finite-to-one (actually bounded-to-one) μ-almost everywhere. Because Γ is an infinite amenable group, finite-to-one extensions do not increase entropy and (7) follows in this case.

□

Remark 5.2. In the proof above we use the fact that Γ is infinite to conclude that finite-to-one extensions do not increase entropy. If Γ is a finite group, Theorem 5.1 fails. To see this, take $\Gamma \curvearrowright X$ to be the action of a finite group Γ on itself by translations, and take $\Gamma \curvearrowright Y$ to be an action of Γ on the cosets of a maximal proper subgroup.

We now consider topologically prime actions of sofic groups. For a measure preserving action $\Gamma \curvearrowright (X, \mu)$ of a sofic group Γ with a sofic approximation Σ, let $h_{\Gamma}^{\Sigma}(X, \mu)$ denote the sofic entropy of the action (for definitions see for instance [5]).

The first-named author and Brandon Seward [2, Proposition 8.8] and independently Ben Hayes [12, Proposition 2.7 (i)] proved the following Abramov-Rokhlin-type inequality:

Proposition 5.3 (Abramov-Rokhlin sub-addition formula for sofic entropy)**.** *Let Γ be a sofic group with sofic approximation Σ, and let $\pi : X \to Y$ be a factor map between the measure preserving actions $\Gamma \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \mu)$. If α is a measurable partition of X such that*

$$(\alpha \vee \pi^{-1}(Borel(Y)))^{\Gamma} \supset Borel(X) \mod \mu, \tag{10}$$

then

$$h_{\Gamma}^{\Sigma}(X, \mu) \leq h_{\Gamma}^{\Sigma}(Y, \nu) + H_{\mu}(\alpha \mid \pi^{-1}(Borel(Y)).$$

The above is a simplified and slightly less general form of the corresponding statements from [2, 12].

The following proposition on sofic entropy of finite-to-one extension follows from the main theorem of Hayes' work [13] relating sofic entropy and spectral properties of actions. That work establishes much more general results. In particular, the statement below will hold with compact extensions instead of finite-to-one. See the discussion following Proposition 5.7 in [6]. The case needed in our exposition allows for a short combinatorial proof, which we include for the sake of self-containment.

Proposition 5.4. *Let Γ be a countably infinite sofic group with sofic approximation Σ, and let $\pi : X \to Y$ be a finite-to-one factor map between the measure preserving actions $\Gamma \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$. Then*

$$h^{\Sigma}_{\Gamma}(X,\mu) \leq h^{\Sigma}_{\Gamma}(Y,\nu). \tag{11}$$

Proof. Because $\pi : X \to Y$ is finite-to-one, by removing null sets we can assume that $|\pi^{-1}(\{y\})|$ is a positive integer for every $y \in Y$. For every $n \in \mathbb{N}$ let

$$Y_n = \{y \in Y : \; \pi^{-1}(\{y\}) = n\} \text{ and } X_n = \pi^{-1}(Y_n).$$

Let us first assume in addition that $\Gamma \curvearrowright (Y,\nu)$ is aperiodic. By the Abramov-Rokhlin sub-addition formula for sofic entropy (Proposition 5.3) it is enough to show that for every $\epsilon > 0$ we can find a measurable partition α of X that satisfies (10) and $H_\mu(\alpha) < \epsilon$. Because $\Gamma \curvearrowright (Y,\nu)$ is aperiodic, so is $\Gamma \curvearrowright (X,\nu)$. So (X,μ) is a standard probability space with no atoms. Let $\phi : X \to [0,1]$ be a Borel bijection.

Because $\Gamma \curvearrowright (Y,\nu)$ is aperiodic whenever $\nu(Y_n) > 0$ we can find arbitrary small $\epsilon_n > 0$ and a Borel measurable set $A_n \subset Y_n$ so that $\bigcup_{g\in\Gamma} g(A_n) = Y_n$ and $\nu(A_n) = \epsilon_n$ (for instance by considering the ergodic decomposition of $\Gamma \curvearrowright (Y,\nu)$). Because $\pi : X \to Y$ is finite-to-one we can define $\psi : X \to \mathbb{N}$ by

$$\psi(x) = \#\left\{x' \in \pi^{-1}\left(\{\pi(x)\}\right) : \; \phi(x') \leq \phi(x)\right\}.$$

Then for every $y \in Y_n$, ψ induces a bijection between $\pi^{-1}\left(\{y\}\right)$ and $\{1,\ldots,n\}$. We denote the inverse by

$$\psi_y^{-1} : \{1,\ldots,n\} \to \pi^{-1}\left(\{y\}\right).$$

Also we have a Borel bijection $\Psi : X \to \bigcup_{i=1}^{\infty} \left(Y_n \times \{1,\ldots,n\}\right)$ given by

$$\Psi(x) = (\pi(x), \psi(x)).$$

For $y \in Y_n$ and $g \in \Gamma$ define a permutation $\Pi_{g,y}$ of $\{1,\ldots,n\}$ by

$$\Pi_{g,y}(i) = \psi\left(g(\psi_y^{-1}(i))\right) \text{ for } i \in \{1,\ldots,n\}.$$

Then the map $(g,y) \mapsto \Pi_{g,y}$ is a Borel map from $\Gamma \times Y_n$ to the set of permutations on $\{1,\ldots,n\}$. For every $n \in \mathbb{N}$ and $i \in \{1,\ldots,n\}$ let

$$B_{n,i} = \pi^{-1}(A_n) \cap \psi^{-1}(\{i\}).$$

Consider the partition α of X given by

$$\alpha = \{B_{n,i} : \; n \in \mathbb{N} \text{ and } 1 \leq i \leq n\} \cup \{X \setminus \bigcup_{n\in\mathbb{N}} \pi^{-1}(A_n)\}.$$

It follows that

$$H_\mu(\alpha) < \sum_{n=1}^{\infty} \epsilon_n \left(\log(\epsilon_n^{-1}) + \log(n)\right) + \log\left(1 - \sum_{n=1}^{\infty} \epsilon_n\right),$$

So if $(\epsilon_n)_{n=1}^{\infty}$ are sufficiently small $H_\mu(\alpha) < \epsilon$. In order to check that α is a relative generator in the sense that it satisfies (10), it suffices to check that $\psi : X \to \mathbb{N}$ is $\alpha^{\Gamma} \vee \pi^{-1}(Borel(Y))$-measurable. Indeed, because $Y_n = \bigcup_{g\in\Gamma} g(A_n)$ for any $x \in X_n$ there exists $g \in \Gamma$ and $i \in \{1,\ldots,n\}$ so that $g(x) \in B_{n,i}$. For such x we have $\psi(g(x)) = i$ so $\psi^{-1}_{g(\pi(x))}(i) = g(x)$ so

$$\psi(x) = \psi(g^{-1}(g(x))) = \psi(g^{-1}(\psi^{-1}_{g(\pi(x))}(i))) = \Pi_{g^{-1},\pi(g\cdot x)}(i).$$

This implies the desirable measurability.

Now let us remove the additional assumption that $\Gamma \curvearrowright (Y,\nu)$ is aperiodic. Let $\Gamma \curvearrowright (Z,\eta)$ be a Bernoulli action. Then $\pi : X \to Y$ naturally induces a finite-to-one factor map from $\Gamma \curvearrowright (X \times Z, \mu \times \eta)$ to $\Gamma \curvearrowright (Y \times Z, \nu \times \eta)$. Since $\Gamma \curvearrowright (Y \times Z, \nu \times \eta)$ is aperiodic we can use the first part to conclude that

$$h_\Gamma^\Sigma(X \times Z, \mu \times \eta) \leq h_\Gamma^\Sigma(Y \times Z, \mu \times \eta).$$

By [4, Theorem 8.1], because (Z,η) is a Bernoulli action

$$h_\Gamma^\Sigma(X \times Z, \mu \times \eta) = h_\Gamma^\Sigma(X,\mu) + h_\Gamma^\Sigma(Z,\eta),$$

and

$$h_\Gamma^\Sigma(Y \times Z, \mu \times \eta) = h_\Gamma^\Sigma(Y,\mu) + h_\Gamma^\Sigma(Z,\eta).$$

This proves (11) without assuming that $\Gamma \curvearrowright (Y,\nu)$ is aperiodic. □

Theorem 5.5. *Suppose Γ is a sofic group with sofic approximation Σ that acts on X and Y and that $\pi : X \to Y$ is a topologically prime extension. Then*

$$h_\Gamma^\Sigma(X) \leq h_\Gamma^\Sigma(Y). \tag{12}$$

In particular, topologically prime actions of sofic groups have non-positive sofic entropy.

Proof. We obtain the proof essentially by repeating the proof of Theorem 5.1, with the following modifications: Instead of applying the Abramov-Rokhlin entropy addition formula we apply the Abramov-Rokhlin sub-addition formula for sofic entropy (Proposition 5.3). To deal with the case where (8) fails, we apply Proposition 5.4 above. □

We note that in general we cannot conclude an equality instead of the equality (12) under the assumptions of Theorem 5.5. From the definition of sofic entropy, if $X \curvearrowright \Gamma$ admits no invariant probability measures then $h_\Gamma^\Sigma(X) = -\infty$. The following is an example of a topologically prime action of the free group on two generators that admits no invariant probability measure:

Example 5.6. Let $T_1 : X \to X$ be a topologically prime homeomorphism that is uniquely ergodic (for instance Kings's example [22] on the cantor set X), and let $T_2 : X \to X$ be a homeomorphism that does not preserve the unique T_1-invariant measure. Consider the action on X of the free group generated by T_1 and T_2. This is a topologically prime action because it has T_1 as a subaction. It admits no invariant probability measure, because the unique T_1-invariant measure is not T_2-invariant.

Remark 5.7. Although finite-to-one extensions can never increase sofic entropy they can certainly decrease it, as in the well known Ornstein-Weiss example of a two-to-one factor map from the Bernoulli 2-shift to the Bernoulli 4-shift over the free group.

Remark 5.8. Using the Abramov-Rokhlin sofic entropy sub-addition formula and similar arguments as in the proof of Proposition 5.4 it is possible to prove the following "atomless" refinement of the Abramov-Rokhlin entropy sub-addition formula as follows: If $\mu = p\mu_c + (1-p)\mu_a$ is the representation of μ as a convex combination of a continuous measure μ_c and a purely atomic measure μ_a and α is a measurable partition of X such that

$$(\alpha \vee \pi^{-1}(Borel(Y)))^\Gamma \supset Borel(X) \mod \mu_c,$$

then

$$h_\Gamma^\Sigma(X,\mu) \le h_\Gamma^\Sigma(Y,\nu) + H_\mu(\alpha \mid \pi^{-1}(Borel(Y)).$$

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