DOI 10.1007/s10958-018-4124-2
Journal of Mathematical Sciences, Vol. 236, No. 4, January, 2019

MULTIPLICITY OF POSITIVE SOLUTIONS TO THE BOUNDARY-VALUE PROBLEMS FOR FRACTIONAL LAPLACIANS

N. S. Ustinov* UDC 517

For the problem $(-\Delta)^s u = u^{q-1}$ in the annulus $\Omega_R = B_{R+1} \setminus B_R \in \mathbb{R}^n$, a so-called "multiplicity effect" is established: for each $N \in \mathbb{N}$ there exists R_0 such that for all $R \geq R_0$ this problem has at least N different positive solutions. $(-\Delta)^s$ in this problem stands either for Navier-type or for Dirichlet-type fractional Laplacian. Similar results were proved earlier for the equations with the usual Laplace operator and with the p-Laplacian operator. Bibliography: 22 titles.

1. Introduction

In the present paper we study the multiplicity of positive solutions of the equation with a fractional Laplacian,

$$(-\Delta)^s u = |u|^{q-2} u \quad \text{in} \quad \Omega_R, \quad u \in \widetilde{H}^s(\Omega_R) \tag{1}$$

in the annulus $\Omega_R = B_{R+1} \backslash B_R \in \mathbb{R}^n$ for $s \in (0,1)$, $2 < q < 2^*_n \equiv \frac{2n}{(n-2s)_+}$. The fractional Laplacian $(-\Delta)^s$ on the left-hand side of equation (1) can be understood in the sense of Dirichlet or Navier, see Sec. 2.

The multiplicity effect was first discovered by C. Coffman [4], who showed that for $n = 2$ the problem

$$-\Delta u = |u|^{q-2} u \quad \text{in} \quad \Omega_R, \qquad u|_{\partial\Omega_R} = 0 \tag{2}$$

has any preassigned number of positive solutions (not obtained from each other by rotation) for $q > 2$ and sufficiently large R.

In [11], the multiplicity of solutions to problem (2) was proved for $n \geq 4$, $2 < q < 2^* \equiv \frac{2n}{(n-2)_+}$, and also the question of existence of non-radial solutions for $q \geq 2^*$ was considered.

The multiplicity of solutions for $n = 3$ was obtained in [1].

Later in [16] and [9], similar results were obtained for an equation with p-Laplacian $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$: the problem

$$-\Delta_p u = |u|^{q-2} u \quad \text{in} \quad \Omega_R, \qquad u|_{\partial\Omega_R} = 0$$

has any preassigned number of different positive solutions for $1 < p < \infty$, $p < q < p^* \equiv \frac{np}{(n-p)_+}$ and sufficiently large R.

We obtain analogous results for problem (1) for $n \neq 3$. We note that the operator of fractional Laplacian is nonlocal, which does not allow us to use the technique presented in the above papers.

The present paper has the following structure: in Sec. 2, the basic definitions used in this paper are given. In Sec. 3, we prove lemmas which helps us to obtain the energy estimates of radial functions in the space $\widetilde{H}^s(\omega_R)$. In Sec. 4, we describe the behavior of energy as $R \to +\infty$. Finally, we prove the main result, Theorem 6, in Sec. 5. Most technical details are in the Appendix.

In what follows, different absolute constants are denoted by C. In the case of the dependence of a constant on a parameter, this parameter is indicated in parentheses. The notation $a \asymp b$ means that the two-sided estimate $C_1 b \leq a \leq C_2 b$ with the constants independent of R is true.

*St.Petersburg State University, St.Petersburg, Russia, e-mail: `ustinns@yandex.ru`.

Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 459, 2017, pp. 104–126. Original article submitted April 25, 2017.

A ball of radius r with center at the point x is denoted by $B_r(x)$. If $x = 0$, then, for brevity, we denote it by B_r. Throughout the paper, the zero vector of dimension m is denoted by $\mathbb{O}_m$.

2. Definitions and basic concepts

We denote by ω_R the annulus in $\mathbb{R}^1$: $\omega_R = [-R-1, -R] \cup [R, R+1]$. A function with support in ω_R or Ω_R is denoted by u_R, emphasizing the dependence on the radius R.

The Fourier transform in the space $\mathbb{R}^n$ is given by the formula

$$\mathcal{F}u(\xi) := \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx.$$

We recall the definition of the spaces $H^s(\mathbb{R}^n)$ and $\widetilde{H}^s(\Omega_R)$ (see, for example, [18, Secs. 2.3.3, 4.3.2]):

$$H^s(\mathbb{R}^n) = \left\{ u \in L_2(\mathbb{R}^n) \mid \int\limits_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 \, d\xi < +\infty \right\},$$

$$\widetilde{H}^s(\Omega_R) = \left\{ u \in H^s(\mathbb{R}^n) \mid \operatorname{supp}(u) \subset \overline{\Omega}_R \right\}.$$

The fractional Laplacian $(-\Delta)^s u$ on the Schwartz class

$$\mathcal{S} = \left\{ u \in \mathcal{C}^\infty(\mathbb{R}^n) \mid \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta u(x)| < +\infty \quad \text{for all } \alpha, \beta \right\}$$

is given by the formula

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u(\xi)).$$

The quadratic form of this operator has the form

$$((-\Delta)^s u, u) = \int\limits_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi. \tag{3}$$

The fractional Dirichlet Laplacian $(-\Delta)^s_D$ in the domain Ω_R, also called the restricted fractional Laplacian, is a self-adjoint operator defined by the quadratic form (3) restricted to the domain $\widetilde{H}^s(\Omega_R)$.

The fractional Navier Laplacian $(-\Delta)^s_N$ is the sth power of the Laplace operator in the sense of the spectral theory, i.e., the self-adjoint operator defined by its quadratic form

$$((-\Delta)^s_N u, u) = \sum_{j=1}^{\infty} \lambda_j^s (u, \phi_j)^2, \tag{4}$$

where the λ_j and ϕ_j are the eigenvalues and orthonormal eigenfunctions of the Laplace operator with the Dirichlet condition in the domain Ω_R. The fractional Navier Laplacian $(-\Delta)^s_N u$ is also called the spectral fractional Laplacian. It is well known (see, for example, [12, Lemma 1]) that for $s \in [0, 1]$, the domain of the quadratic form (4) coincides with $\widetilde{H}^s(\Omega_R)$. We emphasize that both operators are nonlocal for $s \notin \mathbb{Z}$.

The norm in the space $\widetilde{H}^s(\Omega_R)$ is induced by the norm in the space $H^s(\mathbb{R}^n)$,

$$\|u\|^2_{\widetilde{H}^s(\Omega_R)} = \|u\|^2_{L_2(\Omega_R)} + ((-\Delta)^s_D u, u). \tag{5}$$

However, by the Friedrichs inequalities (see Appendix, Lemma 3) in the space $\widetilde{H}^s(\Omega_R)$, quadratic forms (3) and (4) yield for $s \in [0, 1]$ the norm

$$[u]^2_{D,\widetilde{H}^s(\Omega_R)} := ((-\Delta)^s_D u, u) \asymp \|u\|^2_{\widetilde{H}^s(\Omega_R)} \asymp ((-\Delta)^s_N u, u) =: [u]^2_{N,\widetilde{H}^s(\Omega_R)}$$

equivalent to norm (5). We note the following inequality for quadratic forms (3) and (4) (see [12, Theorem 1]): for $s \in (0,1)$ and $u \not\equiv 0$,

$$((-\Delta)^s_N u, u) > ((-\Delta)^s_D u, u). \tag{6}$$

Recall that a quadratic form for the fractional Dirichlet Laplacian can be obtained by means of the Caffarelli–Sylvestre extension [2]. Namely, for $u \in \widetilde{H}^s(\Omega_R)$, $s \in (0,1)$, the minimum of the functional

$$\mathcal{E}^D_s(w) = \int\limits_0^{+\infty} \int\limits_{\mathbb{R}^n} t^{1-2s} |\nabla w(x,t)|^2 \, dx \, dt$$

over the functional space

$$\mathfrak{W}^D = \big\{ w(x,t) \mid \mathcal{E}^D_s(w) < +\infty, \, w|_{t=0} = u \big\}$$

is achieved on a unique function $\widetilde{w}_D$ and gives the value of the square of the Dirichlet norm in $\widetilde{H}^s(\Omega)$ up to the constant $C(s) = \frac{4^s \Gamma(1+s)}{2s \cdot \Gamma(1-s)}$:

$$[u]^2_{D, \widetilde{H}^s(\Omega)} = C(s) \mathcal{E}^D_s(\widetilde{w}_D).$$

Similarly, for the fractional Navier Laplacian, the quadratic form is obtained by the Stinga–Torrea extension [20]: the minimum of the functional

$$\mathcal{E}^N_s(w) = \int\limits_0^{+\infty} \int\limits_{\Omega} t^{1-2s} |\nabla w(x,t)|^2 \, dx \, dt$$

over the functional space

$$\mathfrak{W}^N = \big\{ w(x,t) \mid \mathcal{E}^N_s(w) < +\infty, \, w|_{t=0} = u, \, w|_{x \in \partial\Omega} = 0 \big\}$$

is achieved on a single function $\widetilde{w}_N$, and the following formula is true (see, for example, [14, (2.6)])

$$[u]^2_{N, \widetilde{H}^s(\Omega)} = C(s) \mathcal{E}^N_s(\widetilde{w}_N).$$

For the spaces $\widetilde{H}^s(\Omega)$, the Sobolev inequalities hold true (see, e.g., [18, 2.8.1/15]): for $u \in \widetilde{H}^s(\Omega)$ and $s < \frac{n}{2}$,

$$[u]^2_{D, \widetilde{H}^s(\Omega)} \geq C_s \|u\|^2_{L_{2^*_n}(\Omega)} \quad \text{and} \quad [u]^2_{N, \widetilde{H}^s(\Omega)} \geq C_s \|u\|^2_{L_{2^*_n}(\Omega)} \tag{7}$$

(we recall that $2^*_n \equiv \frac{2n}{(n-2s)_+}$ stands for the critical Sobolov embedding exponent). The exact constant C_s in the inequality for the Dirichlet norm does not depend on the domain; its value was found in [5]. The equality of exact constants for the Navier and Dirichlet norms was obtained in [21] and [7] for $s = 2$, in [6] for $s \in \mathbb{N}$, and in [13] for an arbitrary s.

Inequalities (7) imply the continuity of the embedding of $\widetilde{H}^s(\Omega)$ into $L_q(\Omega)$ for the critical exponent $q = 2^*_n$, that in turn provides the compactness of the embedding for $q < 2^*_n$.

Let G be a closed subgroup of $O(n)$. Denote by $\mathfrak{L}^s_G$ the subspace of G-invariant functions in $\widetilde{H}^s(\Omega_R)$,

$$\mathfrak{L}^s_G = \Big\{ u \in \widetilde{H}^s(\Omega_R) \mid u(x) = u(gx) \, \text{for all} \;\, g \in G \Big\}.$$

The subspace of functions $L_{q,G}(\Omega_R)$ is defined similarly:

$$L_{q,G}(\Omega_R) = \{ u \in L_q(\Omega_R) \mid u(x) = u(gx) \;\; \text{for all } g \in G \}.$$

We follow to the notation introduced in [16]: an **admissible** (m,k)**-decomposition** of the space $\mathbb{R}^n$ is defined to be the decomposition $\mathbb{R}^n = (\mathbb{R}^m)^l \oplus \mathbb{R}^k$, where $l, m \in \mathbb{N}$, $k \in \mathbb{Z}_+$, and

$$ml + k = n, \quad m \geq 2, \quad k = 0 \quad \text{or} \quad k \geq m.$$

For example,

$$\mathbb{R}^7 = (\mathbb{R}^2)^2 \oplus \mathbb{R}^3, \quad \mathbb{R}^7 = (\mathbb{R}^2)^1 \oplus \mathbb{R}^5, \quad \mathbb{R}^7 = (\mathbb{R}^3)^1 \oplus \mathbb{R}^4, \quad \mathbb{R}^7 = (\mathbb{R}^7)^1$$

are admissible decompositions of $\mathbb{R}^7$. In the estimates containing the admissible (m,k)-decompositions, we denote by x the points of the space $\mathbb{R}^n$, by y the points of the space $\mathbb{R}^m$, and by z the points of the space $\mathbb{R}^k$. Thus[1], $X = (y_1, \dots, y_l, z)$. In spherical coordinates, the points are written as $x = (r_x, \theta_x)$, $y = (r_y, \theta_y)$, and $z = (r_z, \theta_z)$, Thus, $x = (r_{y_1}, \dots, r_{y_l}, r_z, \theta_{y_1}, \dots, \theta_{y_l}, \theta_z)$. A function is said to be m**-radial** if it depends only on $r_{y_1}, \dots, r_{y_l}, r_z$, and (m,k)**-radial** if it is m-radial and invariant with respect to all permutations of the vectors $y_1, \dots, y_l$. The group generating the space of all (m,k)-radial functions, is denoted by $G_{m,k}$.

3. Auxiliary statements

The **solution of the equation** (1) is by definition a weak solution $u^* \in \widetilde{H}^s(\Omega_R)$, that is

$$((-\Delta)^s_D u^*, h) \equiv \int\limits_{\mathbb{R}^n} |\xi|^{2s} Re(\mathcal{F}u^* \overline{\mathcal{F}h})\, d\xi = \int\limits_{\Omega_R} |u^*|^{q-2} u^* h\, dx \quad \text{for all } h \in \widetilde{H}^s(\Omega_R) \tag{8}$$

for the fractional Dirichlet Laplacian, and

$$((-\Delta)^s_N u^*, h) = \int\limits_{\Omega_R} |u^*|^{q-2} u^* h\, dx \quad \text{for all } h \in \widetilde{H}^s(\Omega_R) \tag{9}$$

for the fractional Navier Laplacian.

Let us define functionals $J_D(u)$ and $J_N(u)$ by the equalities

$$J_D(u) := \frac{[u]^2_{D,\widetilde{H}^s(\Omega_R)}}{\|u\|^2_{L_q(\Omega_R)}} \quad \text{and} \quad J_N(u) := \frac{[u]^2_{N,\widetilde{H}^s(\Omega_R)}}{\|u\|^2_{L_q(\Omega_R)}}.$$

Lemma 5 (see Appendix) shows that the minimizers of these functionals with respect to the subspaces $\mathfrak{L}^s_G$ for various closed subgroups $G \subset O(n)$ are positive solutions of equation (1).

In the following lemmas we study the Dirichlet norm $[v_R]_{D,\widetilde{H}^s(\omega_R)}$ for functions of one variable. In the first of them, we derive an estimate for the Dirichlet norm of the family of functions $v_R(x)$ which are defined on a line and "run away" as $R \to +\infty$.

Lemma 1. *Let $g_+(x) \in \widetilde{H}^s[0,1]$ be a function and $g_-(x) = g_+(-x)$. Let us define a family of "running away" with respect to R functions by the formula*

$$v_R(x) = g_+(x-R) + g_-(x+R). \tag{10}$$

Then as $R \to +\infty$, we have

$$[v_R]^2_{D,\widetilde{H}^s(\omega_R)} = 2[g_+]^2_{D,\widetilde{H}^s[0,1]} + o(1).$$

Proof. The following series of equalities holds true:

$$\begin{aligned}
[v_R]^2_{D,\widetilde{H}^s(\omega_R)} &= \int\limits_{\mathbb{R}} |\xi|^{2s} |\mathcal{F}v_R|^2\, d\xi = \int\limits_{\mathbb{R}} |\xi|^{2s} \left|\mathcal{F}g_+ \cdot e^{-i\xi R} + \mathcal{F}g_- \cdot e^{i\xi R}\right|^2 d\xi \\
&= \int\limits_{\mathbb{R}} |\xi|^{2s} |\mathcal{F}g_+|^2\, d\xi + \int\limits_{\mathbb{R}} |\xi|^{2s} |\mathcal{F}g_-|^2\, d\xi + \int\limits_{\mathbb{R}} |\xi|^{2s} (\mathcal{F}g_+ \overline{\mathcal{F}g_-} e^{-2i\xi R} + \overline{\mathcal{F}g_+} \mathcal{F}g_- e^{2i\xi R})\, d\xi \\
&\overset{*}{=} [g_+]^2_{D,\widetilde{H}^s[0,1]} + [g_-]^2_{D,\widetilde{H}^s[-1,0]} + o(1) = 2[g_+]^2_{D,\widetilde{H}^s[0,1]} + o(1)
\end{aligned}$$

(the equality $*$ follows from the Riemann–Lebesgue lemma). $\square$

[1]Here and below if $k = 0$, then the coordinate z is omitted.

Lemma 2. *Let $v_R(x)$ be the family from Lemma 1. Then for $a>0$ and as $R\to+\infty$, we have*

$$[v_R r^a]_{D,\widetilde{H}^s(\omega_R)} \asymp R^a [v_R]_{D,\widetilde{H}^s(\omega_R)}.$$

Proof. We need to show that there exist constants C_0 and C_1 that do not depend on R and such that

$$C_0 R^a [v_R]_{D,\widetilde{H}^s(\omega_R)} \ge [v_R r^a]_{D,\widetilde{H}^s(\omega_R)} \ge C_1 R^a [v_R]_{D,\widetilde{H}^s(\omega_R)}. \tag{11}$$

Inequality (26) (see Appendix) for $v=r^a$, implies the left-hand side of inequality (11). Next, we apply inequality (26) to the functions $v=r^{-a}$ and $u=v_R r^a$. Then

$$[v_R r^a]_{D,\widetilde{H}^s(\omega_R)} \ge \frac{[v_R]_{D,\widetilde{H}^s(\omega_R)}}{C\|r^{-a}\|_{C^m(\omega_R)}} \ge C_1 [v_R]_{D,\widetilde{H}^s(\omega_R)} R^a,$$

as required. □

4. Estimating the energy over the subspace of (m,k)-radial functions

First, we estimate the functional J_D on the subspace of radial functions. Any radial function can be identified with a function on a line, which in turn generates a family of "running away" functions by formula (10).

Theorem 1. *Let $v_R\in\widetilde{H}^s(\omega_R)$ be a family of "running away" functions on a line from Lemma 1. We reconstruct the radial function $u_R(x)\in\widetilde{H}^s(\Omega_R)$ from the function $v_R\in\widetilde{H}^s(\omega_R)$ by the formula $u_R(x)=v_R(|x|)$. Then*

$$J_D(u_R) = \frac{[u_R]^2_{D,\widetilde{H}^s(\Omega_R)}}{\|u_R\|^2_{L_q(\Omega_R)}} \asymp \frac{R^{n-1}[v_0]^2_{D,\widetilde{H}^s[0,1]}}{R^{(n-1)\frac{2}{q}}\|v_0\|^2_{L_q[0,1]}} \tag{12}$$

as $R\to+\infty$ and $q\in[2,2_1^*]$.

Proof. The Fourier image of a radial function is radial. We write the Dirichlet norm of the function u_R as follows:

$$[u_R]^2_{D,\widetilde{H}^s(\Omega_R)} = \int\limits_{\mathbb{R}^n} |\xi|^{2s}|\mathcal{F}u_R|^2\,d\xi = C\int\limits_{\mathbb{R}^n}|\xi|^{2s}\Bigg(\int\limits_R^{R+1} v_R(r)r^{n-1}\int\limits_{S^{n-1}} e^{-ir|\xi|(\sigma,\theta_\xi)}\,d\sigma\,dr\Bigg)^2 d\xi.$$

By a property of the Bessel function (see [19, Theorem IV.1.6]),

$$\int\limits_{S^{n-1}} e^{-i|y|(\sigma,\theta)}\,d\sigma = \frac{(2\pi)^{\frac{n}{2}}}{|y|^{\frac{n-2}{2}}}\mathcal{J}_{\frac{n-2}{2}}(|y|), \quad \theta\in S^{n-1},$$

the norm can be transformed to the form

$$[u_R]^2_{D,\widetilde{H}^s(\Omega_R)} = C\int\limits_{\mathbb{R}_+} t^{1+2s}\Bigg(\int\limits_R^{R+1} r^{\frac{n}{2}} v_R(r)\mathcal{J}_{\frac{n-2}{2}}(rt)\,dr\Bigg)^2 dt.$$

To estimate the right-hand side, we divide it into two integrals. Let

$$\varepsilon(R)=\tfrac{1}{\sqrt{R}}.$$

Then

$$[u_R]^2_{D,\widetilde{H}^s(\Omega_R)} = C\Bigg(\int\limits_0^{\varepsilon(R)} + \int\limits_{\varepsilon(R)}^{+\infty}\Bigg) t^{1+2s}\Bigg(\int\limits_R^{R+1} r^{\frac{n}{2}} v_R(r)\mathcal{J}_{\frac{n-2}{2}}(rt)\,dr\Bigg)^2 dt$$
$$=: I_1 + I_2.$$

Now, the integral I_1 is estimated as $o(R^{n-1})$:

$$\int\limits_0^{\varepsilon(R)} t^{1+2s}\Bigg(\int\limits_R^{R+1} r^{\frac{n}{2}} v_R(r)\mathcal{J}_{\frac{n-2}{2}}(rt)\,dr\Bigg)^2 dt \le C\varepsilon(R)^{2+2s}\Bigg(\int\limits_R^{R+1} v_R(r) r^{\frac{n}{2}}\,dr\Bigg)^2$$
$$\le CR^{n-1-s}\|v_R\|^2_{L_2(\omega_R)} \le CR^{n-1-s}\|v_0\|^2_{L_2[0,1]} = o(R^{n-1}).$$

Let us estimate the integral I_2. The Bessel function admits an expansion in the asymptotic series (as $t \to +\infty$) with the remainder $|R_N(t)| \le \frac{C}{t^{2N+\frac{1}{2}}}$ (see [22, p. 199]):

$$\mathcal{J}_{\frac{n-2}{2}}(t) = \sum_{k=0}^{N}(A_k(t) + B_k(t)) + R_N(t), \tag{13}$$

where $A_k(t) = \sqrt{\frac{2}{\pi t}}\frac{\cos(t-\frac{n-1}{4}\pi)}{t^{2k}}$ and $B_k(t) = \sqrt{\frac{2}{\pi t}}\frac{\sin(t-\frac{n-1}{4}\pi)}{t^{2k+1}}$.

It is easy to see that if $t > \varepsilon(R)$ on the support of the function $v_R(r)$, then the expression rt tends to $+\infty$ as $R \to +\infty$. Therefore the asymptotics (13) is applicable. Set

$$\mathfrak{A}_k(t) = \int\limits_R^{R+1} r^{\frac{n}{2}} v_R(r) A_k(rt)\,dr, \quad \mathfrak{B}_k(t) = \int\limits_R^{R+1} r^{\frac{n}{2}} v_R(r) B_k(rt)\,dr,$$
$$\mathfrak{R}_N(t) = \int\limits_R^{R+1} r^{\frac{n}{2}} v_R(r) R_N(rt)\,dr.$$

Thus,

$$I_2 = \int\limits_{\varepsilon(R)}^{+\infty} t^{1+2s}\left(\sum_{k=0}^{N}(\mathfrak{A}_k(t) + \mathfrak{B}_k(t)) + \mathfrak{R}_N(t)\right)^2 dt.$$

As a first approximation to I_2, we use the energy obtained from $\mathfrak{A}_0(t)$:

$$\int\limits_{\varepsilon(R)}^{+\infty} \mathfrak{A}_0^2(t) t^{1+2s} dt = C\int\limits_{\varepsilon(R)}^{+\infty} t^{1+2s}\Bigg(\int\limits_R^{R+1} r^{\frac{n}{2}} v_R(r)\sqrt{\tfrac{2}{\pi rt}}\cos(rt - \tfrac{n-1}{4}\pi)\,dr\Bigg)^2 dt$$
$$= C\int\limits_{\varepsilon(R)}^{+\infty} t^{2s}\Bigg(\int\limits_R^{R+1} r^{\frac{n-1}{2}} v_R(r)\cos(rt - \tfrac{n-1}{4}\pi)\,dr\Bigg)^2 dt.$$

We make the change of variable, $t_1 = t + \frac{(n-1)\pi}{4r}$. Since $t_1 \asymp t$ for $t > \varepsilon(R)$, we have

$$\int\limits_{\varepsilon(R)}^{+\infty} \mathfrak{A}_0^2(t) t^{1+2s}\,dt \asymp \int\limits_{\varepsilon(R)-\frac{n-1}{4R}\pi}^{+\infty} t_1^{2s} \left(\int\limits_R^{R+1} r^{\frac{n-1}{2}} v_R(r)\cos(rt_1)\,dr \right)^2 dt_1$$
$$\asymp \int\limits_{-\infty}^{+\infty} |t_1|^{2s} \left(\int\limits_R^{R+1} r^{\frac{n-1}{2}} v_R(r)\cos(rt_1)\,dr \right)^2 dt_1 + o(R^{n-1}). \tag{14}$$

From equivalence (14) and Lemmas 1 and 2, it follows that

$$\int\limits_{\varepsilon(R)}^{+\infty} \mathfrak{A}_0^2(t) t^{1+2s}\,dt \asymp [r^{\frac{n-1}{2}} v_R]^2_{D,\widetilde{H}^s(\omega_R)} \asymp R^{n-1}[v_R]^2_{D,\widetilde{H}^s(\omega_R)} \asymp R^{n-1}[v_0]^2_{D,\widetilde{H}^s[0,1]}.$$

Similarly,[2] one can estimate the energy associated with $\mathfrak{A}_k$ and $\mathfrak{B}_k$ for $k \le N = \lceil s+1 \rceil$. The asymptotics of these terms are powers of R with smaller exponents, i.e., $o(R^{n-1})$. Finally, the estimate $R_N(t)$ allows us to estimate the term with $\mathfrak{R}_N$:

$$\int\limits_{\varepsilon(R)}^{+\infty} \mathfrak{R}_N^2(t) t^{1+2s}\,dt \le C \int\limits_{\varepsilon(R)}^{+\infty} t^{-2} \left(\int\limits_R^{R+1} r^{\frac{n-1}{2}-\lceil s+1 \rceil} |v_R(r)|\,dr \right)^2 dt$$
$$\le C R^{n-1-2s-2+\frac{1}{2}} \left(\int\limits_0^1 |v_0(r)|\,dr \right)^2 = o(R^{n-1}) \|v_0\|^2_{L_2[0,1]}.$$

The equivalence $I_2 \asymp R^{n-1}[v_0]^2_{D,\widetilde{H}^s[0,1]}$ follows from the equivalence

$$I_2 \asymp \int\limits_{\varepsilon(R)}^{+\infty} t^{1+2s} \left(\sum_{k=0}^{N} \left(\mathfrak{A}_k^2(t) + \mathfrak{B}_k^2(t) \right) + \mathfrak{R}_N^2(t) \right) dt. \qquad \square$$

Corollary 1. *The minimum of the functional J_D over the subspace of radial functions is equivalent to $R^{(n-1)\left(1-\frac{2}{q}\right)}$,*

$$\min_{u_R \in \mathfrak{L}^s_{O(n)}} J_D(u_R) \asymp R^{(n-1)\left(1-\frac{2}{q}\right)} \tag{15}$$

as $R \to +\infty$ and $q \in [2, 2_1^*]$.

Proof. The upper bound in (15) obviously follows from the equivalence (12). The lower bound follows from (12) and the boundedness of the embedding operator $\widetilde{H}^s[0,1] \hookrightarrow L_q[0,1]$. $\square$

To study the behavior of energy on the subspaces $\mathfrak{L}^s_G$, a two-sided estimate is required. The following theorem gives a lower bound for (m,k)-radial functions.

Theorem 2. *Let $u_R(x) \in \widetilde{H}^s(\Omega_R)$ be an (m,k)-radial function, and also $m \ne n$. Then for $q \in [2, 2^*_{n-m+1}]$, the following inequalities hold:*

$$[u_R]^2_{D,\widetilde{H}^s(\Omega_R)} \ge C R^{(m-1)(1-\frac{2}{q})} \|u_R\|^2_{L_q(\Omega_R)},$$
$$[u_R]^2_{N,\widetilde{H}^s(\Omega_R)} \ge C R^{(m-1)(1-\frac{2}{q})} \|u_R\|^2_{L_q(\Omega_R)}. \tag{16}$$

[2]By the reduction formula, $\sin(r\rho - \frac{n-1}{4}\pi) = \cos(r\rho - \frac{n+1}{4}\pi)$.

Proof. Let T_0 be the identity operator on the space $L_{2,G_{m,k}}$:

$$T_0 : L_{2,G_{m,k}}(\Omega_R) \to L_{2,G_{m,k}}(\Omega_R).$$

The norm of this operator is equal to one. Let T_1 be the operator embedding the space $\mathfrak{L}^1_{G_{m,k}}$ into the space $L_{p,G_{m,k}}(\Omega_R)$ for $p \in [2, \frac{2(n-m+1)}{(n-m-1)_+}]$,

$$T_1 : \mathfrak{L}^1_{G_{m,k}} \to L_{p,G_{m,k}}(\Omega_R).$$

According to papers [11] for $m = 2$ and $k = n-2$, and [16] for arbitrary (m,k)-expansions, there exists a constant C_0 such that for any $v \in \mathfrak{L}^1_{G_{m,k}}$,

$$[v]^2_{\widetilde{H}^1(\Omega_R)} \geq C_0 R^{(m-1)(1-\frac{2}{p})} \|v\|^2_{L_p(\Omega_R)}.$$

Thus, the operator T_1 is continuous and has an estimate for the norm,

$$\|T_1\| = \sup_{v \in \mathfrak{L}^1_{G_{m,k}}} \frac{\|T_1 v\|_{L_p(\Omega_R)}}{\|v\|_{\mathfrak{L}^1_{G_{m,k}}}} \leq C_0^{-\frac{1}{2}} R^{(m-1)(\frac{1}{p}-\frac{1}{2})}.$$

Equality (27) from Lemma 6 describes the spaces $\widetilde{H}^s(\Omega_R)$ as an interpolation scale,

$$[\widetilde{H}^k(\Omega_R), \widetilde{H}^{k+1}(\Omega_R)]_\delta = \widetilde{H}^{k+\delta}(\Omega_R).$$

As is well known, the Lebesgue spaces $L_p(\Omega_R)$ also form an interpolation scale: for $\frac{1}{p} = \frac{1-\delta}{p_0} + \frac{\delta}{p_1}$,

$$[L_{p_0}(\Omega_R), L_{p_1}(\Omega_R)]_\delta = L_p(\Omega_R).$$

Using averaging over a group with Haar measure, one can construct projections into the spaces of $G_{m,k}$-invariant functions. Actually, we decompose the function h into the sum of the functions h_1 and h_2 as follows (denote by $\mathfrak{G}(y)$ the orbit of the point y with respect to the action of the group G; this group has the Haar measure μ_y invariant with respect to the action of the group):

$$h_1(y) = \frac{1}{\mu_y(\mathfrak{G}(y))} \int\limits_{\mathfrak{G}(y)} h d\mu_y, \quad h_2(y) = h(y) - h_1(y), \quad \int\limits_{\mathfrak{G}(y)} h_2 d\mu_y = 0. \tag{17}$$

It is easy to see that the function h_1 is $G_{m,k}$-invariant, and formulas (17) define continuous projections from the spaces $L_{p_0}(\Omega_R)$ and $L_{p_1}(\Omega_R)$ into the spaces $L_{p_0,G_{m,k}}(\Omega_R)$ and $L_{p_1,G_{m,k}}(\Omega_R)$, respectively. The subspace $L_{p_0,G_{m,k}}(\Omega_R)$ is complementable. Therefore, by virtue of [18, Theorem 1.17.1.1],

$$[L_{p_0,G_{m,k}}(\Omega_R), L_{p_1,G_{m,k}}(\Omega_R)]_\delta = L_{p,G_{m,k}}(\Omega_R).$$

Similarly, the formula (17) defines a continuous projector from $L_2(\Omega_R)$ into $L_{2,G_{m,k}}(\Omega_R)$ (also continuous as a projector from $\widetilde{H}^1(\Omega_R)$ to $\mathfrak{L}^1_{G_{m,k}}$); the space is complementable and

$$[L_{2,G_{m,k}}(\Omega_R), \mathfrak{L}^1_{G_{m,k}}]_\delta = \mathfrak{L}^\delta_{G_{m,k}}.$$

Thus, we can interpolate the embedding operator between the operators T_0 and T_1. The resulting operator is denoted by T_s,

$$T_s : \mathfrak{L}^s_{G_{m,k}} \to L_{q,G_{m,k}}(\Omega_R) \quad \text{for} \quad \frac{1}{q} = \frac{s}{p} + \frac{1-s}{2}. \tag{18}$$

The norm of this operator is estimated with the help of the interpolation inequality,

$$\|T_s\| \leq \|T_0\|^{1-s} \|T_1\|^s \leq C_0^{-\frac{s}{2}} R^{(m-1)(\frac{s}{p}-\frac{s}{2})} = C_0^{-\frac{s}{2}} R^{(m-1)(\frac{1}{q}-\frac{1}{2})}. \tag{19}$$

For $p \in [2, \frac{2(n-m+1)}{(n-m-1)_+}]$, the exponent q runs over the interval $[2, 2^*_{n-m+1}]$, and inequality (19) gives an estimate for the interpolation norm (which coincides with the standard norm in $H^s(\mathbb{R}^n)$),

$$\|v\|^2_{\widetilde{H}^s(\Omega_R)} \geq C_0^s R^{(m-1)(1-\frac{2}{q})} \|v\|^2_{L_q(\Omega_R)}.$$

Now from the Friedrichs inequality (see Lemma 3, Appendix), we obtain inequality (16) for the Dirichlet norm,

$$2[v]^2_{D,\widetilde{H}^s(\Omega_R)} \geq \|v\|^2_{\widetilde{H}^s(\Omega_R)} \geq CR^{(m-1)(1-\frac{2}{q})} \|v\|^2_{L_q(\Omega_R)}.$$

Inequality (16) for the Navier norm follows from estimate (6),

$$[v]^2_{N,\widetilde{H}^s(\Omega_R)} \geq [v]^2_{D,\widetilde{H}^s(\Omega_R)} \geq CR^{(m-1)(1-\frac{2}{q})} \|v\|^2_{L_q(\Omega_R)}. \qquad \square$$

Remark 1. The condition $m \neq n$ is essentially used in the proof: for $m = n$, the limit exponent q equals $2^*_1 = \frac{2}{1-2s}$, and even in the case $s < \frac{1}{2}$ it cannot be obtained from the interpolation in the Lebesgue spaces $L_p(\Omega_R)$; equality (18) provides the exponents $q \leq \frac{2}{1-s}$, which is less than 2^*_1. However, Theorem 1 shows, that the assertion of the theorem is also true in this case.

The following theorem shows that the estimate from Theorem 2 is sharp.

Theorem 3. *For any R and $q \in [2, 2^*_{n-m+1}]$, there exists an (m,k)-radial function $\widetilde{u}_R$ such that*

$$\begin{aligned} [\widetilde{u}_R]^2_{D,\widetilde{H}^s(\Omega_R)} &\leq CR^{(m-1)(1-\frac{2}{q})} \|\widetilde{u}_R\|^2_{L_q(\Omega_R)}, \\ [\widetilde{u}_R]^2_{N,\widetilde{H}^s(\Omega_R)} &\leq CR^{(m-1)(1-\frac{2}{q})} \|\widetilde{u}_R\|^2_{L_q(\Omega_R)}. \end{aligned} \tag{20}$$

Proof. In accordance with [11] (for $m = 2$, $k = n-2$, and for $m = n$) and [16] (for arbitrary (m,k)-expansions), there exists $\widetilde{u}_R \in \mathfrak{L}^1_{G_{m,k}}$ such that

$$[\widetilde{u}_R]^2_{\widetilde{H}^1(\Omega_R)} \leq CR^{(m-1)(1-\frac{2}{q})} \|\widetilde{u}_R\|^2_{L_q(\Omega_R)}.$$

Note that $[2, 2^*_{n-m+1}] \subset [2, \frac{2(n-m+1)}{(n-m-1)_+}]$. Therefore Lemma 4 (see Appendix) gives the required estimate for the Navier and Dirichlet norms:

$$[\widetilde{u}_R]^2_{\widetilde{H}^s(\Omega_R)} \leq [\widetilde{u}_R]^2_{\widetilde{H}^1(\Omega_R)} \leq CR^{(m-1)(1-\frac{2}{q})} \|\widetilde{u}_R\|^2_{L_q(\Omega_R)}. \qquad \square$$

To prove the multiplicity of solutions, it is required to estimate the energy in the spaces $\mathfrak{L}^s_{O(n-2)\times O(2)}$.

Corollary 2. *Let $n \geq 4$. Then the minima J_D and J_N with respect to the subspace $\mathfrak{L}^s_{O(n-2)\times O(2)}$ are equivalent to $R^{1-\frac{2}{q}}$,*

$$\min_{u_R \in \mathfrak{L}^s_{O(n-2)\times O(2)}} J_D(u_R) \asymp R^{1-\frac{2}{q}} \quad \textit{and} \quad \min_{u_R \in \mathfrak{L}^s_{O(n-2)\times O(2)}} J_N(u_R) \asymp R^{1-\frac{2}{q}}. \tag{21}$$

Proof. For $n \geq 4$, the function of the spaces $\mathfrak{L}^s_{O(n-2)\times O(2)}$ are $(2, n-2)$-radial, and the estimates follow from inequalities (16) and (20). $\square$

5. The theorems on existence and multiplicity

Theorem 1 gives a two-sided estimate for the Dirichlet norm of a radial function in $\widetilde{H}^s(\Omega_R)$ in terms of the restriction norm in the space $\widetilde{H}^s(\omega_R)$. This means that for the subspace $\mathfrak{L}^s_{O(n)}$ with the Dirichlet norm, the compactness of the embedding holds for $q \in [1, 2^*_1)$. Also, in view of the fact that the Dirichlet and Navier norms are equivalent, the compactness of the embedding for $q \in [1, 2^*_1)$ is also valid for the subspace $\mathfrak{L}^s_{O(n)}$ with the Navier norm. Using Lemma 5 (see Appendix), we get the following theorem.

Theorem 4 (**Existence of radial solutions**). *For $q \in [1, 2^*_1)$, $q \neq 2$, there exist positive radial solutions to problems* (1) *with fractional Dirichlet and Navier Laplacians.*

Let $n \geq 4$. We consider an admissible (m, k)-expansion. Theorem 2 provides an embedding $\mathfrak{L}^s_{G_{m,k}}(\Omega_R) \hookrightarrow L_q(\Omega_R)$ for the Dirichlet norm with $q = 2^*_{n-m+1}$. This means that it holds and is compact for $q \in [1, 2^*_{n-m+1})$. For the Navier norm, the embedding is valid by virtue of the equivalence of norms. Lemma 5 shows the existence of a generalized solution $u_R \in \mathfrak{L}^s_{G_{m,k}}$. For $q > 2$ and large R, the minima of the functional J_D with respect to the subspaces $\mathfrak{L}^s_{O(n)}$ and $\mathfrak{L}^s_{G_{m,k}}$ are different because of estimates(15), (16), and (20); a similar statement for the functional J_N is obtained by using inequality (6). Thus the solution is not radial, and the following theorem holds.

Theorem 5 (**Existence of a non-radial solutions at exponents** $q \geq 2^*_n$). *For $n \geq 4$ and $q \in (2, 2^*_{n-m+1})$, there is exist a radius R_0 such that for $R > R_0$, there are positive (m, k)-radial solutions in Ω_R (for different m, the solutions are different) to problems* (1) *with Dirichlet and Navier fractional Laplacians.*

Remark 2. The maximum exponent is obtained for the largest admissible m, i.e., for $m = \lfloor \frac{n}{2} \rfloor$.

Theorem 6 (**Multiplicity of solutions for** $n \neq 3$). *Let $n \neq 3$, $s \in (0, 1)$, $q \in (2, 2^*_n)$, and let N be a positive integer. Then there exists $R_1(N)$ such that for any $R \geq R_1$ there exist at least N positive solutions to problems* (1) *with Dirichlet and Navier fractional Laplacians which cannot be obtained by rotation from each other.*

Proof. Let us consider a family of groups

$$T_\ell \times O(n-2), \quad \ell = 1, 2, 3 \ldots, N,$$

where T_ℓ is the group of all rotations by an angle multiple of $\frac{2\pi}{\ell}$. The minimizers with respect to the invariant subspaces $\mathfrak{L}^s_{T_\ell \times O(n-2)}$ are weak solutions to problem (1) for $q < 2^*_n$.

Let us consider a positive function $\phi(x) \in \mathcal{C}^\infty_0(B_{\frac{1}{2}}(\mathbb{O}_n))$ satisfying the equality

$$\phi(x) = \phi(y, z) = \phi(|y|, |z|), \quad y \in \mathbb{R}^2, \;\; z \in \mathbb{R}^{n-2}, \quad |y| + |z| \leq \tfrac{1}{2}.$$

Denote by y^0_i the vertices of a regular ℓ-gon in the plane with center at the origin and $y^0_1 = (R + \frac{1}{2}, 0)$. Define a function u_ℓ by the equality

$$u_\ell(y, z) = \sum_{i=1}^{\ell} \phi\,(y - y^0_i, z).$$

Lemma 7 ensures that the values $J_D(u_\ell)$ and $J_N(u_\ell)$ are uniformly bounded for large R. From estimates (21), it follows that there exists a level R_1 such that the minima of the functionals J_D and J_N over $\mathfrak{L}^s_{O(2)\times O(n-2)}$ for $R > R_1$ are greater than the constant found above. It remains to show that the minimizers with respect to $\mathfrak{L}^s_{T_\ell \times O(n-2)}$, $\ell = 1, 2, 3, \ldots, N$, are pairwise distinct.

The invariance of the function $u(x)$ with respect to the action of the group $T_\ell \times O(n-2)$ is transferred to the minimizer: if $\widetilde{w}(x,t)$ is the Caffarelli–Silvestre (respectively, the Stinga–Torrea) extension[3] for the function $u(x)$, then $\widetilde{w}(gx,t)$ is the C-S (respectively, S-T) extension for the function $u(gx)=u(x)$ for all $g \in T_\ell$. These are extensions of the same function, and by virtue of the uniqueness they coincide,

$$\widetilde{w}(x,t) = \widetilde{w}(gx,t) \quad \text{for all } \ g \in T_\ell \times O(n-2).$$

Thus, $\mathcal{E}(w)$ can be minimized over the subspace of $T_\ell \times O(n-2)$-invariant functions in $\mathfrak{W}$. Let $\ell_1, \ell_2 \in [1:N]$, $\ell_1 > \ell_2$. We consider two cases.

The first case (ℓ_1 is divisible by ℓ_2). Let u_{ℓ_1} and u_{ℓ_2} be minimizers over $\mathfrak{L}^s_{T_{\ell_1}\times O(n-2)}$ and $\mathfrak{L}^s_{T_{\ell_2}\times O(n-2)}$ with the unit norm in $L_q(\Omega)$; they correspond to the extensions w_{ℓ_1} and w_{ℓ_2}. Consider the function $v = u_{\ell_1}(r_y, \frac{\ell_2}{\ell_1}\theta_y, z)$. Obviously, $v \in \mathfrak{L}^s_{T_{\ell_2}\times O(n-2)}$, $\|v\|_{L_q(\Omega)} = 1$, and the extension for v satisfies the equality $w = w_{\ell_1}(r_y, \frac{\ell_2}{\ell_1}\theta_y, z, t)$. It is easy to see that

$$\begin{aligned}
[u_{\ell_2}]^2_{\widetilde{H}^s(\Omega)} &\le [v]^2_{\widetilde{H}^s(\Omega)} = C(s)\mathcal{E}(w) \\
&= C(s)|S^{n-3}| \int_0^{+\infty}\int_0^{+\infty}\int_0^{2\pi}\int_0^{+\infty} t^{1-2s} r_y |z|^{n-3} \big(w^2_{r_y} + \tfrac{1}{r_y^2} w^2_{\theta_y} + w_z^2 + w_t^2\big)\, dz d\theta_y\, dr_y\, dt \\
&= C(s)|S^{n-3}| \int_0^{+\infty}\int_0^{+\infty}\int_0^{2\pi}\int_0^{+\infty} t^{1-2s} r_y |z|^{n-3} \\
&\qquad \times \big((w_{\ell_1})^2_{r_y} + \tfrac{1}{r_y^2}\tfrac{\ell_2^2}{\ell_1^2}(w_{\ell_1})^2_{\theta_y} + (w_{\ell_1})^2_z + (w_{\ell_1})^2_t\big)\, dz d\theta_y\, dr_y\, dt \\
&< C(s)|S^{n-3}| \int_0^{+\infty}\int_0^{+\infty}\int_0^{2\pi}\int_0^{+\infty} t^{1-2s} r_y |z|^{n-3} \\
&\qquad \times \big((w_{\ell_1})^2_{r_y} + \tfrac{1}{r_y^2}(w_{\ell_1})^2_{\theta_y} + (w_{\ell_1})^2_z + (w_{\ell_1})^2_t\big)\, dz d\theta_y\, dr_y\, dt \\
&= C(s)\mathcal{E}(w_{\ell_1}) = [u_{\ell_1}]^2_{\widetilde{H}^s(\Omega)},
\end{aligned}$$

the strict inequality follows from the fact that for $R \ge R_1$, the function u_{ℓ_1} does not belong to $\mathfrak{L}^s_{O(2)\times O(n-2)}$. Thus the energy value of u_{ℓ_1} is strictly greater than that of u_{ℓ_2}.

The second case (ℓ_1 is not divisible by ℓ_2). If the minimizers over $\mathfrak{L}^s_{T_{\ell_1}\times O(n-2)}$ and $\mathfrak{L}^s_{T_{\ell_2}\times O(n-2)}$ are equal, then this minimizer belongs to $\mathfrak{L}^s_{T_{\mathrm{LCM}(\ell_1,\ell_2)}\times O(n-2)}$. Applying the first case to the numbers ℓ_1 and $\mathrm{LCM}(\ell_1,\ell_2)$, we obtain the required result. $\square$

Remark 3. For the Laplace operator and p-Laplacian, Theorem 6 is true in the case $n=3$, as indicated in the Introduction. The proofs of these assertions known to the author require more advanced methods using the concentration of solutions. Therefore for fractional Laplacians, the question of the existence of such solutions remains open.

Appendix

Lemma 3 (Friedrichs inequalities). *For any function*

$$u \in \widetilde{H}^s(\Omega_R) \quad \textit{for} \quad s \in (0,1),$$

[3]For brevity, C-S and S-T, respectively.

the following inequalities hold:

$$((-\Delta)^s_D u, u) \geq \|u\|^2_{L_2(\Omega_R)} \quad \textit{and} \quad ((-\Delta)^s_N u, u) \geq \|u\|^2_{L_2(\Omega_R)}. \tag{22}$$

Proof. The inequality for the Navier norm can be obtained directly from the definition of the fractional Navier Laplacian,

$$((-\Delta)^s_N u, u) = \sum_{j=1}^{\infty} \lambda_j^s (u, \phi_j)^2 \geq \lambda_1^s \|u\|^2_{L_2(\Omega_R)},$$

where $\lambda_1 > 1$ by the Friedrichs inequality in the domain of width 1 for $u \in \widetilde{H}^1(\Omega_R)$:

$$\|\nabla u\|^2_{L_2(\Omega_R)} \geq \|u\|^2_{L_2(\Omega_R)}. \tag{23}$$

The inequality for the Dirichlet norm is sufficient to prove for $u \in C_0^\infty(\Omega_R)$; the initial inequality is obtained by closure with respect to the norm of the space $H^s(\mathbb{R}^n)$. We define a family of norms in the space $\widetilde{H}^s(\Omega_R)$ that are indexed by the parameter ε, equivalent to the norm in the space $H^s(\mathbb{R}^n)$, and given by the formula

$$\|u\|^2_{\widetilde{H}^s(\Omega_R)} \equiv \int\limits_{\mathbb{R}^n} (\varepsilon + |\xi|)^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi.$$

We consider the embedding operator

$$A : \widetilde{H}^s(\Omega_R) \hookrightarrow L_2(\Omega_R).$$

The adjoint operator acts as

$$A^* : L_2(\Omega_R) \to (\widetilde{H}^s(\Omega_R))'.$$

The norms of A and A^* are the same. The induced norm in the dual simple space is given by the formula

$$\|v\|^2_{(\widetilde{H}^s(\Omega_R))'} \equiv \int\limits_{\mathbb{R}^n} (\varepsilon + |\xi|)^{-2s} |\mathcal{F}v(\xi)|^2 \, d\xi.$$

By the Hölder inequality, we have

$$\begin{aligned} \|v\|^2_{(\widetilde{H}^s(\Omega_R))'} &\equiv \int\limits_{\mathbb{R}^n} (\varepsilon + |\xi|)^{-2s} |\mathcal{F}v(\xi)|^2 \, d\xi \\ &\leq \left(\int\limits_{\mathbb{R}^n} (\varepsilon + |\xi|)^{-2} |\mathcal{F}v(\xi)|^2 \, d\xi \right)^s \left(\int\limits_{\mathbb{R}^n} |\mathcal{F}v(\xi)|^2 \, d\xi \right)^{1-s} = \|v\|^{2s}_{(\widetilde{H}^1(\Omega_R))'} \|v\|^{2-2s}_{L_2(\Omega_R)}. \end{aligned} \tag{24}$$

Using estimate (24) and the Friedrichs inequality (23), we obtain a chain of inequalities:

$$\begin{aligned} \|A\| = \sup \frac{\|u\|_{L_2(\Omega_R)}}{\|u\|_{\widetilde{H}^s(\Omega_R)}} = \sup \frac{\|v\|_{(\widetilde{H}^s(\Omega_R))'}}{\|v\|_{L_2(\Omega_R)}} \leq \sup \frac{\|v\|^s_{(\widetilde{H}^1(\Omega_R))'} \|v\|^{1-s}_{L_2(\Omega_R)}}{\|v\|_{L_2(\Omega_R)}} \\ \leq \sup \left(\frac{\|v\|_{(\widetilde{H}^1(\Omega_R))'}}{\|v\|_{L_2(\Omega_R)}} \right)^s = \sup \left(\frac{\|u\|_{L_2(\Omega_R)}}{\|u\|_{\widetilde{H}^1(\Omega_R)}} \right)^s \leq 1. \end{aligned}$$

It follows that given $\varepsilon > 0$,

$$\int\limits_{\mathbb{R}^n} (\varepsilon + |\xi|)^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi \geq \|u\|^2_{L_2(\Omega_R)}.$$

Now inequality (3) is obtained by passing to the limit as $\varepsilon \to 0$. □

Lemma 4. *For any function $u \in \widetilde{H}^1(\Omega_R)$, the following inequalities*

$$[u]_{D,\widetilde{H}^s(\Omega_R)} \le [u]_{D,\widetilde{H}^1(\Omega_R)} \quad \textit{and} \quad [u]_{N,\widetilde{H}^s(\Omega_R)} \le [u]_{N,\widetilde{H}^1(\Omega_R)}$$

hold for $s \in (0,1)$.

Proof. First, we prove the statement for the Dirichlet norm. In view of the Hölder inequality and Friedrichs inequality (22), we have

$$\begin{aligned}
[u]^2_{D,\widetilde{H}^s(\Omega_R)} &= \int\limits_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi \le \left(\int\limits_{\mathbb{R}^n} |\mathcal{F}u(\xi)|^2 \, d\xi \right)^{1-s} \left(\int\limits_{\mathbb{R}^n} |\xi|^2 |\mathcal{F}u(\xi)|^2 \, d\xi \right)^{s} \\
&= [u]^{2-2s}_{D,L_2(\Omega_R)} [u]^{2s}_{D,\widetilde{H}^1(\Omega_R)} \le [u]^{2-2s}_{D,\widetilde{H}^1(\Omega_R)} [u]^{2s}_{D,\widetilde{H}^1(\Omega_R)} = [u]^2_{D,\widetilde{H}^1(\Omega_R)}.
\end{aligned}$$

The statement for the Navier norm is also obtained from the Hölder inequality and Friedrichs inequality (22),

$$\begin{aligned}
[u]^2_{N,\widetilde{H}^s(\Omega_R)} &= \sum_{j=1}^{\infty} \lambda_j^s (u,\phi_j)^2 \le \left(\sum_{j=1}^{\infty} (u,\phi_j)^2 \right)^{1-s} \left(\sum_{j=1}^{\infty} \lambda_j (u,\phi_j)^2 \right)^{s} \\
&= [u]^{2-2s}_{N,L_2(\Omega_R)} [u]^{2s}_{N,\widetilde{H}^1(\Omega_R)} \le [u]^{2-2s}_{N,\widetilde{H}^1(\Omega_R)} [u]^{2s}_{N,\widetilde{H}^1(\Omega_R)} = [u]^2_{N,\widetilde{H}^1(\Omega_R)}.
\end{aligned}$$ □

Lemma 5. *Let the embedding $\mathfrak{L}^s_G \hookrightarrow L_q(\Omega_R)$ be compact. Then the minimizers of the functionals $J_D(u)$ and $J_N(u)$ over the space $\mathfrak{L}^s_G$ exist and are positive solutions to problem* (1) *with fractional Navier and Dirichlet Laplacians.*

Proof. By virtue of the homogeneity of the functionals $J_D(u)$ and $J_N(u)$, their denominators can be regarded as unit ones. Thus the problem is reduced to minimizing the norms $W_D(u) = [u]^2_{D,\widetilde{H}^s(\Omega_R)}$ and $W_N(u) = [u]^2_{N,\widetilde{H}^s(\Omega_R)}$ over the level surface $V(u) = \|u\|^q_{L_q(\Omega_R)} = 1$, which, due to the compact embedding, is weakly closed. The existence of the minimizers follows from the existence theorem for minimizer of a weakly lower semicontinuous coercive functional on a weakly closed set (see [10, Theorem 26.8]). After multiplication by suitable constants, the Euler equations turn into identities (8) and (9) for generalized solutions on increments of $h \in \mathfrak{L}^s_G$:

$$\exists \lambda_1, \lambda_2 \ \forall h \in \mathfrak{L}^s_G : DV(u^*_D)h = \lambda_1 DW_D(u^*_D)h, \quad DV(u^*_N)h = \lambda_2 DW_N(u^*_N)h. \tag{25}$$

We make use of **the principle of symmetric criticality**, see [17, Theorem 1.1]: since both functionals are invariant with respect to the action of a compact closed Lie group G, equalities (25) for the increments $h \in \mathfrak{L}^s_G$ imply analogous equalities for variations $h \in \widetilde{H}^s(\Omega_R)$.

To complete the proof, it remains to show the positivity of the minimizers. Their nonnegativity is ensured by the following statement.

Proposition [13, Theorem 3]. Let $u(x) \in \widetilde{H}^s(\Omega)$, where $s \in (0,1)$. Then the function $|u(x)|$ belongs to the space $\widetilde{H}^s(\Omega)$, and

$$[u]_{D,\widetilde{H}^s(\Omega)} \ge [|u|]_{D,\widetilde{H}^s(\Omega)} \quad \text{and} \quad [u]_{N,\widetilde{H}^s(\Omega)} \ge [|u|]_{N,\widetilde{H}^s(\Omega)}.$$

Furthermore, if the positive and negative parts of the function $u(x)$ are not degenerate, then the inequalities are strict.

The positivity of the minimizers follows from the nonnegativity using the strong maximum principle.

Proposition ([3, Lemma 2.6], [8, Theorem 2.5]). Let a function $u(x) \in \widetilde{H}^s(\Omega) \setminus \{0\}$ satisfy the inequality $(-\Delta)^s u \ge 0$ for fractional Dirichlet or Navier Laplacians. Then $u > 0$ on any compact subset $K \subset \Omega$. □

Remark 4. For $q < 2_n^*$, the conditions of Lemma 5 are satisfied for every closed subgroup $G \subset O(n)$ because of the compactness of the embedding

$$\widetilde{H}^s(\Omega_R) \hookrightarrow L_q(\Omega_R).$$

Lemma 6. *Let Ω be a bounded domain in $\mathbb{R}^n$, $m \in \mathbb{Z}_+$, and $s = m + \delta \in [m, m+1]$. Then for the functions $u \in \widetilde{H}^s(\Omega)$ and $v \in C^{m+1}(\overline{\Omega})$, we have $uv \in \widetilde{H}^s(\Omega)$ and*

$$[uv]_{D,\widetilde{H}^s(\Omega)} \le C[u]_{D,\widetilde{H}^s(\Omega)} \|v\|^{1-\delta}_{C^m(\overline{\Omega})} \|v\|^{\delta}_{C^{m+1}(\overline{\Omega})}. \tag{26}$$

Proof. For the integer $s = m$, the required statement follows from the obvious inequality

$$\sum_{|\alpha|=m} \|D^\alpha(uv)\|_{L_2(\Omega)} \le C \sum_{|\alpha|=m} \|D^\alpha u\|_{L_2(\Omega)} \|v\|_{C^m(\overline{\Omega})}.$$

The statement for $\delta > 0$ is obtained by interpolation: by the above, we have (in the case $m = 0$, the space $\widetilde{H}^0(\Omega)$ should be understood as $L_2(\Omega)$):

$$[uv]_{D,\widetilde{H}^m(\Omega)} \le C[u]_{D,\widetilde{H}^m(\Omega)} \|v\|_{C^m(\overline{\Omega})},$$
$$[uv]_{D,\widetilde{H}^{m+1}(\Omega)} \le C[u]_{D,\widetilde{H}^{m+1}(\Omega)} \|v\|_{C^{m+1}(\overline{\Omega})}.$$

Therefore the multiplier operator of the function v is continuous in the spaces $\widetilde{H}^m(\Omega)$ and $\widetilde{H}^{m+1}(\Omega)$. In accordance with [18, Theorem 4.3.2/2],

$$[\widetilde{H}^m(\Omega), \widetilde{H}^{m+1}(\Omega)]_\delta = \widetilde{H}^{m+\delta}(\Omega). \tag{27}$$

This implies the continuity of the multiplier operator of the function v in the space $\widetilde{H}^{m+\delta}(\Omega)$, and the interpolation inequality coincides with required estimate (26). ☐

Remark 5. For $s \in [0, 1]$, the conclusion of Lemma 6 holds also for the Navier norms.

Lemma 7. *Let $u_i(x) \in \widetilde{H}^s(\Omega)$, $i = 1, \dots, k$. Denote by $U(x)$ the sum*

$$U(x) = u_1(x) + \dots + u_k(x).$$

Then

$$[U]^2_{D,\widetilde{H}^s(\Omega)} \le k \sum_{i=1}^k [u_i]^2_{D,\widetilde{H}^s(\Omega)} \quad \textit{and} \quad [U]^2_{N,\widetilde{H}^s(\Omega)} \le k \sum_{i=1}^k [u_i]^2_{N,\widetilde{H}^s(\Omega)}.$$

Proof. It is an obvious consequence of the inequality on the arithmetic mean and the quadratic mean. ☐

It is a pleasant duty of the author to thank A. I. Nazarov for precise comments, patience, and his help in editing the text of the paper.

The research was supported by the RFBR grant No. 17-01-00678A.

Translated by I. Ponomarenko.

REFERENCES

1. J. Byeon, "Existence of many nonequivalent nonradial positive solutions of semilinear elliptic equations on three-dimensional annuli," *J. Diff. Eqs.*, **136**, No. 1, 136–165 (1997).
2. L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian," *Comm. PDE's*, **32**, No. 7–9, 1245–1260 (2007).
3. A. Capella, J. Dávila, L. Dupaigne, and Y. Sire, "Regularity of radial extremal solutions for some non-local semilinear equations," *Comm. PDE's*, **36**, No. 8, 1353–1384 (2011).

4. C. V. Coffman, “A non-linear boundary-value problem with many positive solutions,” *J. Diff. Eqs.*, **54**, No. 3, 429–437 (1984).
5. A. Cotsiolis and N. K. Tavoularis, “Best constants for Sobolev inequalities for higher order fractional derivatives,” *J. Math. Anal. Appl.*, **295**, No. 1, 225–236 (2004).
6. F. Gazzola, H.-C. Grunau, and G. Sweers, “Optimal Sobolev and Hardy–Rellich constants under Navier boundary conditions,” *Ann. Mat. Pura Appl. (4)*, **189**, No. 3, 475–486 (2010).
7. Y. Ge, “Sharp Sobolev inequalities in critical dimensions,” *Michigan Math. J.*, **51**, No. 1, 27–45 (2003).
8. A. Iannizzotto, S. Mosconi, and M. Squassina, “H^s versus C^0-weighted minimizers,” *NoDEA Nonlinear Diff. Eqs Appl.*, **22**, No. 3, 477–497 (2015).
9. S. B. Kolonitskii, “Multiplicity of solutions of the Dirichlet problem for an equation with the p-Laplacian in a three-dimensional spherical layer,” *Algebra Analiz*, **22**, No. 3, 206–221 (2010).
10. A. Kufner and S. Fuchik, *Nonlinear Differential Equations* [Russian translation], Nauka, Moscow (1988).
11. Y. Y. Li, “Existence of many positive solutions of semilinear elliptic equations on annulus,” *J. Diff. Eqs.*, **83**, No. 2, 348–367 (1990).
12. R. Musina and A. I. Nazarov, “On fractional Laplacians,” *Comm. PDE’s*, **39**, No. 9, 1780–1790 (2014).
13. R. Musina and A. I. Nazarov, “On the Sobolev and Hardy constants for the fractional Navier Laplacian,” *Nonlinear Analysis*, **121**, 123–129 (2015).
14. R. Musina and A. I. Nazarov, “On fractional Laplacians–3,” *J. ESAIM*, **22**, No. 3, 832–841 (2016).
15. R. Musina and A. I. Nazarov, “Variational inequalities for the spectral fractional Laplacian,” *Comp. Math. Math. Phys.*, **57**, No. 3, 373–386 (2017).
16. A. I. Nazarov, “On solutions of the Dirichlet problem for an equation involving the p-Laplacian in a spherical layer,” *Trudy St. Peterburg. Mat. Obshch.*, **10**, 33–62 (2004).
17. R. S. Palais, “The principle of symmetric criticality,” *Comm. Math. Phys.*, **69**, No. 1, 19–30 (1979).
18. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators* [Russian translation], Mir, Moscow (1980).
19. E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean spaces* [in Russian], Mir, Moscow (1974).
20. P. R. Stinga and J. L. Torrea, “Extension problem and Harnack’s inequality for some fractional operators,” *Comm. PDE’s*, **35**, No. 11, 2092–2122 (2010).
21. R. C. A. M. Van der Vorst, “Best constant for the embedding of the space $H^2 \cap H^1_0(\Omega)$ into $L^{2N/(N-4)}$,” *Diff. Integral Eqs*,” **6**, No. 2, 259–276 (1993).
22. G. N. Watson, *A Treatise on the Theory of Bessel Functions. I* [Russian translation], Moscow, Izd. Inostr. Lit. (1949).